Linear and Combinatorial Optimization

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Integer linear programming (ILP)

In integer linear programming, we study linear optimization problems with an integer constraint added:

$$\text{maximize} \quad z = \mathbf{c}^{\mathsf{T}}\mathbf{x},$$
$$\text{subject to} \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \quad \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} \in \mathbb{Z}^n. \end{cases}$$

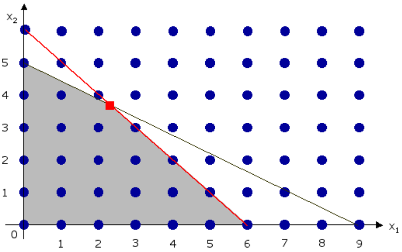

Figure 1: How does ILP differ from LP?

Integer linear programming (ILP)

How does LP differ from ILP?

- We can start by solving the corresponding LP problem, which is obtained from (ILP) by removing the integer constraint. If we are lucky, then the optimal solution to this LP problem belongs to $\mathbb{Z}^n$. Then the obtained solution must also be optimal for (ILP).
- What if the optimal solution of (LP) does not belong to $\mathbb{Z}^n$? What do we do? One guess one might have is to round off too the closest feasible point. Unfortunately this does not always work.
- Sometimes, the special structure of the problem can be used to guarantee that (LP) has an integer optimal solution.

Integer linear programming (ILP)

Theorem (Kruskaal–Hoffman)

Suppose that $\mathbf{A}$ *contains only* 0*'s,* 1*'s and* -1*'s and that* $\mathbf{b}$ *has integer entries. Then (LP) has an optimal integer solution, which is an extreme point of the feasible set.*

- The theorem implies that if $\mathbf{A}$ and $\mathbf{b}$ are as in the theorem, then we can solve (ILP) with the simplex method.
- Note that no assumptions are made on $\mathbf{c}$, but the maximum value may not be an integer unless $\mathbf{c}$ has only integer entries.

Example (The transportation problem)

A manufacturer has m factories and n warehouses. Warehouse number j demands d_j of a product. Factory number i can supply s_i of the same product. The shipping cost from factory number i to warehouse number j is c_{ij}. Determine the amount x_{ij} that should be shipped from factory number i to warehouse number j in order to minimize the shipping cost.
We assume that

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j \qquad \textit{(total supply} \geq \textit{total demand).}$$

Integer linear programming (ILP)

Example (The transportation problem, Cont.)

The transportation problem can then be formulated as the LP problem

$$
\begin{aligned}
&\textit{minimize} \quad z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \\
&\textit{subject to} \begin{cases}
\displaystyle\sum_{j=1}^{n} x_{ij} \leq s_i, & i = 1, \ldots, m, \\
\displaystyle\sum_{i=1}^{m} x_{ij} \geq d_j & j = 1, \ldots, n, \\
x_{ij} \geq 0, & i = 1, \ldots, m, j = 1, \ldots, n, \\
x_{ij} \in \mathbb{Z}^n, & i = 1, \ldots, m, j = 1, \ldots, n.
\end{cases}
\end{aligned}
$$

Integer linear problems (ILP)

Example (The transportation problem, Cont.)

- *If we rewrite this problem into an ILP in standard form, then we will see that* $\mathbf{A}$ *has only the entries* 1, 0 *and* -1.
- *If* s_j *and* d_j *are integers, then (LP) and (ILP) have the same solution. We can solve the problem with the simplex method.*
- *On the other hand, there are special algorithms developed for this problem. We will learn the* transportation algorithm *later in this course.*

Integer linear problems (ILP)

Example (The knapsack problem)

A hiker is packing for a field trip and has to decide what to bring. She can carry at most k kg. She chooses between n items. Assign values c_j to the items, with the most important item having the highest value. Let a_j be the weight of item number j. The problem can then be phrased as maximization of the total value subject to weight limitation, i.e.

$$\text{maximize} \quad z = \sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{n} a_j x_j \leq k, \ \text{where} \\ \qquad x_j = \begin{cases} 0 & \text{(if item } j \text{ is chosen)} \\ 1 & \text{(if item } j \text{ is not chosen),} \end{cases} \\ \qquad j = 1, \ldots, n. \end{cases}.$$

Example (The knapsack problem, Cont.)

- *This type of problem, where the variables only take values* 0 *and* 1 *is called a* zero–one *programming problem.*
- *For this problem we won't automatically get an integer solution when solving with the simplex method even if k is an integer. (The Kruskaal–Hoffman theorem is not applicable unless all the a_j's are* 1*.)*

- We can handle general ILP's with the *cutting plane method* (which we will do today) or the *branch and bound method* (which we will do next time).

The cutting plane method

- Start by solving the corresponding LP problem (which is called the LP relaxation problem) by removing the $\mathbf{x} \in \mathbb{Z}^n$-constraint and solving with the simplex method. If, by chance, the optimal solution $\mathbf{x_0}$ belongs to $\mathbb{Z}^n$, then $\mathbf{x_0}$ is also a solution of (ILP). (Why?)
- If $\mathbf{x_0} \notin \mathbb{Z}^n$, we have a situation as in the figure, that the optimal solution is not a lattice point.

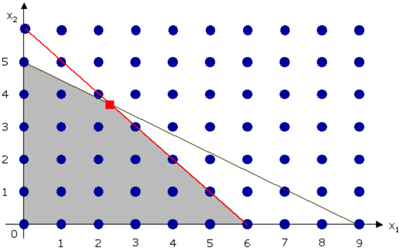

Figure 2: The optimal solution of (LP) is not feasible for (ILP).

The cutting plane method (Cont.)

- In this situation, we add an extra constraint (a cutting plane) in such a way that this solution is removed, but without removing any integer solutions.
- We will show in an explicit example how to construct a cutting plane.

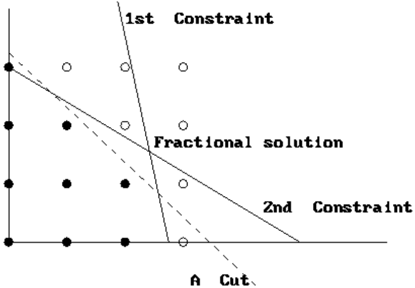

Figure 3: The cutting plane removes the fractional solution found with the simplex method, but keeps all the lattice points within the feasible set.

The cutting plane method (Cont.)

Example (p. 265)

Assume that we ended up with the following system after applying the simplex method. (The original problem must have integer coefficients to start with, but after running the simplex method, this may no longer be the case.)

$$
\begin{array}{c|ccccccccccc}
\hline
x_1 & x_1 & + & & + & \frac{1}{8}x_3 & - & \frac{1}{8}x_4 & & & = & \frac{17}{4} \\
x_2 & & & x_2 & - & \frac{1}{12}x_3 & + & \frac{5}{12}x_4 & & & = & \frac{19}{6} \\
\hline
z & & & & & \frac{1}{8}x_3 & + & \frac{15}{8}x_4 & + & z & = & \frac{161}{4}
\end{array}
$$

As you can see, x_1 and x_2 are basic variables, x_3 and x_4 are nonbasic variables. $x_1 = \frac{17}{4}$, $x_2 = \frac{19}{6}$, $x_3 = x_4 = 0$ is optimal for (LP) but not feasible for (ILP).

The cutting plane method

Example (Cont.)

We add a cutting plane as follows:

- *Choose one of the rows with a noninteger RHS. We'll take the first row. Write it as*

$$\underbrace{x_1 + 0 \cdot x_3 - 1 \cdot x_4}_{A \in \mathbb{Z} \text{ (integer part)}} \quad + \quad \underbrace{\frac{1}{8}x_3 + \frac{7}{8}x_4}_{B \geq 0 \text{ (fractional part)}} \quad = 4 + \tfrac{1}{4}.$$

- *Note that we round the coefficients down in A so that $\lfloor -\frac{1}{8} \rfloor = -1$ for example.*
- *B carries all the fractional part, and so $B \geq +\frac{1}{4}$. (Possible values for B are $\frac{1}{4}, 1 + \frac{1}{4}, 2 + \frac{1}{4}, \ldots$)*
- *This means that we can safely add the constraint*

$$\tfrac{1}{8}x_3 + \tfrac{7}{8}x_4 \geq \tfrac{1}{4}.$$

The cutting plane method (Cont.)

Example (Cont.)

- *We introduce a new slack variable u_1:*

$$\frac{1}{8}x_3 + \frac{7}{8}x_4 - u_1 = \frac{1}{4} \quad \Longleftrightarrow \quad -\frac{1}{8}x_3 - \frac{7}{8}x_4 + u_1 = -\frac{1}{4}.$$

- **Key point:** *$u_1 \in \mathbb{Z}$ and $u_1 \geq 0$.*
  Why? *It is clear that $u_1 \geq 0$. B carries all the fractional part, so B can be $\frac{1}{4}$, $1 + \frac{1}{4}$, $2 + \frac{1}{4}$, etc, and so it follows that $u_1 \in \mathbb{Z}$.*

The cutting plane method (Cont.)

Example (Cont.)

We get the new system

$$
\begin{array}{c|ccccccccccc}
x_1 & x_1 & + & & + & \frac{1}{8}x_3 & - & \frac{1}{8}x_4 & & & & = & \frac{17}{4} \\
x_2 & & & x_2 & - & \frac{1}{12}x_3 & + & \frac{5}{12}x_4 & & & & = & \frac{19}{6} \\
u_1 & & & & - & \frac{1}{8}x_3 & - & \frac{7}{8}x_4 & + & u_1 & & = & -\frac{1}{4} \\
\hline
z & & & & & \frac{1}{8}x_3 & + & \frac{15}{8}x_4 & & & + \quad z & = & \frac{161}{4}
\end{array}
$$

- *We have a basic solution* $(x_1 = \frac{17}{4}, x_2 = \frac{19}{6}, x_3 = x_4 = 0, u_1 = -\frac{1}{4})$ *which is* not *feasible. This shouldn't come as a surprise, since the basic solution is represented by the optimum of the LP problem, that is now outside the feasible set.*

The cutting plane method (Cont.)

Example (Cont.)

- *Now we can initialize using the two phase method, but there is another, more convenient way, using the dual problem.*
- *Rewrite the tableau into a problem in standard form, using the basic variables x_1, x_2 and u_1 as slack variables:*

$$\begin{array}{ll} \textit{maximize} & z = -\frac{1}{8}x_3 - \frac{15}{8}x_4 + \frac{161}{4} \\ \textit{subject to} & \begin{cases} \frac{1}{8}x_3 - \frac{1}{8}x_4 \le \frac{17}{4}, \\ -\frac{1}{12}x_3 + \frac{5}{12}x_4 \le \frac{19}{6}, \\ -\frac{1}{8}x_3 - \frac{7}{8}x_4 \le -\frac{1}{4} \\ x_3, x_4 \ge 0. \end{cases} \end{array}$$

The cutting plane method (Cont.)

Example (Cont.)

- *Now, we will look at the dual problem. For making notation as simple as possible, it is convenient to index the dual variables in the same way as the slack variables of the primal problem. The dual problem is then*

$$\textit{minimize} \quad w = \frac{17}{4}y_1 + \frac{19}{6}y_2 - \frac{1}{4}v_1 + \frac{161}{4}$$

$$\textit{subject to} \quad \begin{cases} \frac{1}{8}y_1 - \frac{1}{12}y_2 - \frac{1}{8}v_1 \geq -\frac{1}{8}, \\ -\frac{1}{8}y_1 + \frac{5}{12}y_2 - \frac{7}{8}v_1 \geq -\frac{15}{8}, \\ y_1, y_2, v_1 \geq 0. \end{cases}$$

The cutting plane method (Cont.)

Example (Cont.)

- *Note that the primal problem is, strictly speaking, not an LP problem because of the constant term $\frac{161}{4}$ in the objective function. It does, however have the same optimal solution as the LP problem obtained by removing that term, and the optimal value will be the $\frac{161}{4}$ higher than the optimal value of that LP problem.*
- *When forming the dual, the term $\frac{161}{4}$ carries over to the objective function of the dual problem. This is sensible, because then the primal and dual problems will have the same optimal value just as is the case for LP problems. Indeed, this follows from applying the strong duality theorem on the LP problem.*

The cutting plane method (Cont.)

Example (Cont.)

- *Put the dual problem into canonical form. Then we need two new slack variables, y_3 and y_4:*

$$
\begin{aligned}
&\textit{maximize} \quad z = -\frac{17}{4}y_1 - \frac{19}{6}y_2 + \frac{1}{4}v_1 - \frac{161}{4} \\
&\textit{subject to} \begin{cases} -\frac{1}{8}y_1 + \frac{1}{12}y_2 + y_3 + \frac{1}{8}v_1 = \frac{1}{8}, \\ \frac{1}{8}y_1 - \frac{5}{12}y_2 + y_4 + \frac{7}{8}v_1 = \frac{15}{8}, \\ y_1, y_2, y_3, y_4, v_1 \geq 0. \end{cases}
\end{aligned}
$$

- *We see that the right hand side of the constraint equations are non-negative, and so $y_3 = \frac{1}{8}$, $y_4 = \frac{15}{8}$, $y_1 = y_2 = v_1 = 0$ is a basic feasible solution.*

The cutting plane method (Cont.)

Example (Cont.)

- *We solve this system with the simplex method, and then translate back to the primal problem. We get*

$$
\begin{array}{c|ccccccccccc}
\hline
x_1 & x_1 & & & - & x_4 & + & u_1 & & & = & 4 \\
x_2 & & x_2 & & + & x_4 & - & \frac{2}{3}u_1 & & & = & \frac{10}{3} \\
x_3 & & & x_3 & + & 7x_4 & - & 8u_1 & & & = & 2 \\
\hline
z & & & & & x_4 & + & u_1 & + & z & = & 40
\end{array}
$$

- *The optimal solution for LP is not feasible for ILP since* $x_2 = \frac{10}{3} \notin \mathbb{Z}$.

The cutting plane method (Cont.)

Example (Cont.)

- *We add another cutting plane, now using the second row (since its RHS is not an integer):*

$$\underbrace{x_2 + x_4 - 1 \cdot u_1}_{A \text{ (integer part)}} \quad + \quad \underbrace{\tfrac{1}{3}u_1}_{B \geq 0 \text{ (fractional part)}} \quad = 3 + \frac{1}{3}.$$

- *B can be $\frac{1}{3}$, $1 + \frac{1}{3}$, $2 + \frac{1}{3}$, etc, i.e. $B = u_2 + \frac{1}{3}$, where u_2 is a nonnegative integer.*
- *We will add the equation*

$$\frac{1}{3}u_1 = u_2 + \frac{1}{3} \quad \Longleftrightarrow \quad -\frac{1}{3}u_1 + u_2 = -\frac{1}{3}.$$

The cutting plane method

Example (Cont.)

- *The new system is then*

$$
\begin{array}{cccccccccccc}
\hline
x_1 & & & - & x_4 & + & u_1 & & & & = & 4 \\
& x_2 & & + & x_4 & - & \frac{2}{3}u_1 & & & & = & \frac{10}{3} \\
& & x_3 & + & 7x_4 & - & 8u_1 & & & & = & 2 \\
& & & & & - & \frac{1}{3}u_1 & +u_2 & & & = & -\frac{1}{3} \\
\hline
& & & & x_4 & + & u_1 & & + & z & = & 40
\end{array}
$$

- *The system is solved using the dual problem as in the previous step. We obtain the solution $x_1 = 3$, $x_2 = 4$ (original variables) and $z = 39$.*
- *See the homepage for a script which can be used for converting from the primal to the dual problem.*