Linear and Combinatorial Optimization

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1 Duality

The weak duality theorem

Recall the weak duality theorem and some of its consequences from last time:

Theorem (The weak duality theorem)

If $\mathbf{x}$ *is a feasible solution of (P) and* $\mathbf{y}$ *is a feasible solution of (D), then* $\mathbf{c}^{\mathsf{T}}\mathbf{x} \leq \mathbf{b}^{\mathsf{T}}\mathbf{y}$.

Corollary

If the primal problem is unbounded, then the dual problem is infeasible.

Figure 1: Weak duality theorem

The weak duality theorem

Theorem

If $\mathbf{x}$ and $\mathbf{y}$ are feasible solutions of the primal and dual problem, respectively, and if $\mathbf{c}^\mathbf{T}\mathbf{x} = \mathbf{b}^\mathbf{T}\mathbf{y}$, then both $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions of their respective problems.

- Note: It can happen that both the primal and dual problems are infeasible.
- If the dual problem is unbounded (to $-\infty$), then the primal problem is infeasible (since the dual of the dual is the primal!).

The strong duality theorem

Theorem (The strong duality theorem)

If $\hat{\mathbf{x}}$ *is optimal and feasible for (P), then there exists a* $\hat{\mathbf{y}}$ *which is optimal and feasible for (D), and* $\mathbf{c}^{\mathsf{T}}\hat{\mathbf{x}} = \mathbf{b}^{\mathsf{T}}\hat{\mathbf{y}}$.

Proof.

Introduce slack variables to put (P) into canonical form, and solve the problem with the simplex algorithm. There exists an optimal solution $\widehat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}' \end{bmatrix}$, where $\mathbf{x}'$ is the vector of slack variables. Let $\widehat{\mathbf{c}} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}$.

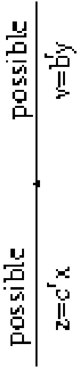

Figure 2: Strong duality theorem

The strong duality theorem (Cont.)

Proof (Cont.)

We decompose $\widehat{\mathbf{x}}$ into its basic and nonbasic variables, $\widehat{\mathbf{x}}_{\mathbf{B}}$ and $\widehat{\mathbf{x}}_{\mathbf{N}} = \mathbf{0}$, and change the order of the variables so that $\widehat{\mathbf{x}} = \begin{bmatrix} \widehat{\mathbf{x}}_{\mathbf{N}} \\ \widehat{\mathbf{x}}_{\mathbf{B}} \end{bmatrix}$.

At the same time, we need to change the order of the columns in $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$ and in $\widehat{\mathbf{c}}$. Then $\widehat{\mathbf{x}}_{\mathbf{B}} = \mathbf{B}^{-\mathbf{1}}\mathbf{b}$, where $\mathbf{B}$ is the (permuted!) submatrix of $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$ corresponding to the basic variables of the optimal solution. Also, we decompose $\widehat{\mathbf{c}}$ into $\widehat{\mathbf{c}}_{\mathbf{B}}$ and $\widehat{\mathbf{c}}_{\mathbf{N}}$, and let $\mathbf{y} = (\mathbf{B}^{-\mathbf{1}})^T \widehat{\mathbf{c}}_{\mathbf{B}}$.

The strong duality theorem (Cont.)

Proof (Cont.)

Then $\mathbf{y}^{\mathbf{T}} = \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-\mathbf{1}}$, and so

$$z = \widehat{\mathbf{c}}^{\mathbf{T}}\widehat{\mathbf{x}} = \widehat{\mathbf{c}}_{\mathbf{N}}^{\mathbf{T}} \cdot \mathbf{0} + \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}}\widehat{\mathbf{x}}_{\mathbf{B}} = \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-\mathbf{1}}\mathbf{b} = \mathbf{y}^{\mathbf{T}}\mathbf{b} = \mathbf{b}^{\mathbf{T}}\mathbf{y}.$$

By the weak duality theorem, we are done if we can show that $\mathbf{y}$ is a feasible solution of (D).

Last time, we used the simplex method for a problem in canonical form:

$$\begin{cases} \text{maximize} & z = \mathbf{c}^{\mathbf{T}}\mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases} \end{cases}$$

The strong duality theorem (Cont.)

Proof (Cont.)

We decomposed $\mathbf{A}$ into $\begin{bmatrix}\mathbf{A_N} & \mathbf{A_B}\end{bmatrix}$, $\mathbf{x}$ into $\begin{bmatrix}\mathbf{x_N} \\ \mathbf{x_B}\end{bmatrix}$ and $\mathbf{c}$ into $\begin{bmatrix}\mathbf{c_N} \\ \mathbf{c_B}\end{bmatrix}$ and solved for $\mathbf{x_B}$ in terms of $\mathbf{x_N}$. We got the tableau

$$\begin{array}{c|rcrcrcr} \hline \mathbf{x}_B & & & \mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N & + & \mathbf{x}_B & & & = & \mathbf{A}_B^{-1}\mathbf{b} \\ \hline z & (\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_N - \mathbf{c}_N^T)\mathbf{x}_N & & & & & + & z & = & \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b} \\ \hline \end{array}$$

If we do the same for our problem, and denote the (permuted) submatrix of the original matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix}$ corresponding to the nonbasic variables of the optimal solution $\widehat{\mathbf{x}}$ by $\mathbf{N}$, the tableau becomes

$$\begin{array}{c|rcrcrcr} \hline \widehat{\mathbf{x}}_B & & & \mathbf{B}^{-1}\mathbf{N}\widehat{\mathbf{x}}_N & + & \widehat{\mathbf{x}}_B & & & = & \mathbf{B}^{-1}\mathbf{b} \\ \hline z & (\widehat{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{N} - \widehat{\mathbf{c}}_N^T)\widehat{\mathbf{x}}_N & & & & & + & z & = & \widehat{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{b} \\ \hline \end{array}$$

The strong duality theorem (Cont.)

Proof (Cont.)

The solution $\widehat{\mathbf{x}}$ is optimal if and only if

$$\widehat{\mathbf{c}}_B^T \mathbf{B}^{-1}\mathbf{N} - \widehat{\mathbf{c}}_N^T \geq 0,$$

according to the optimality criterion in the simplex algorithm.
But note that we also have

$$\widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} \mathbf{B}^{-\mathbf{1}}\mathbf{B} - \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} = \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} - \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} = \mathbf{0} \geq \mathbf{0} \text{ (trivially!)}$$

Together, this gives

$$\widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} \mathbf{B}^{-\mathbf{1}} \begin{bmatrix} \mathbf{N} & \mathbf{B} \end{bmatrix} - \begin{bmatrix} \widehat{\mathbf{c}}_{\mathbf{N}}^{\mathbf{T}} & \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} \end{bmatrix} \geq \mathbf{0}.$$

But $\begin{bmatrix} \mathbf{N} & \mathbf{B} \end{bmatrix}$ is just the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$ with permuted columns, and $\begin{bmatrix} \widehat{\mathbf{c}}_{\mathbf{N}}^{\mathbf{T}} & \widehat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}} \end{bmatrix}$ is the same permutation of the row vector $\begin{bmatrix} \mathbf{c}^{\mathbf{T}} & \mathbf{0} \end{bmatrix}$.

The strong duality theorem (Cont.)

Proof.

Hence

$$\hat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-1}\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix} - \begin{bmatrix}\mathbf{c}^{\mathbf{T}} & \mathbf{0}\end{bmatrix} \geq \mathbf{0}$$

which is equivalent to

$$\begin{cases} \hat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{c}^{\mathbf{T}}, \\ \hat{\mathbf{c}}_{\mathbf{B}}^{\mathbf{T}}\mathbf{B}^{-1} \geq \mathbf{0}. \end{cases} \qquad (*)$$

Recall that $\mathbf{y} = (\mathbf{B}^{-1})^T\hat{\mathbf{c}}_{\mathbf{B}}$. So (*) is equivalent to

$$\begin{cases} \mathbf{y}^{\mathbf{T}}\mathbf{A} \geq \mathbf{c}^{\mathbf{T}}, \\ \mathbf{y}^{\mathbf{T}} \geq \mathbf{0}. \end{cases} \iff \begin{cases} \mathbf{A}^{\mathbf{T}}\mathbf{y} \geq \mathbf{c}, \\ \mathbf{y} \geq \mathbf{0}, \end{cases}$$

which shows that $\mathbf{y}$ is feasible for (D). □

Complementary slackness

Definition

Let $\mathbf{x}$ *and* $\mathbf{y}$ *be feasible solutions for (P) and (D), respectively. Then* $\mathbf{x}$ *and* $\mathbf{y}$ *are said to satisfy the complementary slackness condition (CS) if*

$$\mathbf{y}^{\mathbf{T}}(\mathbf{A}\mathbf{x}-\mathbf{b})=0 \qquad \textit{and} \qquad \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{y}-\mathbf{c})=0.$$

What does this mean?

Recall that $\mathbf{y} \geq \mathbf{0}$ and the slack variables $\mathbf{x}' = \mathbf{b} - \mathbf{A}\mathbf{x} \geq \mathbf{0}$. We have

$$\mathbf{y}^{\mathbf{T}}(\mathbf{A}\mathbf{x}-\mathbf{b})=\mathbf{0} \iff -\mathbf{y}^{\mathbf{T}}\mathbf{x}'=0 \iff \mathbf{y}^{\mathbf{T}}\mathbf{x}'=0 \iff$$
$$y_1x_1'+y_2x_2'+\cdots+y_mx_m'=0 \iff y_jx_j'=0 \text{ for all } j.$$

Complementary slackness

Since all the terms $y_j x'_j \geq 0$, this is equivalent to $y_j x'_j = 0$ for all $j = 1, \ldots, m$, i.e. if and only if $y_j = 0$ or $x'_j = 0$ for all $j = 1, \ldots, m$. This proves that

$$\mathbf{y}^\mathbf{T}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0} \quad \Longleftrightarrow \quad y_j = 0 \text{ or } (\mathbf{A}\mathbf{x})_j = \mathbf{b}_j \text{ for every } j = 1, \ldots, m.$$

If for some $j \in \{1, \ldots, m\}$, $(\mathbf{A}\mathbf{x})_j = \mathbf{b}_j$, we say that the jth constraint is active.
In the same way as above, we can show that

$$\mathbf{x}^\mathbf{T}(\mathbf{A}^\mathbf{T}\mathbf{y} - \mathbf{c}) = \mathbf{0} \quad \Longleftrightarrow \quad x_i = 0 \text{ or } (\mathbf{A}^\mathbf{T}\mathbf{y})_i = \mathbf{c}_i \text{ for every } i = 1, \ldots, n.$$

Complementary slackness

Lemma

If $\mathbf{x}$, $\mathbf{y}$ *satisfy (CS), then* $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{b}^\mathsf{T}\mathbf{y}$
(and so $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal for (P) and (D), respectively, by the weak duality theorem).*

Proof.

$$\text{(CS)} \quad \Longrightarrow \quad \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} = \mathbf{y}^\mathsf{T}\mathbf{b} = \mathbf{b}^\mathsf{T}\mathbf{y},$$

but also

$$\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} = \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{y} = \mathbf{x}^\mathsf{T}\mathbf{c} = \mathbf{c}^\mathsf{T}\mathbf{x},$$

and so

$$\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{b}^\mathsf{T}\mathbf{y}.$$

□

Complementary slackness

Theorem

If $\mathbf{x}$ is optimal for (P) and $\mathbf{y}$ is optimal for (D), then (CS) holds.

Proof.

By the strong duality theorem, we have $\mathbf{c}^\mathbf{T}\mathbf{x} = \mathbf{b}^\mathbf{T}\mathbf{y}$. Introduce slack variables for (P) so that

$$\mathbf{x}' = \mathbf{b} - \mathbf{A}\mathbf{x} \quad \Longleftrightarrow \quad \mathbf{x}'^\mathbf{T} = \mathbf{b}^\mathbf{T} - \mathbf{x}^\mathbf{T}\mathbf{A}^\mathbf{T}$$

which implies that

$$\mathbf{x}'^\mathbf{T}\mathbf{y} = \mathbf{b}^\mathbf{T}\mathbf{y} - \mathbf{x}^\mathbf{T}\mathbf{A}^\mathbf{T}\mathbf{y} \le \mathbf{c}^\mathbf{T}\mathbf{x} - \mathbf{x}^\mathbf{T}\mathbf{c} = 0.$$

Hence $(\mathbf{b} - \mathbf{A}\mathbf{x})^T y \le 0$. But $\mathbf{b} - \mathbf{A}\mathbf{x} \ge 0$ and $\mathbf{y} \ge \mathbf{0}$, and so equality holds. In the same way, it can be proved that $\mathbf{x}^\mathbf{T}(\mathbf{A}^\mathbf{T}\mathbf{y} - \mathbf{c}) = 0$. □

The diet problem

Note that we can solve whichever problem is easier to solve of (P) and (D), and then we automatically get a solution also for the other problem.

Example (Diet problem, p. 46–47 in Kolman–Beck)

2 foods, F_1 and F_2 contain nutrients N_1, N_2 and N_3. The nutrient content and price per unit of food is given in the table below together with the minimal amounts required for each unit.

	N_1	N_2	N_3	*Price*
F_1	2	1	4	20
F_2	3	3	3	25
min amount	18	12	24	

The diet problem

Example (Diet problem, Cont.)

How do we compose a meal satisfying the nutrient requirements for the smallest possible cost?
This can be formulated as an LP problem as follows: Let x_1, x_2 *be the amounts of the foods* F_1 *and* F_2 *that goes into the meal. The LP problem is then the minimization problem*

$$\begin{aligned} &\textit{minimize} \quad z = 20x_1 + 25x_2, \\ &\textit{subject to} \begin{cases} 2x_1 + 3x_2 \geq 18, \\ x_1 + 3x_2 \geq 12, \\ 4x_1 + 3x_2 \geq 24, \\ x_1, x_2 \geq 0. \end{cases} \end{aligned}$$

Example (Diet problem, Cont.)

Now, let's say that there is a manufacturer of artificial foods P_1, P_2, P_3, where one unit of P_j contains one unit of N_j. How should the manufacturer set the prices y_1, y_2, y_3 of the foods P_1, P_2, P_3? The cost of the substitute for F_j cannot be higher than the cost of F_j (otherwise nobody would buy it). This gives the constraints

$$\begin{cases} 2y_1 + y_2 + 4y_3 \leq 20, \\ 3y_1 + 3y_2 + 3y_3 \leq 25, . \\ y_1, y_2, y_3 \geq 0. \end{cases}$$

The profit should be maximized, so the problem is to

$$\textit{maximize } v = 18y_1 + 12y_2 + 24y_3$$

subject to the above constraints.

The diet problem

Example (Diet problem, Cont.)

- *Note that this is precisely the dual problem of the diet problem. We can choose to solve either of them. We notice that in the dual problem, phase 1 of the two-phase method is not required since the right hand side of the constraint vector has only positive entries.*
- *Solving this with the simplex method, we get* $y_1 = \frac{20}{3}$, $y_2 = 0$, $y_3 = \frac{5}{3}$ *and slack variables* $y_4 = 0$, $y_5 = 0$.
- *The complementary slackness condition implies that constraint number* 1 *and* 3 *are active in (P). Hence*

$$\begin{cases} 2x_1 + 3x_2 = 18 \\ 4x_1 + 3x_2 = 24, \end{cases}$$

The diet problem

Example (Diet problem, Cont.)

- *The above system has the solution* $x_1 = 3$ *and* $x_2 = 4$.
- *The optimal value for (P) is* $20 \cdot 3 + 25 \cdot 4 = 160$, *and for (D) it is* $18 \cdot \frac{20}{3} + 24 \cdot \frac{5}{3} = 6 \cdot 20 + 8 \cdot 5 = 160$, *which are the same as expected.*