



Linear and Combinatorial Optimization

Sara Maad Sasane

Center for Mathematical Sciences, Lund University

1 Summary of the simplex method

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The Simplex method

As last time, we write $\mathbf{A} = [\mathbf{A}_N, \mathbf{A}_B]$, and recall that

$$\mathbf{Ax} = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{A}_N \mathbf{x}_N + \mathbf{A}_B \mathbf{x}_B = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x}_B = \mathbf{A}_B^{-1}(\mathbf{b} - \mathbf{A}_N \mathbf{x}_N).$$

If $\mathbf{x}_N = \mathbf{0}$ then $\mathbf{x}_B = \mathbf{A}_B^{-1} \mathbf{b}$ and we have a *basic solution*. If in addition, $\mathbf{x}_B \geq \mathbf{0}$, then we have a *basic feasible solution*.

We express the objective function in terms of only the non-basic variables:

$$\begin{aligned} z = \mathbf{c}^T \mathbf{x} &= \mathbf{c}_N^T \mathbf{x}_N + \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_N^T \mathbf{x}_N + \mathbf{c}_B^T \mathbf{A}_B^{-1}(\mathbf{b} - \mathbf{A}_N \mathbf{x}_N) \\ &= (\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N) \mathbf{x}_N + \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b}. \end{aligned}$$

Linear programming

It is convenient to organize the computations in a table as follows:

x_B	$\mathbf{A}_B^{-1}\mathbf{A}_N x_N + x_B$	$=$	$\mathbf{A}_B^{-1}\mathbf{b}$
z	$(\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N - \mathbf{c}_N^T) x_N$	$+ z =$	$\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b}$

Or, equivalently, we use tableaus where the variables are omitted:

$\mathbf{A}_B^{-1}\mathbf{A}_N$	\mathbf{I}	0	$\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N - \mathbf{c}_N^T$	$\mathbf{0}$	1	$\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b}$

Choosing the incoming variable

Recall the saw mill example that we completed last time, and check that you agree with the following principles for one step in the simplex method:

- ▶ As the incoming variable, we need to choose a variable with the property that if that variable is increased while the other nonbasic variables are kept at 0, the objective value z will increase.
- ▶ This happens precisely when the coefficient in front of that variable in the objective row is negative, i.e. when there is a negative entry in the vector $\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N$.
- ▶ If there is more than one negative entry in the vector $\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N$, then either one works. Pick one of them randomly, or according to some criterion.
- ▶ If we cannot choose an incoming variable, then the solution is optimal, as is shown in the following theorem:

The optimality condition

Theorem

Suppose that $\bar{\mathbf{x}}$ is a basic feasible solution, and that $\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N - \mathbf{c}_N^T \geq \mathbf{0}$. Then $\bar{\mathbf{x}}$ is an optimal solution.

Proof.

Assume without loss of generality that the variables are ordered so that $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_N, \bar{\mathbf{x}}_B)^T = (\mathbf{0}, \bar{\mathbf{x}}_B)^T$. We denote the corresponding objective value by \bar{z} . Let $\mathbf{x} = (\mathbf{x}_N, \mathbf{x}_B)^T$ be *any other* feasible solution, and denote its objective value by z . Note that we are still using the splitting of the variables according to which variables are basic and nonbasic for the particular solution $\bar{\mathbf{x}}$. Since \mathbf{x} is feasible, in particular $\mathbf{x}_N \geq \mathbf{0}$.

The optimality condition

Proof (Cont.)

It follows that

$$\begin{aligned} z &= (\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N) \mathbf{x}_N + \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} \\ &\leq 0 + \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} \\ &= (\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N) \bar{\mathbf{x}}_N + \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} = \bar{z}, \end{aligned}$$

and so $\bar{\mathbf{x}}$ is optimal. □

Choosing the outgoing variable

As the outgoing variable, we need to choose the (basic) variable that becomes 0 first, when the incoming variable is increasing (while the other nonbasic variables are kept at 0). In the tableau, we see that we need to maintain the (column) vector inequality $\mathbf{A}_B^{-1}\mathbf{A}_i\mathbf{x}_i \leq \mathbf{A}_B^{-1}\mathbf{b}$, where \mathbf{A}_i denotes the column of \mathbf{A} which corresponds to the incoming variable \mathbf{x}_i . For each entry, we will have one of the following two scenarios, where we note that all entries of $\mathbf{A}_B^{-1}\mathbf{b}$ are nonnegative, since the inequality above holds when $\mathbf{x}_i = 0$:

1. If entry number j of $\mathbf{A}_B^{-1}\mathbf{A}_i$ is nonpositive, then the inequality corresponding to that entry will always be satisfied no matter how big \mathbf{x}_i becomes.

Choosing the outgoing variable

2. If entry number j in $\mathbf{A}_B^{-1}\mathbf{A}_i$ is positive, then the inequality of that row is equivalent to $x_i \leq [\mathbf{A}_B^{-1}\mathbf{b}]_j / [\mathbf{A}_B^{-1}\mathbf{A}_i]_j$, where the index j denotes the index of the entry of the relevant vector.

From the above, we conclude that the index of the outgoing variable should be the j for which $[\mathbf{A}_B^{-1}\mathbf{b}]_j / [\mathbf{A}_B^{-1}\mathbf{A}_i]_j$ is the smallest among all the j 's for which $[\mathbf{A}_B^{-1}\mathbf{A}_i]_j$ is positive.

If there is more than one choice for the smallest of those numbers, we pick a random one or choose one of them according to some other criterion.

Unbounded problems

If we cannot choose an outgoing variable, i.e. if $\mathbf{A}_B^{-1}\mathbf{A}_i \leq 0$, then the incoming variable can be increased to an arbitrarily large number while the constraints still hold. In this case there is no optimal solution and there exist feasible solutions with arbitrarily large values.

In this situation, we say that the problem is unbounded.

① Summary of the simplex method

② **The two phase method**

③ The dual of an LP problem

An example

- ▶ Our description of the simplex method requires that we have a basic feasible solution to start with. We have seen in the example that this is easy if the problem comes from a problem in standard form, which is put in canonical form using slack variables, *and* if the vector \mathbf{b} in the constraint has nonnegative entries.
- ▶ If this is the case, we can choose the slack variables as the initial basic variables, and $\mathbf{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}$ can be taken as an initial basic feasible solution (with objective value 0).
- ▶ If $\mathbf{b} \not\geq \mathbf{0}$, then the above does not give rise to a basic feasible solution, and so we need some other way to find an initial feasible solution. One way of doing this is by the *two phase method* that we will describe now.

An example

Example

$$\begin{array}{ll} \text{maximize} & z = 2x_1 + 5x_2, \\ \text{subject to} & \left\{ \begin{array}{l} 2x_1 + 3x_2 \leq 6, \\ -2x_1 + x_2 \leq -2, \\ x_1 - 6x_2 = -2, \\ x_1, x_2 \geq 0. \end{array} \right. \end{array}$$

As you can see, this LP problem is neither in standard nor canonical form. We start by rewriting the constraints so that the RHS vector is non-negative, by multiplying some constraints by -1 , and remembering to reverse the inequalities at the same time:

An example (Cont.)

Example

$$\begin{array}{ll} \text{maximize} & z = 2x_1 + 5x_2, \\ \text{subject to} & \left\{ \begin{array}{l} 2x_1 + 3x_2 \leq 6, \\ 2x_1 - x_2 \geq 2, \\ -x_1 + 6x_2 = 2, \\ x_1, x_2 \geq 0. \end{array} \right. \end{array}$$

Next, we insert one slack variable for each inequality constraint. In this case we have two inequalities, and so we need two slack variables, x_3 and x_4 . Note carefully the sign in front of the slack variables, and convince yourself that the systems of the LP problem of this and the following page are equivalent.

An example (Cont.)

Example

$$\begin{array}{ll} \text{maximize} & z = 2x_1 + 5x_2, \\ \text{subject to} & \left\{ \begin{array}{ll} 2x_1 + 3x_2 + x_3 & = 6, \\ 2x_1 - x_2 - x_4 & = 2, \\ -x_1 + 6x_2 & = 2, \\ x_1, x_2, x_3, x_4 & \geq 0. \end{array} \right. \end{array}$$

Note that if we would now choose the slack variables x_3 and x_4 as our basic variables, we would get $x_3 = 6$ and $x_4 = -2 < 0$, so we will not obtain a basic feasible solution in this way.

An example (Cont.)

Example

We need a trick to find a first basic feasible solution. We introduce artificial variables y_1 , y_2 and y_3 (one for each constraint) and study an auxiliary problem, which is another LP problem with constraint set

$$\begin{cases} 2x_1 + 3x_2 + x_3 + y_1 = 6, \\ 2x_1 - x_2 - x_4 + y_2 = 2, \\ -x_1 + 6x_2 + y_3 = 2, \\ x_1, x_2, x_3, x_4, y_1, y_2, y_3 \geq 0. \end{cases}$$

We will soon write down what the objective function should be. If the auxiliary problem has a solution with $y_1 = y_2 = y_3 = 0$, then we have also found a basic feasible solution to our original problem, since we can just remove the y -variables from the above system.

Artificial variables

As in the example, we add one artificial variable to each of the constraints, and study the auxiliary problem.

- ▶ How do we find a solution with all $y_j = 0$?
- ▶ Minimize $\sum_{j=1}^m y_j$ subject to the constraint set of the auxiliary problem. If this LP problem has optimal value 0, then all y_j must be zero for the optimal solution, and we have found our initial basic feasible solution of our original problem.
- ▶ Note that

$$\sum_{j=1}^m y_j = 0 \iff y_j = 0 \text{ for } j = 1, \dots, m \text{ since } y_j \geq 0.$$

- ▶ Note that it is easy to find a basic feasible solution for the auxiliary problem, since we can use the auxiliary variables as basic variables.

The auxiliary problem

- ▶ Use the simplex method until an optimal solution of the auxiliary problem is found. (Remember to convert $\min \sum_{j=1}^m y_j$ to $\max(-\sum_{j=1}^m y_j)$).
- ▶ If the maximum value is less than 0, then there is no basic feasible solution to the original problem. In that case, the feasible set of the original problem is empty.
- ▶ If the maximum value is 0, then we have found a basic feasible solution for the original problem, and we can start the simplex algorithm on that problem originating for this solution.
- ▶ It is straight-forward to complete these steps for the above example (although a computer is probably needed). I leave this as an exercise.

① Summary of the simplex method

② The two phase method

③ **The dual of an LP problem**

The dual of an LP problem in standard form

Let

$$\left\{ \begin{array}{l} \text{maximize} \quad z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} \quad \left\{ \begin{array}{l} \mathbf{Ax} \leq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \end{array} \right. \end{array} \right. \quad (P)$$

be an LP problem in standard form, which will be referred to as the primal problem (P).

Definition (Dual problem of a problem in standard form)

The dual of an LP problem (P) in standard form is

$$\left\{ \begin{array}{l} \text{minimize} \quad v = \mathbf{b}^T \mathbf{y}, \\ \text{subject to} \quad \left\{ \begin{array}{l} \mathbf{A}^T \mathbf{y} \geq \mathbf{c}, \\ \mathbf{y} \geq \mathbf{0}. \end{array} \right. \end{array} \right. \quad (D)$$

Dual of the dual

After rewriting the dual problem as a maximization problem with \leq constraints in the usual way, we can define its dual. We then have the following theorem:

Theorem

The dual of the dual problem is the primal problem.

Proof.

The dual problem is equivalent to

$$\begin{cases} \text{maximize} & -v = -\mathbf{b}^T \mathbf{y}, \\ \text{subject to} & \begin{cases} -\mathbf{A}^T \mathbf{y} \leq -\mathbf{c}, \\ \mathbf{y} \geq \mathbf{0}. \end{cases} \end{cases}$$

Dual of the dual (Cont.)

Proof.

The dual of this problem is by definition,

$$\begin{cases} \text{minimize} & -z = -\mathbf{c}^T \mathbf{x}, \\ \text{subject to} & \begin{cases} (-\mathbf{A}^T)^T \mathbf{x} \geq -\mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \end{cases} \end{cases}$$

which can be simplified to

$$\begin{cases} \text{maximize} & z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \end{cases} \end{cases}$$

which we identify as the primal problem (P). □

The dual problem of an LP problem in canonical form

Theorem

The dual of an LP problem in canonical form

$$\left\{ \begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \right. \left\{ \begin{array}{l} z = \mathbf{c}^T \mathbf{x}, \\ \mathbf{Ax} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{array} \right. \quad (\text{P})$$

is the minimization problem

$$\left\{ \begin{array}{l} \text{minimize} \\ \text{subject to} \end{array} \right. \left\{ \begin{array}{l} v = \mathbf{b}^T \mathbf{y}, \\ \mathbf{A}^T \mathbf{y} \geq \mathbf{c}, \\ \mathbf{y} \in \mathbb{R}^m \quad (\mathbf{y} \text{ unconstrained}). \end{array} \right. \quad (\text{D})$$

The dual of an LP problem in canonical form (Cont.)

Proof idea.

The plan is to convert (P) into standard form using the operations from Lecture 2. Then construct the dual of this problem, using the definition of the dual of an LP problem in standard form. Finally, simplify the resulting LP problem.

The primal problem converted into standard form is

$$\left\{ \begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \right. \left\{ \begin{array}{l} z = \mathbf{c}^T \mathbf{x}, \\ \mathbf{Ax} \leq \mathbf{b}, \\ -\mathbf{Ax} \leq -\mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \end{array} \right.$$

or, equivalently,

The dual of an LP problem in canonical form (Cont.)

Example

$$\left\{ \begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \right. \left\{ \begin{array}{l} z = \mathbf{c}^T \mathbf{x}, \\ \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}, \\ \mathbf{x} \geq \mathbf{0}. \end{array} \right.$$

Its dual problem is therefore

$$\left\{ \begin{array}{l} \text{minimize} \\ \text{subject to} \end{array} \right. \left\{ \begin{array}{l} v = [\mathbf{b}^T \quad -\mathbf{b}^T] \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \\ \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \geq \mathbf{c}, \\ \mathbf{u}, \mathbf{w} \geq \mathbf{0}. \end{array} \right.$$

The dual of an LP problem in canonical form, Cont.

Example

$$\begin{cases} \text{minimize} & v = \mathbf{b}^T(\mathbf{u} - \mathbf{w}) \\ \text{subject to} & \begin{cases} \mathbf{A}^T(\mathbf{u} - \mathbf{w}) \geq \mathbf{c}, \\ \mathbf{u}, \mathbf{w} \geq \mathbf{0}, \end{cases} \end{cases}$$

After putting $\mathbf{y} = \mathbf{u} - \mathbf{w}$, and noting that all $\mathbf{y} \in \mathbb{R}^m$ can be expressed as a difference between two nonnegative vectors in \mathbb{R}^m , we see that this dual problem is equivalent to (D).

Dual problems for other types of LP problems

- ▶ Similarly as above, we can form dual problems for other types of LP problems.
- ▶ Always convert the problem into a problem in standard form, and then construct the dual of that problem. Simplify the dual problem if possible.
- ▶ Once you have practised on a few problems, you can look at the primal–dual pairs on p. 160 in the book by Kolman–Beck.

Shadow prices

Example (Shadow prices in the saw mill example)

The primal and dual problems of the saw mill example are

$$\begin{array}{ll} \text{maximize} & z = 120x_1 + 100x_2, \\ \text{subject to} & \left\{ \begin{array}{l} 2x_1 + 2x_2 \leq 8, \\ 5x_1 + 3x_2 \leq 15, \\ x_1, x_2 \geq 0. \end{array} \right. \end{array} \quad (\text{P})$$

and

$$\begin{array}{ll} \text{minimize} & v = 8y_1 + 15y_2, \\ \text{subject to} & \left\{ \begin{array}{l} 2y_1 + 5y_2 \geq 120, \\ 2y_1 + 3y_2 \geq 100, \\ y_1, y_2 \geq 0, \end{array} \right. \end{array} \quad (\text{D})$$

respectively.

Shadow prices

Example

- ▶ *In the primal problem, x_1 , x_2 represent the amount that the saw mill produces of each type of lumber.*
- ▶ *In the dual problem, y_1 , y_2 represent the (fictitious) value of using the saw for one hour and for using the plane for one hour, respectively.*
- ▶ *y_1 and y_2 are called shadow prices or fictitious prices.*
- ▶ *We can solve the dual problem with the simplex method. The optimal solution is $y_1 = 35$, $y_2 = 10$. The optimal value is $8 \cdot 35 + 15 \cdot 10 = 430$.*
- ▶ *Recall that the primal problem also has the optimal value 430. This is not a coincidence, and we will see why soon.*

Shadow prices

- ▶ We will see next time that if \mathbf{x} is optimal for (P) and \mathbf{y} is optimal for (D), then $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$.
- ▶ If b_i is increased by one unit, the profit will increase by y_i . Thus, y_i represents the marginal value of the i 'th input.
- ▶ For example, if the saw becomes available for one more hour per day (b_1 increases from 8 to 9), then the profit will increase with $1 \cdot y_1 = y_1$.

The weak duality theorem

Theorem (The weak duality theorem)

If \mathbf{x} is a feasible solution of (P) and \mathbf{y} is a feasible solution of (D), then $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$.

Proof.

Since $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ (and $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$), we have

$$\mathbf{c}^T \mathbf{x} \leq (\mathbf{A}^T \mathbf{y})^T \mathbf{x} = \mathbf{y}^T \mathbf{Ax} \leq \mathbf{y}^T \mathbf{b} = \mathbf{b}^T \mathbf{y},$$

where the last equality follows since $\mathbf{y}^T \mathbf{b}$ is a number, and hence equal to its transpose. \square

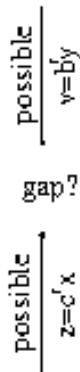


Figure 1:
Weak duality
theorem

The weak duality theorem (Cont.)

Corollary

If the primal problem is unbounded, then the dual problem is infeasible.

- ▶ Note: It can happen that both the primal and dual problems are infeasible.
- ▶ If the dual problem is unbounded (to $-\infty$), then the primal problem is infeasible (since the dual of the dual is the primal!).

The weak duality theorem

Theorem

If \mathbf{x} and \mathbf{y} are feasible solutions of the primal and dual problem, respectively, and if $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$, then both \mathbf{x} and \mathbf{y} are optimal solutions of their respective problems.

Proof.

Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ be feasible solutions of (P) and (D), respectively. By the weak duality theorem we have $\mathbf{c}^T \hat{\mathbf{x}} \leq \mathbf{b}^T \hat{\mathbf{y}}$ and $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \hat{\mathbf{y}}$.

Hence

$$\mathbf{c}^T \hat{\mathbf{x}} \leq \mathbf{b}^T \hat{\mathbf{y}} = \mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \hat{\mathbf{y}}.$$

