Exercises

1. Let \( m = m(x) \) be a continuous function defined on \([a, b]\) and define
\[
J[y] = \int_a^b \frac{1}{2} (y'(x) - m(x))^2 \, dx
\]
Determine the extremals of this functional (Using, for instance, Euler’s equation in integrated form by Du Bois Reymond). Find an example (i.e., a choice of \( a, b, m \) and possibly end-point conditions) which shows that the extremals may belong to \( C^1 \) but not to \( C^2 \).

2. During most of the 19th century many mathematicians seemed to believe that any functional \( J \) with a non-negative integrand must, self-evidently, possess an admissible extremal \( y_0 \) which minimizes the functional. Around 1870 Karl Weierstrass, then professor in Berlin, challenged this belief with the following problem: Minimize the integral
\[
J[y] = \int_{-1}^1 x^2 y'(x)^2 \, dx
\]
over the set of functions \( y \in C^1 \) which satisfies the end point conditions \( y(-1) = -1 \) and \( y(1) = 1 \).

   a) Verify that \( J \) is bounded below by zero, i.e., \( J[y] \geq 0 \).

   b) Compute (as Weierstrass did) the values \( J[y_\epsilon] \) where
   \[
y_\epsilon(x) = \frac{\arctan(x/\epsilon)}{\arctan(1/\epsilon)}, \quad \epsilon > 0.
   \]
   Determine the greatest lower bound on \( J[y] \) (denoted \( \inf_y J[y] \)) when \( y \) ranges over the set of admissible functions.

   c) Does there exist an admissible function \( y_0 \) such that \( J[y_0] = \inf_y J[y] \)?

   d) The sequence of functions \( \{y_\epsilon\}_{\epsilon > 0} \) in b) is an example of a so-called minimizing sequence for \( J \). Verify that the following construction also gives a minimizing sequence: Take any non-decreasing function \( \phi \in C^1(\mathbb{R}) \) which satisfies \( \phi(x) = -1 \) for \( x \leq -1 \) and \( \phi(x) = 1 \) for \( x \geq 1 \) (draw a figure!) and set
   \[
   \phi_\epsilon(x) = \phi(x/\epsilon), \quad 0 < \epsilon \leq 1.
   \]

3. Derive Euler’s equations for the problem of minimizing curve length
\[
L[r] = \int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} \, dt
\]
for curves \( r : [0, 1] \to \mathbb{R}^3, \; r(t) = (x(t), y(t), z(t)) \) lying on the unit sphere \( S^2 : x^2 + y^2 + z^2 = 1 \).

4. Show that the functional defined in Problem 3 (and any other curve length functional, for that matter) satisfies

\[ L[r] = L[r \circ \varphi], \]

where \( \varphi : [0, 1] \to [0, 1] \) is any strictly increasing continuously differentiable function satisfying \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). The curve \( r \circ \varphi \) is just a reparametrized version of \( r \). What is the geometric significance of this property of the length functional \( L \)?