Calculus of Variations

Lecture 8

2016-03-23/2018-03-23

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Experimental problems and
Euler's rule.

Added: New (alternative) complete solution
of Dido's problem.
Isoperimetric problems

General formulation Problems where we are asked to minimize the functional

\[ J[y] = \int_a^b F(x, y, y') \, dx \]

subject to

\[ I[y] = \int_a^b G(x, y, y') \, dx = h, \]

for some fixed value \( h \), as well as the usual boundary conditions \( y(a) = \alpha \) and \( y(b) = \beta \), are called isoperimetric problems.

Example The prototype example, and the one giving this class of problems its name, is Dido’s problem:

\[ \min J[y] = -\int_a^b y \, dx \quad (*) \]

subject to \( y \geq 0 \), \( y(a) = y(b) = 0 \), and

\[ I[y] = \int_a^b \sqrt{1+y'^2} \, dx = L \]

*) \( \min -\int_a^b y \, dx = -\max \int_a^b y \, dx \), so Dido’s problem is to maximize the area of the region enclosed by string of fixed length and with endpoint lying on a straight "shore line".
Tool: Optimization with constraints

- **Lagrange multipliers**

Given two $C^1$-functions $f(x, y)$ and $g(x, y)$ defined in an open region $D$ of $\mathbb{R}^2$. Consider the problem

\[(\ast) \min \left\{ f(x, y) \mid (x, y) \in D, \; g(x, y) = h_0 \right\}.
\]

That is, we want to minimize $f(x, y)$ under the constraint that $g(x, y) = h_0$.

Then suppose $(x_0, y_0) \in D$ solves $(\ast)$. If $\nabla g(x_0, y_0) \neq 0$ then there exists a number $\lambda \in \mathbb{R}$ such that

\[\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).\]

The number $\lambda$ is called a **Lagrange multiplier**.
Euler's Rule for Isoperimetrical Problems

We want to establish the following result:

Suppose \( y = \phi(x) \) solves the following isoperimetrical problem:

\[
\text{Minimize } J[y] = \int_a^b F(x, y, y') \, dx
\]

subject to \( y(a) = \alpha, \ y(b) = \beta \) and

\[
I[y] = \int_a^b G(x, y, y') \, dx = h.
\]

Assume that \( \phi \) is not an extremal of \( J[y] \). Then there exists a constant \( k \) such that \( y = \phi(x) \) is an extremal of

\[
\int_a^b \Phi(x, y, y') \, dx
\]

where \( \Phi(x, y, y') = F(x, y, y') - kG(x, y, y') \).

Remark: This means that \( \phi \) satisfies

\[
F\phi - \frac{d}{dx} F\phi' = -1 \left\{ G\phi - \frac{d}{dx} G\phi' \right\},
\]

which reminds us of Lagrange's rule in finite dimensional optimization.
Proof: Let us pick two admissible variations \( \eta_1 \) and \( \eta_2 \) (in \( C^2 \) with \( \eta_1(a) = \eta_1(b) = 0 \)). Define

\[
\tilde{J}(\varepsilon_1, \varepsilon_2) = I[\phi + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2] \quad \text{and} \\
\hat{J}(\varepsilon_1, \varepsilon_2) = I[\phi + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2].
\]

Clearly \( (\varepsilon_1, \varepsilon_2) = (0, 0) \) minimizes \( \tilde{J} \) subject to \( \hat{J}(\varepsilon_1, \varepsilon_2) = \). The assumption that \( \phi \) is not an extremal of \( I \) means that we may choose \( \eta_2 \) such that

\[
\frac{\partial \hat{J}(0,0)}{\partial \varepsilon_2} = \int_a^b \frac{\partial}{\partial \varepsilon_2} \left( \frac{\partial H}{\partial \eta_2} \right) \eta_2 \, dx \neq 0.
\]

Therefore \( \nabla \hat{J}(0,0) = \left( \frac{\partial \hat{J}(0,0)}{\partial \varepsilon_1}, \frac{\partial \hat{J}(0,0)}{\partial \varepsilon_2} \right) \neq (0, 0) \).

By Lagrange's rule there exists \( \lambda \in \mathbb{R} \) such that

\[
\nabla \tilde{J}(0,0) = \lambda \nabla \hat{J}(0,0).
\]

\[
\begin{align*}
\int_a^b \left\{ \frac{\partial}{\partial \varepsilon_1} \left( \frac{\partial H}{\partial \eta_1} \right) \eta_1 + \frac{\partial}{\partial \varepsilon_2} \left( \frac{\partial H}{\partial \eta_2} \right) \eta_2 \right\} \, dx &= \lambda \int_a^b \left\{ \frac{\partial}{\partial \varepsilon_1} \left( \frac{\partial H}{\partial \eta_1} \right) \eta_1 + \frac{\partial}{\partial \varepsilon_2} \left( \frac{\partial H}{\partial \eta_2} \right) \eta_2 \right\} \, dx \\
\int_a^b \frac{\partial}{\partial \varepsilon_1} \left( \frac{\partial H}{\partial \eta_1} \right) \eta_1 \, dx &= \lambda \int_a^b \frac{\partial}{\partial \varepsilon_1} \left( \frac{\partial H}{\partial \eta_1} \right) \eta_1 \, dx \\
\int_a^b \frac{\partial}{\partial \varepsilon_2} \left( \frac{\partial H}{\partial \eta_2} \right) \eta_2 \, dx &= \lambda \int_a^b \frac{\partial}{\partial \varepsilon_2} \left( \frac{\partial H}{\partial \eta_2} \right) \eta_2 \, dx
\end{align*}
\]
Observe that $\eta_1$ in (1) is an arbitrary admissible variation whereas $\eta_2$ is fixed such that

$$\int_a^b \left\{ g \phi - \frac{d}{dx} (F-1\lambda) \phi \right\} \eta_2 \, dx \neq 0,$$

and therefore determines the value of $\lambda$. We find that

$$\int_a^b \left\{ (F-1\lambda) \phi - \frac{d}{dx} (F-1\lambda) \phi \right\} \eta \, dx = 0$$

for all admissible variations $\eta$. If we set $\Phi = F-1\lambda$ we find that

$$\Phi \phi - \frac{d}{dx} \Phi \phi' = 0,$$

as claimed.

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**Example (Solving Dido's problem)**

By the theorem we need to determine extremals of $I(x,y,y') = y - \frac{1}{\sqrt{1+y'^2}}$ which satisfy $y(a)=y(b)=0$, $y(a) > 0$ and $I'(y) = 1$.

(Euler) $0 = \Phi y - \frac{d}{dx} \Phi y'$ has the first integral $y' y'' - \Phi = \text{const}$. Therefore

$$\lambda = \text{const} = y' - \frac{\Phi y'}{\sqrt{1+y'^2}} + y + \sqrt{1+y'^2}$$

$$= \frac{1}{\sqrt{1+y'^2}} + y.$$
An alternative solution to Dido's problem

Problem: We want to solve the following variant of the isoperimetric problem:

$$\max_{a} \int_{a}^{b} y(x) \, dx$$

subject to the isoperimetric condition

$$\int_{a}^{b} \sqrt{1 + y'(x)^2} \, dx = L, \quad L > 0.$$ 

Here $y \in C_1$, $y(a) = y(b) = 0$ and $y(x) > 0$ for $a < x < b$.

Solution: We use Euler's rule and introduce the auxiliary Lagrangian

$$\Phi = y - \lambda \sqrt{1 + y'^2},$$

where $\lambda \in \mathbb{R}$ is the (Lagrange-) multiplier.

The solution of our problem, should it exist, must satisfy Euler's equation

$$\Phi_y - \frac{d}{dx} \Phi_{y'} = 0.$$ 

Since $\Phi_y = 1$ and $\Phi_{y'} = -\lambda y' / \sqrt{1 + y'^2}$, we get

$$1 = \frac{d}{dx} \left( \frac{-\lambda y'}{\sqrt{1 + y'^2}} \right)$$

so

$$x - k = \frac{-\lambda y'}{\sqrt{1 + y'^2}}, \quad k \in \mathbb{R}.$$
Before we go on, notice that the identity (\(*)\) implies that \(\lambda > 0\). In fact, \(\lambda = 0\) is impossible since the left hand side is nonzero. Moreover, since \(y > 0\) in \((a, b)\) we have \(y'(a) \geq 0\) and \(y'(b) \leq 0\). Hence \(y'(\xi) = 0\) for some \(\xi, a < \xi < b\), by the intermediate value theorem.

Clearly \(k = \xi\). Setting \(x = a\) (or \(x = b\)) in (\(*)\) and using \(a - k < 0\) implies \(\lambda > 0\) (and \(y'(a) > 0\)).

Now, \(\xi \mapsto \xi / \sqrt{1 + \xi^2}, \xi \in \mathbb{R}\), is strictly monotone, so we may solve (\(*)\) for \(y'\) and yield

\[
y' = \frac{(k-x)/\lambda}{\sqrt{1 - (x-k)^2/\lambda^2}}
\]

\[
= \frac{k-x}{\sqrt{\lambda^2 - (x-k)^2}}
\]

Now one more integration yields the solution

\[
y - l = \sqrt{\lambda^2 - (x-k)^2}
\]

where \(l \in \mathbb{R}\) is a second constant of integration.
Since \((y-l)^2 + (x-k)^2 = l^2\) we see that the graph of \(y(x)\) is the arc of a circle with center at \((k,l)\) and radius \(l\).

Using the end point conditions\(y(a) = y(b) = 0\) we get

\[
\begin{align*}
-\ell &= \sqrt{(a-k)^2 - l^2}, \\
-\ell &= \sqrt{(b-k)^2 - l^2},
\end{align*}
\]

so \(\ell \leq 0\), and we proceed as follows:

\[
\begin{align*}
(a-k)^2 + \ell^2 &= l^2, \\
(b-k)^2 + \ell^2 &= l^2, \\
\ell \leq 0.
\end{align*}
\]

\[
\begin{align*}
(a-k)^2 - (b-k)^2 &= 0, \\
(b-k)^2 + \ell^2 &= l^2, \\
\ell \leq 0.
\end{align*}
\]

\[
\begin{align*}
k &= \frac{a+b}{2}, \\
\ell^2 &= l^2 - \frac{(b-a)^2}{2}, \\
\ell \leq 0.
\end{align*}
\]
It remains to determine the radius $\lambda$ such that $y$ satisfies the isoperimetrical condition. In terms of the angle $\theta$ in the figure, the length of the circular arc is $2\lambda \theta$, hence we want to find $\lambda > 0$ s.t.

$$2\lambda \theta = L$$

We have $\lambda \sin \theta = (b-a)/2$, and there is a one-to-one correspondence between $\lambda \in (\frac{b-a}{2}, \infty)$ and $\theta \in (0, \frac{\pi}{2})$. Therefore it suffices to determine $\theta$ which satisfies

$$\frac{\theta}{\sin \theta} = \frac{L}{b-a}, \quad \theta \in (0, \frac{\pi}{2}).$$

The range of $f(\theta) = \theta/\sin \theta$ equals $(1, \frac{\pi}{2}]$. To see this, notice that

$$f'(\theta) = \frac{\sin \theta - \theta \cos \theta}{\sin^2 \theta} = \frac{\cos \theta}{\sin^2 \theta} (\tan \theta - \theta) > 0,$$

so $f$ is strictly increasing, and

$$\lim_{\theta \to 0^+} f(\theta) = 1 \quad \text{and} \quad f(\frac{\pi}{2}) = \frac{\pi}{2}.$$  

Our conclusion is that we have a unique admissible sol. $y(x)$ when $1 \leq \frac{L}{b-a} \leq \frac{\pi}{2}$.

Here $1=\frac{L}{b-a}$ corresponds to $y \equiv 0$. 