Calculus of Variations
Lecture 7
2015-01-27
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H. Kot: Ch 1, p. 1-25 +
Ch 2, p. 27-31.

(H. Mesterton-Gibbons: Lecture 7 +
part of Lecture 2, p. 1-8.)

Separate sheet with recommended exercises.
The Classical Problems

We begin with a number of examples that will help us understand which types of problems are studied in the calculus of variations.

**Shortest path** Find the curve of minimum length among all curves connecting two points \( A: (a,\alpha) \) and \( B: (b,\beta) \) in the plane.

\[
\begin{align*}
\text{We consider curves } & \gamma \text{ which are graphs of some function } u: [a, b] \to \mathbb{R} \\
\gamma &= \{ (x, y) | y = u(x), a \leq x \leq b \} \\
\text{We usually write } y &= u(x) \text{ or simply } y = u(x).
\end{align*}
\]

\[
\text{length } (\gamma) = \int_a^b \sqrt{1 + (u'(x))^2} \, dx
\]

We require \( u(a) = \alpha \) and \( u(b) = \beta \). (Ancient origin)

**Minimal surface of rotation** Let \( y: [a, b] \to \mathbb{R} \) be a function such that \( y(a) = \alpha \), \( y(b) = \beta \) and \( y'(x) > 0 \) for \( a \leq x \leq b \). If the curve \( y = y(x) \) is rotated around the \( x \)-axis, then a surface \( S \) (a surface of rotation) is generated, whose area is

\[
\text{area}(S) = 2\pi \int_a^b y(x) \sqrt{1 + y'(x)^2} \, dx
\]

Problem: Determine the function \( y \) corresponding to the surface of rotation with the smallest area. (Euler, 1743)
**Fermat's Principle** According to this principle, a beam of light will travel from a point \( A \) to another point \( B \) along the path where the time of transit is least possible.

![Diagram showing light path and variables](image)

Optically inhomogeneous medium: \( c = c(x, y) \) speed of light at \((x, y)\). \( n(x, y) = \frac{1}{c(x, y)} \) is the index of refraction.

\[
\frac{dt}{c} = n \, ds \\
\text{Transit time } (\gamma) = \int_{a}^{b} \frac{ds}{c(x, y)} = \int_{a}^{b} \frac{\sqrt{1+(y')^2}}{c(x, y)} \, dx,
\]

where \( \gamma : y = y(x) \) and \( y(a) = \alpha, y(b) = \beta \).

**The Brachistochrone** Find the function \( y(x) \) which minimizes the transit time for a pearl rolling along a frictionless slide in the form of the curve \( y = y(x) \) from \( A : (0,0) \) to \( B : (\beta, \beta) \).

![Diagram showing pearl path and variables](image)

\[
\frac{dt}{\nu} = \\
\nu = \nu(x) = \text{speed of pearl when passing the vertical at } x
\]

Conservation of energy:

\[
\frac{1}{2} mv^2 - mg y = \text{const} = 0
\]

since \( \nu(0) = 0, y(0) = 0 \).

Transit time \( (\gamma) = \frac{1}{2g} \int_{a}^{b} \frac{\sqrt{1+(y')^2}}{\sqrt{y}} \, dx \) (J. Bernoulli, 1696)
The Isoperimetric Problem

Given a string of a fixed length $l$. In what shape should the string be laid out on the ground in order that the enclosed area is maximum?

Maximize

\[
\text{Area} = \int_a^b y(x) \, dx
\]

Subject to $y(a) = y(b) = 0$ and

\[
\text{Length} = \int_a^b \sqrt{1 + y'(x)^2} \, dx = l.
\]

(Dido ~ 850 B.C.)

Problem with a variable end point

Consider a cross-country race where all runners start at the point $A$ and the first person to cross the finish line $l$ is proclaimed winner.

Due to variations in topography the maximum running speed at $(x, y)$ is $v(x, y) > 0$.

Along a given path $y = w(x)$ the best finish time one may achieve is

\[
J[w] = \int_{\alpha}^{\beta} \frac{ds}{v} = \int_{\alpha}^{\beta} \frac{\sqrt{1 + w'(x)^2}}{v(x, w(x))} \, dx.
\]

Problem Minimize $J[w]$ subject to $w(a) = \alpha$.

(Determining the abscissa $\beta$ of the point where the winner crosses $l$ is now a part of the problem.)
Vector valued Problems

Consider a point particle of mass \( m \) which moves in space, under the influence of a conservative force field, from a point A to another point B during fixed time interval.

Let \( \mathbf{x}(t) = (x(t), y(t), z(t)) \) denote the position of the particle at time \( t \) with respect to some Cartesian coordinate system. Let \( U(\mathbf{x}) = U(x, y, z) \) denote the potential function associated with the force field.

Hamilton's principle asserts that the particle will move along the trajectory which minimizes the integral

\[
I = \int_{t_0}^{t_f} T - U \, dt
\]

\[
= \int_{t_0}^{t_f} \frac{1}{2} m \left( \frac{d}{dt} \mathbf{x}(t) \right)^2 - U(\mathbf{x}(t)) \, dt.
\]

This is a vector valued problem.

\( T = \frac{1}{2} m \left( \frac{d}{dt} \mathbf{x} \right)^2 \) kinetic energy

\( U = U(\mathbf{x}) \) potential energy

\( L = T - U \) Lagrangian of the system.
Pointwise constrained problems

Suppose we want to determine the shortest path on a surface \( S : g(x, y, z) = 0 \) connecting two points \( A \) and \( B \) on the surface.

Assuming that we may write \( \gamma : (x, y, z) = (x, y(x), z(x)) \) we get the constrained minimization problem:

Minimize \( I = \int_a^b \sqrt{1 + y'(x)^2 + z'(x)^2} \, dx \)

Subject to \( (a, y(a), z(a)) = A \), \( (b, y(b), z(b)) = B \) and

\( (*) \quad g(x, y(x), z(x)) = 0 \)

The constraint \( (*) \) is pointwise and corresponds to the requirement \( y \in S \).
**Function Classes**

What kind of functions are allowed to compete in the above optimization problems? The functions that we are going to consider are required to have certain regularity properties (i.e., certain degrees of differentiability and continuity).

For real numbers $a, b$ let $(a, b)$ denote the open interval $\{ x \mid a < x < b \}$ and $[a, b]$ denote the closed interval $\{ x \mid a \leq x \leq b \}$.

$$C(a, b) = \text{The class of continuous functions on } (a, b).$$

$$C_1(a, b) = \text{The class of continuous functions with a continuous derivative on } (a, b). \text{ [Also denoted } C_1'(a, b)\text{.]}$$

$$C = C([a, b]) = \text{The class of continuous functions on the closed interval } [a, b].$$

$$C_1 = C_1([a, b]) = \text{The class of continuous functions on } [a, b] \text{ whose derivative (one-sided derivatives at the end points } x = a, x = b \text{) are also continuous on } [a, b]. \text{ That is, } f \in C_1 \text{ if } f \in C \text{ and } f' \text{ exists and is in } C, \text{ where } f'(a) \text{ and } f'(b) \text{ are one-sided derivatives.} \text{ [C]_1}$$
\[ D_1 = D_1([a,b]) = \text{The class of piecewise differentiable functions. That is, } f \in D_1 \text{ if } f \in C \text{ and there exists a natural number } N \text{ and reals } a = x_0 < x_1 < \ldots < x_N = b \text{ such that } f \in C_i([x_{i-1}, x_i]) \text{ for } i = 1, \ldots, N. \] (The number } N \text{ depends on } f.\]

We shall also encounter } C_2(a, b) \text{ and } C_2([a, b]), \text{ which are defined in a similar manner as } C_1(a, b) \text{ and } C_1.

**Examples** \[ J = [a, b] = [-1, 1]. \]

a) \[ f(x) = x^2 \in C_2 \]

b) \[ f(x) = \begin{cases} x^2, & x > 0 \\ 0, & x \leq 0 \end{cases} \in C_1 \]

but \[ f \notin C_2. \]

c) \[ f(x) = |x| \in D_1 \]

but \[ f \notin C_1. \]

d) \[ f(x) = \begin{cases} \sqrt{x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \in C \]

but \[ f \notin D_1 \text{ (Why?)} \]

e) \[ f(x) = \frac{1}{1-x^2} \in C_1(a, b) \]

but \[ f \notin C_1 \text{ (Why?)} \]

nor \[ f \in C \text{ (Why?)} \].

**Generally:** \[ C_2 \subset C_1 \subset D_1 \subset C. \]
Functionals
Let $K$ denote a class of functions, e.g. $K = C$, $K = C_1$ or something similar.

A function $J : K \to \mathbb{R}$ is called a functional

Hence a functional is "a function of a function" $J$ takes an entire function $y \in K$ as argument and delivers a real number $J[y]$. (The use of square brackets for functionals is traditional.)

Examples

a) Curve length:

$J : C_1 \to \mathbb{R}$, $J[u] = \int_a^b \sqrt{1 + u'(x)^2} \, dx$.

b) Area of surface of rotation:

$K = \{ y \in C_1 : y(x) > 0, \ a \leq x \leq b \}$

$J : K \to \mathbb{R}$, $J[y] = 2\pi \int_a^b y \sqrt{1 + y'^2} \, dx$.

c) Brachistochrone:

$K = \{ y \in C_1(a,b) \cap C : y(x) > 0, \ a < x < b \}$

$J : K \to \mathbb{R}$, $J[y] = (2g)^{-1/2} \int_a^b \frac{\sqrt{1 + y'^2}}{y} \, dx$.

d) $J : C \to \mathbb{R}$, $J[y] = \int_a^b p(x)y(x) \, dx$, where $p \in C$, is a linear functional.

e) $J : C_1 \to \mathbb{R}$, $J[y] = \int_a^b p(x)y(x) + q(x)y'(x) \, dx$, where $p, q \in C$, is also a linear functional, i.e. $J[c_1 y_1 + c_2 y_2] = c_1 J[y_1] + c_2 J[y_2]$ for $c_1, c_2 \in \mathbb{R}$ and $y_1, y_2 \in C_1$. 

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(Examples, contd.)

f) \( J : C \rightarrow \mathbb{R} \)

\[
J[y] = \int_a^b \left( f(x)y'(x)^2 + 2g(x)x(y(x) + r(x)y(x)^2) \right) \, dx,
\]

where \( f, g, r \in C \), is a quadratic functional. We are going to study linear and quadratic functionals later on.

A more general type of functional

Let \( U \subset \mathbb{R}^3 \) an open set

\( F : U \rightarrow \mathbb{R} \) a given function of three variables ; \( F = F(x, y, z) \).

Then the functional

\[
J[y] = \int_a^b F(x, y(x), y'(x)) \, dx
\]

is defined on

\[
\mathcal{X} = \left\{ y \in C_1 \mid (x, y(x), y'(x)) \in U \right\}
\]

Remark. All earlier examples are of this form. For instance the area functional

\[
J[u] = 2\pi \int_0^6 \sqrt{1+u'^2} \, dx \quad \text{has}
\]

\[
F(x, y, z) = 2\pi y \sqrt{1+z^2} \quad \text{(indep. of } x)\]

and the quadratic functional above has

\[
F(x, y, z) = p(x)z^2 + 2r(x)yz + q(x)y^2.
\]

\( F \) is called the Lagrange function, or the Lagrangian, of the functional \( J \), or integrand of \( J \).
The "simplest" variational problem

Find the function \( y \) which minimizes

\[
J[y] = \int_a^b F(x, y, y') \, dx
\]

subject to \( y \in \mathcal{E} \) and \( y(a) = \alpha, y(b) = \beta \).

The admissible functions

for the problem.

We shall study this (general) problem theoretically in the coming lectures.

**Trial Functions** We propose to minimize

\[
J[y] = \int_0^1 \sqrt{1 + y'^2} \, dx
\]

subject to \( y(0) = 0, y(1) = 1 \)

by considering trial functions of the form

\[
y_\varepsilon(x) = x^\varepsilon, \quad \varepsilon > 0.
\]

Substituting \( y_\varepsilon \) for \( y \) in \( J[y] \) gives

\[
j(\varepsilon) = J[y_\varepsilon] = \int_0^1 \frac{1 + \varepsilon^2 x^{2\varepsilon-2}}{x^\varepsilon} \, dx
\]

which may be evaluated numerically

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j(\varepsilon) )</td>
<td>2.690</td>
<td>2.634</td>
<td>2.602</td>
<td>2.587</td>
<td>2.589</td>
<td>2.608</td>
<td>2.647</td>
</tr>
</tbody>
</table>

In fact, \( j(\varepsilon) \) is minimized for \( \varepsilon^* \approx 0.539726 \)

with \( j(\varepsilon^*) \approx 2.58598 \).

Since we only consider a subset of the admissible functions, we, of course, miss the true minimum \( j^* = 2.5819045 \), but the approximation is not awfully bad!
Straight lines give shortest paths

We return to the shortest path problem; to minimize

\[ J[u] = \int_a^b \sqrt{1 + u'(x)^2} \, dx \]

subject to \( u(a) = \alpha \) and \( u(b) = \beta \).

Guess: The straight line \( y = \phi(x) \)
where \( \phi(x) = \alpha + \frac{\beta - \alpha}{b - a} (x - a) \), \( a \leq x \leq b \),
is shortest.

The guess is correct and \( \phi \) is the unique solution.

To prove this we need Cauchy-Schwarz' ineq.: If \( \bar{x} = (x_1, x_2, \ldots, x_n) \) and \( \bar{y} = (y_1, y_2, \ldots, y_n) \) are vectors in \( \mathbb{R}^n \), then

\[ \bar{x} \cdot \bar{y} \leq |\bar{x}| |\bar{y}| \]

with equality only if \( \bar{x} = \lambda \bar{y} \) for some non-negative real number \( \lambda \). (We assume \( \bar{y} \neq 0 \).)

Proof of Th. Clearly \( J[\phi] = \int_a^b \sqrt{1 + (\frac{\beta - \alpha}{b - a})^2} \, dx = \sqrt{(b - a)^2 + (\beta - \alpha)^2} \). We have to show that \( J[\phi] \leq J[u] \) for any admissible \( u \) and that equality holds only if \( \phi = u \). Notice that

\( \beta - \alpha = u(b) - u(a) = \int_a^b u'(x) \, dx \) and \( b - a = \int_a^b dx \).

We get

\[ (b - a)^2 + (\beta - \alpha)^2 = \int_a^b (b - a) \, dx + (\beta - \alpha) \int_a^b u'(x) \, dx \]
\[ = \int_a^b (b - a) \cdot 1 + (\beta - \alpha) \cdot u'(x) \, dx \]
\[ \leq \int_a^b \sqrt{(b - a)^2 + (\beta - \alpha)^2} \sqrt{1 + u'(x)^2} \, dx \]

so

\[ \sqrt{(b - a)^2 + (\beta - \alpha)^2} \leq \int_a^b \sqrt{1 + u'(x)^2} \, dx \]
This is the same as \( J[\phi] \leq J[u] \), hence \( \phi \) is optimal (as we guessed!).

Suppose an admissible function \( u_0 \) satisfies \( J[u_0] = J[\phi] \). Uniqueness will be proved if we can show that \( u_0 = \phi \). The assumption on \( u_0 \) implies that equality must hold (all \( x \)) in the application of Cauchy-Schwarz inequality. Therefore there exists a function \( \lambda(x) \) such that for \( a \leq x \leq b \):

\[
(1, u'_0(x)) = \lambda(x) (b-a)(\beta-\alpha).
\]

It follows that \( \lambda(x) = \frac{1}{b-a} \), i.e. constant, and

\[
u_0' = \frac{\beta-\alpha}{b-a}.
\]

In view of the condition \( u_0(a) = \alpha \) we get

\[
u_0(x) = \alpha + \frac{\beta-\alpha}{b-a} (x-a) = \phi(x),
\]

and the proof is complete.