



# Linear and Combinatorial Optimization

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# 1 Optimization

2 Discrete and continuous optimization

3 Convexity

4 Linear optimization (Linear programming)

# Optimization

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- ▶ "Optimization = the best"
- ▶ Let  $D \subset \mathbb{R}^n$  ( $D = \mathbb{R}^n$  is allowed).

*The problem: Find the largest or smallest value of a function  $f : D \rightarrow \mathbb{R}$  (if they exist).*

- ▶ This general problem has applications in many diverse fields such as mechanics, economics, electrical engineering, control theory, etc.
- ▶ As soon as we would like to find "the best" way of doing something, optimization can be useful.

# Optimization

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- ▶ We define

$$f_{\min} = \min_{x \in D} f(x),$$

$$x_{\min} = \operatorname{argmin}_{x \in D} f(x) \quad (= x \in D \text{ such that } f(x) = f_{\min}).$$

Note that  $x_{\min}$  is not necessarily unique.

- ▶ In the same way, we define  $f_{\max}$  and  $x_{\max}$ .
- ▶ We distinguish between discrete (combinatorial) and continuous optimization.

① Optimization

② **Discrete and continuous optimization**

③ Convexity

④ Linear optimization (Linear programming)

# Existence of minimizers

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D discrete	D not discrete
<ul style="list-style-type: none"><li>• Combinatorial or discrete optimization</li><li>• <math>D</math> is finite <math>\Rightarrow f_{\min}</math> and <math>f_{\max}</math> exist. (obvious)</li></ul>	<ul style="list-style-type: none"><li>• Continuous optimization (<math>f</math> is required to be continuous on <math>D</math>).</li><li>• <math>D</math> is compact (= closed and bounded) <math>\Rightarrow f_{\min}</math> and <math>f_{\max}</math> exist. (Weierstrass's extreme value theorem)</li></ul>

- ▶ How do we find  $x_{\max}$  and  $x_{\min}$ ?
- ▶ One way is to reduce the number of possible candidates by studying local minima and maxima (which may be easier to find than the global one).

## Local minimizers/maximizers

D discrete	D not discrete
<ul style="list-style-type: none"><li>• To each point <math>x \in D</math>, we identify a set of neighbours, <math>G_x</math>.</li><li>• A point <math>x_{loc}</math> is said to be a <i>local minimum</i> if<math display="block">f(x_{loc}) \leq f(y)</math>for all <math>y \in G_x</math>.</li><li>• A point <math>x_{loc}</math> is said to be a <i>local maximum</i> if<math display="block">f(x_{loc}) \geq f(y)</math>for all <math>y \in G_x</math>.</li></ul>	<ul style="list-style-type: none"><li>• <math>D \subset \mathbb{R}^n</math>.</li><li>• A point <math>x_{loc}</math> is said to be a <i>local minimum</i> if there exists a <math>\delta &gt; 0</math> s t<math display="block">f(x_{loc}) \leq f(y)</math>for all <math>y \in D</math> s t <math>\ y - x_{loc}\  &lt; \delta</math>.</li><li>• A point <math>x_{loc}</math> is said to be a <i>local maximum</i> if there exists a <math>\delta &gt; 0</math> s t<math display="block">f(x_{loc}) \geq f(y)</math>for all <math>y \in D</math> s t <math>\ y - x_{loc}\  &lt; \delta</math>.</li></ul>

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# Convexity

Convex optimization is an important special case of continuous optimization.

## Definition

A set  $C \subset \mathbb{R}^n$  is said to be convex if

$$\mathbf{x}, \mathbf{y} \in C, \quad \lambda \in [0, 1] \quad \implies \quad \lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in C.$$

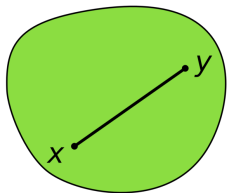


Figure 1: A convex set

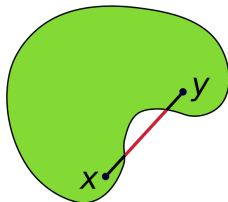


Figure 2: A non-convex set

# Convexity

- ▶ Examples of convex sets:
  - ▶ a square,
  - ▶ a disc,
  - ▶  $\mathbb{R}^n$ .

## Proposition

*The half space  $\{(x_1, x_2, \dots, x_n); x_1 \geq 0\}$  is convex.*

## Proof.

Let the half space be denoted by  $C$ . Take  $\mathbf{x}, \mathbf{y} \in C$  and  $\lambda \in [0, 1]$ .  
Then

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

$$\mathbf{y} = (y_1, y_2, \dots, y_n)$$

where  $x_1, y_1 \geq 0$ .

# Convexity

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## Proof (cont.).

We then have

$$\begin{aligned}\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} &= \lambda(x_1, x_2, \dots, x_n) + (1 - \lambda)(y_1, y_2, \dots, y_n) \\ &= (\lambda x_1 + (1 - \lambda)y_1, \dots, \lambda x_n + (1 - \lambda)y_n),\end{aligned}$$

whose first entry is  $\lambda x_1 + (1 - \lambda)y_1 \geq 0$  since  $\lambda$ ,  $(1 - \lambda)$ ,  $x_1$  and  $y_1$  are all non-negative. Hence  $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in C$ . Since  $\mathbf{x}, \mathbf{y} \in C$  were arbitrary, this shows that  $C$  is convex.  $\square$

## Example

*A general halfspace*

$$\{(x_1, \dots, x_n); a_1x_1 + a_2x_2 + \dots + a_nx_n \geq b\}$$

*is convex. You will prove this in one of the assigned exercises.*

## Proposition

*The intersection of finitely many convex sets is convex.*

Try to prove this yourself or together with a friend before you look at my proof. Note that it suffices to prove the statement for two sets,  $A$  and  $B$ . (Why?)

# Convexity

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## Proof.

Suppose that  $A, B \subset \mathbb{R}^n$  are convex. The intersection of  $A$  and  $B$  is, by definition,

$$A \cap B = \{\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \in A \text{ and } \mathbf{x} \in B\}.$$

Let  $\mathbf{x}, \mathbf{y} \in A \cap B$  and let  $\lambda \in [0, 1]$ . We need to prove that

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in A \cap B.$$

$$\mathbf{x}, \mathbf{y} \in A \text{ and } A \text{ is convex} \implies \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in A.$$

$$\mathbf{x}, \mathbf{y} \in B \text{ and } B \text{ is convex} \implies \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in B.$$

Hence  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in A \cap B$ , as required. □

# Convexity

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The boundary of a half-space is called a hyperplane:

$$\{(x_1, \dots, x_n); a_1x_1 + \dots + a_nx_n = b\}.$$

## Proposition

*Every hyperplane is convex.*

In one of the assigned exercises, you will be asked to prove this proposition. Do this in two ways:

- ▶ Directly from the definition of convexity,
- ▶ By writing the hyperplane as an intersection of two sets that you have already proved are convex, and using a previous proposition.

## Proposition

*The solution set of a linear system*

$$\mathbf{Ax} = \mathbf{b}$$

*is convex.*

## Proof.

Each of the equations in the system describes a hyperplane in  $\mathbb{R}^n$ . The solution set of the system is the intersection of all these hyperplanes (since all the equations have to be satisfied at the same time). The result follows since all hyperplanes are convex, and the intersection of finitely many convex sets is convex.  $\square$

# Convexity

## Definition

*The intersection of a finite number of half spaces (in  $\mathbb{R}^n$ ) is called a convex polyhedron.*

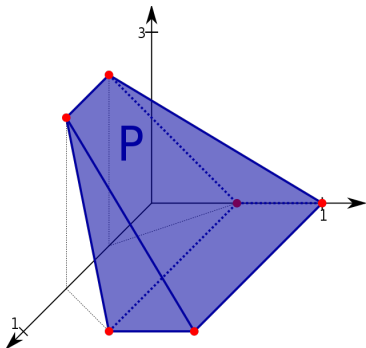


Figure 3: A convex polyhedron. Can you identify the half-spaces?



# Convexity

## Definition

Let  $C \subset \mathbb{R}^n$  be a convex set. A function  $f : C \rightarrow \mathbb{R}$  is said to be convex if

$$\mathbf{x}, \mathbf{y} \in C, \quad \lambda \in [0, 1] \quad \implies \quad f(\lambda \mathbf{x} + (1-\lambda) \mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).$$

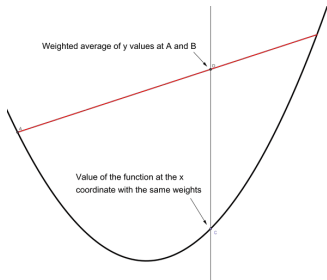


Figure 4: A convex function

# Convexity

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Note that the domain of a convex function needs to be a convex set. This is to ensure that  $f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y})$  is defined for all  $\lambda \in [0, 1]$ , i.e. for all points on the line segment between  $\mathbf{x}$  and  $\mathbf{y}$ .

## Definition

*A convex optimization problem is a problem of the form*

$$\begin{array}{ll} \text{Minimize} & z = f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in C, \end{array}$$

*where  $C \subset \mathbb{R}^n$  is a closed convex set and  $f : C \rightarrow \mathbb{R}$  is a convex function.*

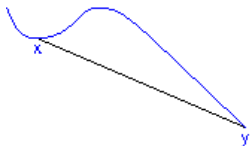
Note that a convex optimization problem is always a minimization problem.

# Convexity

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## Theorem

*For convex optimization problems, every local minimizer is a global minimizer (but it can happen that a minimizer doesn't exist).*



**Figure 5:** Proof idea: Suppose that  $\mathbf{x}$  is a local minimizer that is not a global minimizer. So there exists a point  $\mathbf{y}$  such that  $f(\mathbf{y}) < f(\mathbf{x})$ . But then there are points on the line segment between  $(\mathbf{x}, f(\mathbf{x}))$  and  $(\mathbf{y}, f(\mathbf{y}))$  which lies below the graph of  $f$ , contradicting the convexity of  $f$ .

# Convexity

## Proof.

Suppose that  $\mathbf{x}$  is a local minimizer of  $f$  which is not a global minimizer. Then there exists a point  $\mathbf{y}$  such that  $f(\mathbf{y}) < f(\mathbf{x})$ . Then for every  $\lambda \in [0, 1]$ , we have

$$\lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) < \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{x}) = f(\mathbf{x}),$$

but since  $\mathbf{x}$  is a local minimizer, there exists  $\lambda_0 \in (0, 1)$  such that

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \geq f(\mathbf{x})$$

for all  $\lambda \in [\lambda_0, 1]$ . Hence, for every  $\lambda \in [\lambda_0, 1]$  we have

$$\lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) < f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}),$$

contradicting the assumption that  $f$  is convex. The proof is complete. □

# Convexity

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## Definition

Let  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \subset \mathbb{R}^n$  be a finite set. A convex combination of  $\mathbf{x}_1, \dots, \mathbf{x}_n$  is a point  $\mathbf{y} \in \mathbb{R}^n$  which can be written as

$$\mathbf{y} = \sum_{j=1}^n \lambda_j \mathbf{x}_j,$$

where  $\sum_{j=1}^n \lambda_j = 1$  and  $\lambda_j \in [0, 1]$ .

## Theorem

*The set of all convex combinations of a finite set is a bounded convex polyhedron.*

# Convexity

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## Definition

*Let  $C$  be a convex set. A point  $x \in C$  is called an extreme point of  $C$  if it is not the interior point of any line segment in  $C$ .*

## Example

- ▶ *The set of extreme points of a disc is a circle (the boundary of the disc).*
- ▶ *The set of extreme points of a rectangle is the set of the four corner points of the rectangle.*

## Example

*In linear programming we optimize over convex polyhedra. A convex polyhedron has finitely many extreme points (if any).*

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# Linear programming

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## The general problem of linear programming.

$$\begin{array}{ll} \text{Maximize or minimize} & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \left\{ \begin{array}{l} \mathbf{A}_1 \mathbf{x} \leq \mathbf{b}_1, \\ \mathbf{A}_2 \mathbf{x} \geq \mathbf{b}_2, \\ \mathbf{A}_3 \mathbf{x} = \mathbf{b}_3, \end{array} \right. \end{array}$$

where  $\mathbf{c}$ ,  $\mathbf{A}_j$ ,  $\mathbf{b}_j$  are given vectors or matrices of size  $n \times 1$ ,  $m_j \times n$ ,  $m_j \times 1$ , respectively (but all of them don't have to be there). The relations  $\leq$ ,  $\geq$  and  $=$  are taken componentwise.  $\mathbf{x} \in \mathbb{R}^n$  is the unknown vector that we wish to find together with the optimal value  $z$ . □



# Linear programming

## Theorem

*A linear programming (LP) problem is a convex optimization problem and the domain is a convex polyhedron.*

## Proof sketch.

The constraints describe the intersection of finitely many half-spaces, and so the domain is a convex polyhedron by definition. If the LP problem is a maximization problem, it can be rephrased as a minimization problem by using

$$z = \max \mathbf{c}^T \mathbf{x} \quad \iff \quad -z = \min(-\mathbf{c}^T \mathbf{x}).$$

It is easy to check (Do it!) that all linear functions are convex. Hence the optimization problem is a convex optimization problem. □

## Theorem (Extreme point theorem for LP problems)

*Let  $S$  be the domain of a general LP problem (so  $S$  is a convex polyhedron), and let  $E$  be the set of extreme points of  $S$ .*

- ▶ If  $S \neq \emptyset$  and  $S$  is bounded, then there exists an optimal solution which belongs to  $E$ .*
- ▶ If  $E \neq \emptyset$  and  $S$  is unbounded, and if an optimal solution exists, then there is an optimal solution which belongs to  $E$ .*
- ▶ If an optimal solution does not exist, then  $S$  is either  $\emptyset$  or unbounded.*

# Linear programming

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- ▶ The extreme point theorem for LP problems implies that if an optimal solution exists and if there exists at least one extreme point, then there has to be an optimal solution at an extreme point of the feasible set. Since the feasible set of an LP problem is a convex polyhedron and the set of extreme points of a convex polyhedron is finite (or empty), we obtain a combinatorial optimization problem.
- ▶ For this reason, linear programming problems can be regarded as convex optimization problems (which is a special case of continuous optimization problems) as well as combinatorial (discrete) optimization problems.
- ▶ In this course, we will mainly study linear programming problems as a combinatorial optimization problem.