Calculus of Variations: Assignment 2.

This is the second part of the assignments used in the examination of the course. The quality of the solutions may be taken into account, if necessary. *You are allowed to talk to friends and people on the course about the problems (indicate this on the cover sheet). However, each individual has to hand in his or her own personal solutions* Clearly written and edited solutions are appreciated. The solutions should be handed in to the teacher or the secretaries on the fifth floor of the Math. Building **no later than Friday 17 May 2019 at noon (12:00 AM).**

Problem 1. (An isoperimetrical problem) Find an admissible extremal to problem of minimizing the functional

$$J[y] = \int_0^2 y'^2 \, dx$$

subject to the constraint $I[y] = 6$, where

$$I[y] = \int_0^2 y \, dx,$$

and the endpoint conditions $y(0) = y(2) = 1$.

Problem 2. (Transversality conditions) Determine the extremals of the functional

$$I[u] = \int_0^b u'(x)^2 \, dx$$

which satisfies the end point conditions $u(0) = 0$ and $u(b) = \psi(b)$, for some $b > 0$, where ψ is the function given by $\psi(x) = x^2/3 + 1$.

Problem 3. (The fundamental sufficiency condition) Consider the problem of minimizing the functional

$$J[y] = \int_0^1 \frac{y'(x)^2}{y(x)} \, dx$$

subject to $y(x) > 0$ for $0 < x < 1$, and the boundary conditions $y(0) = 1$ and $y(1) = 4$.

(a) Find an admissible extremal y_0 for the problem.

(b) Describe in words and pictures what is meant by a *strong local minimum* of a variational problem.

(c) Choose a field of extremals into which y_0 is embedded and compute its direction field ρ as well as the corresponding Weierstrass' excess function $\mathscr{E}$.

(d) Use the fundamental sufficiency condition to show that y_0 is a strong local minimum for the problem. Does it follow from this condition that y_0 is a global minimum of the problem?

Problem 4. (Sufficient conditions for problems with a free endpoint) Consider the problem of minimizing the functional

$$I[y] = \int_0^1 \frac{1}{2} y'^2 + y \, dx$$

subject to $y(0) = 0$ and with a free endpoint at $x = 1$.

(a) Determine the extremals of I and single out the admissible extremals of the problem.

(b) Construct a field of extremals such that (i) the admissible extremal is embedded in the field, and (ii) all extremals in the field are perpendicular to the line $x = 1$. Also, compute the corresponding direction field.

(c) Write down the Hilbert invariant integral as a line integral of a differential form,[1]

$$K[\Gamma] = \int_\Gamma P\,dx + Q\,dy.$$

Give a direct verification of the invariance of K by finding a potential function $U(x, y)$ such that $\nabla U = (P,Q)$. What is the value of U along the vertical line $x = 1$?

(d) Follow the proof of the fundamental sufficiency condition given in the course and show that the admissible extremal found in (a) is a strong local minimum. Is the minimum a global one? (Hint: Use the information in (c) to get further invariance properties of K.)

Problem 5. (An obstacle problem.) Minimize the following functional,

$$E[u] = \int_0^4 \frac{1}{2}\tau u'(x)^2 + \rho g u(x)\,dx$$

over continuously differentiable functions u satisfying $u(x) \geq 0$ for $0 \leq x \leq 4$ and the boundary conditions $u(0) = 2$ and $u(4) = 0$. For simplicity we may assume that $\tau/\rho g = 1$. Provide an argument for the optimality of the found solution. What would happen if the right end point is moved leftwards?

(The problem may be interpreted as a simplified mechanical model for the minimum energy configuration for at heavy spring of density ρ (mass/length) and modulus of elasticity τ which hangs in a vertical plane, where it is attach to the wall ($u(0) = 2$) at one end and at ground level ($u(4) = 0$) at the other end, and which is not allowed to pass through the floor ($u \geq 0$). Here g denotes the gravitational constant.)

[1] Recall that if Γ is the graph $y = y(x)$ of a function y, then $K[\Gamma] = K[y] = \int_0^1 P + y'Q\,dx$.