

## Calculus of Variations: Assignment 1.

The assignments are *a part of the examination* of the course in calculus of variations. This is the first part of two. A second set of approximately the same size is handed out later (in April). To pass this part of the examination you are required to solve approximately 75 per cent of the problems *correctly*. The quality of your solutions may be taken into account. *You are allowed to talk to friends and fellow students about the problems.* However, each individual has to hand in his or her own personal solutions. *For accountability reasons, list the persons you have been sharing ideas with.* Clearly presented and *well edited* solutions are appreciated. The solutions (written or printed out on paper!) should be handed in to the teacher or to the secretaries on the fifth floor of the Mathematics Building **no later than at noon, Friday March 15, 2019.**

**Problem 1.** (Warm-up!) Determine the admissible extremals to the problem of minimizing the functional

$$J[y] = \int_0^3 \frac{\sqrt{1+y'^2}}{y} dx,$$

subject  $y(x) > 0$  on  $0 < x < 3$ , and the end-point conditions  $y(0) = 2$  and  $y(3) = 1$ . Sketch the graphs of the found extremals and describe them in geometric terms.

**Problem 2.** Minimize the functional

$$J[y] = \int_1^2 \frac{1}{2} x^2 y'^2 + x^2 y' dx,$$

subject to  $y \in C^1$  with  $y(1) = 0$  and  $y(2) = 1$ . In particular, use a direct verification to show optimality of the admissible extremal as well as uniqueness of the solution.

**Problem 3.** Consider the following variant of du Bois Reymond's lemma: Suppose  $M : [a, b] \rightarrow \mathbf{R}$  is a piecewise continuous function such that

$$\int_a^b M(x) h'(x) dx = 0,$$

for all  $h \in D_1$  with  $h(a) = h(b) = 0$ . Then there exists a real number  $C$  such that  $M(x) = C$  for all  $a \leq x \leq b$ .

Prove this theorem using test functions  $h$  of the form indicated in the figure on next page. Your solution should recall the definition of the classes of piecewise continuous functions ( $D$ ) and piecewise continuously differentiable functions ( $D_1$ ). For convenience we assume that  $M$  is right continuous:  $M(x) = \lim_{\epsilon \rightarrow 0^+} M(x + \epsilon)$  for all  $x \in [a, b)$ . The mean value theorem for integrals may come in handy!

**Problem 4.** (Corners) Decide which of the following functionals admit broken extremals or not:

$$I[y] = \int_0^1 y'^6 - 5y'^4 + 15y'^2 dx, \quad J[y] = \int_0^1 y'^3 dx, \quad \text{and} \quad K[y] = \int_0^1 \cos(y') dx.$$

Either construct an example of a broken extremal or rule out its existence using available theory. (The geometrical interpretation of necessary conditions are allowed tools.) Moreover, for each of the functionals, determine an admissible extremal among functions  $y \in D_1$  which satisfies  $y(0) = 0$  and  $y(1) = \pi/2$ .

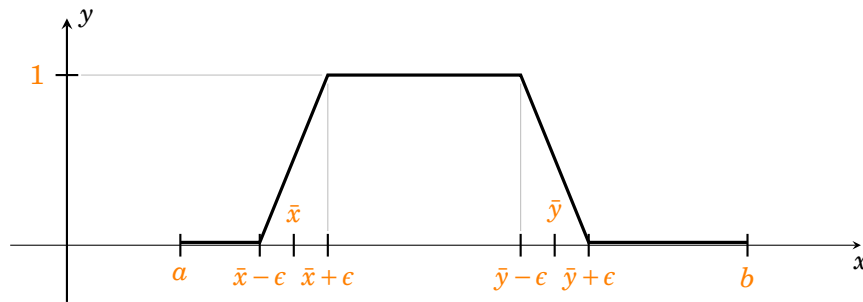


Figure 1: A piecewise linear test function. Here  $\bar{x}$  and  $\bar{y}$  are arbitrary numbers in the interval  $[a, b]$  and  $\epsilon > 0$  is a sufficiently small positive number.

**Problem 5.** (Snell's law) Consider the functional:

$$T[y] = \int_a^b n(x) \sqrt{1 + y'(x)^2} dx$$

where  $y \in D_1$  satisfies the end point conditions  $y(a) = \alpha$  and  $y(b) = \beta$ . The function  $n : [a, b] \rightarrow (0, +\infty)$  is given by:

$$n(x) = \begin{cases} n_a & \text{for } a \leq x < c, \text{ and} \\ n_b & \text{for } c \leq x \leq b, \end{cases}$$

where  $c, a < c < b$ , is a fixed number. This problem uses Fermat's principle—that a light follows the path with shortest transit time—to model the situation when light emanating from a source at  $(a, y_a)$  in one medium (air) is received by a sensor at  $(b, y_b)$  in another medium (water). The function  $n(x)$  is the refractive index at  $(x, y(x))$ . Sketch the light ray (no detailed calculations needed) and relate the first Weierstrass-Erdmann condition to Snell's law of refraction.

**Problem 6.** Solve the following free end-point problem: Minimize the integral

$$L[x] = \int_0^1 \frac{1}{2} \dot{x}^2 + x dt,$$

over functions  $x \in C^1$  such that  $x(0) = 0$ . Don't forget to verify optimality of the obtained extremal. (Hint: What do we know about the admissible variations of  $x(t)$  in this case?)

**Problem 7.** Minimize the functional

$$J[y] = \int_1^2 \frac{1}{2} x^2 y'^2 + x^2 y' dx,$$

subject to  $y \in C^1$  and  $y(1) = 0$ .

**Problem 8.** (Higher order variational problems) Suppose  $F = F(x, y, z, w)$  is a sufficiently smooth function and consider the problem of minimizing the functional

$$J[y] = \int_a^b F(x, y(x), y'(x), y''(x)) dx$$

over the set of functions  $y \in C^2$  which satisfies the end point-conditions  $y(a) = \alpha$ ,  $y'(a) = \bar{\alpha}$ ,  $y(b) = \beta$  and  $y'(b) = \bar{\beta}$ .

**a)** Assume that  $\phi(x)$  is a solution to the above minimization problem. Derive the corresponding Euler equation for  $\phi$ . You may assume that  $\phi \in C^4$ , i.e., that  $\phi$  is at least four times continuously differentiable.

**b)** Use the above theory to determine the admissible extremals of the functional

$$E[y] = \int_{-\ell}^{\ell} \frac{1}{2} \mu y''(x)^2 + \rho y(x) dx,$$

satisfying  $y(\ell) = y(-\ell) = 0$  and  $y'(\ell) = y'(-\ell) = 0$ . This is a model for the deflection of a (horizontal) narrow beam (a spline) of length  $2\ell$  under the influence of gravity. The numbers  $\mu$  and  $\rho$  are positive physical constants. (Make a *qualitatively* precise sketch of the admissible extremal.)

**c)** Modify the above theory in order to find admissible extremals for the spline functional in b) satisfying the boundary conditions  $y(\ell) = y(-\ell) = 0$  and  $y'(-\ell) = 0$  (no assignment of  $y'$  at  $x = \ell$  this time.) (Again, sketch your solution.)

**Problem 9.** (Several independent variables) Let  $U = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$  denote the (open) unit disc in the plane and let  $\partial U = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$  be the boundary of  $U$  (the unit circle) and set  $\bar{U} = \partial U \cup U$ . The set of continuously differentiable functions  $u(x, y)$  on  $\bar{U}$  is denoted  $C^1(\bar{U})$ . Derive and solve Euler's equation corresponding to the minimization of the integral

$$I[u] = \iint_U \frac{1}{2} |\nabla u(x, y)|^2 - \sqrt{x^2 + y^2} u(x, y) dx dy$$

over the set of functions  $u$  in  $C^1(\bar{U})$  with  $u = 0$  on  $\partial U$ . Show that the extremal is the unique minimizer. (In deriving Euler's equation you may assume that the minimizer is  $C^2$ , and the multi-dimensional version of the Fundamental lemma of the calculus of variations may be taken for granted. The divergence theorem of Gauss and the formula  $\nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \nabla^2 u$  may turn out to be useful. Here  $\nabla^2 u = u_{xx} + u_{yy} = \Delta u$  are various notations used for the Laplacian of  $u$ . What about optimality of the found extremal?

**Problem 10.** (Optimal constant) It can be shown that the value at the origin of any real-valued function  $u \in C^1(0, 1) \cap C([0, 1])$  can be bounded by integrals of  $u$  and  $u'$  in the following manner,

$$\frac{1}{2} u(0)^2 \leq \int_0^1 u(x)^2 dx + \int_0^1 u'(x)^2 dx.$$

However, the constant  $1/2$  in the left hand-side of this inequality is not optimal. Indeed, one can show that the inequality

$$C u(0)^2 \leq \int_0^1 u(x)^2 dx + \int_0^1 u'(x)^2 dx. \quad (1)$$

holds with  $C = 2/(1 + \sqrt{5}) \approx 0.61$ , but this constant is not optimal either!

Determine the optimal constant  $C$  in the inequality (1) by formulating and solving an appropriate variational problem.

**Problem 11.** (Vector-valued arguments) Let  $\mathbf{u}(t) = (x(t), y(t))$  denote a curve in the plane parametrized with a parameter  $t$ , where  $0 \leq t \leq 1$ . For all such curves, which does not pass through the origin, define the functional:

$$A[\mathbf{u}] = \frac{1}{2} \int_0^1 \frac{|\dot{\mathbf{u}}(t)|^2}{|\mathbf{u}(t)|^2} dt = \frac{1}{2} \int_0^1 \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{x(t)^2 + y(t)^2} dt.$$

where  $\dot{\cdot} = d/dt$ .

**a)** Determine all extremals of  $A$ . (Hint: express the curve in polar coordinates  $(r(t), \varphi(t))$ .) Describe the solutions geometrically.

**b)** Find all the admissible extremals satisfying  $\mathbf{u}(0) = (0, 2)$  and  $\mathbf{u}(1) = (0, 1)$ .