LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Written Examination
Algebraic Structures
Saturday March 23, 2013
Duration: 09.00–14.00

Solutions may of course be presented in Swedish! The sheets of the department should be used. Write your initials and at most one solution on each sheet handed in. Give careful motivations to your solutions!

1. Let $a, b \in \mathbb{Z}_n$, $a \neq 0$. Prove that if

$$ax = b$$

has no solutions in $\mathbb{Z}_n$, then a is a zero-divisor.

2. Let $a, b \in \mathbb{Z}$, not both 0, and set $d = gcd(a, b)$. Show that the ideal (a, b) in $\mathbb{Z}$ generated by a and b equals (d).

3. Let F be a field and let $c \in F$. Prove that

$$\phi : F[x] \to F[x]$$
$$\phi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i (x + c)^i$$

is a ring isomorphism.

4. Let

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R}, a \neq 0, c \neq 0 \right\}$$

be a multiplicative group and let

$$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$$

be a subgroup of G. Show that N is a normal subgroup of G. Show that G/N is isomorphic to $\mathbb{R}^* \times \mathbb{R}^*$.

5. (i) Show that both $\sqrt{2}$ and i are in $\mathbb{Q}(i + \sqrt{2})$ and deduce that $[\mathbb{Q}(i + \sqrt{2}) : \mathbb{Q}] = 4$. (3p)
(ii) Find the minimal polynomial of $i + \sqrt{2}$ over $\mathbb{Q}$. (2p)

6. Prove that the permutations (12) and $(12 \ldots n)$ generate S_n for $n \geq 2$.