Seminarium. | Fö 11 | 1

12.33 b | Bestäm alla primitiver till $\frac{1}{\sin^3 x}$ med variabelbytet $t = \tan\frac{x}{2}$.

Enligt Fö 7 är $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$.

Vi har $\int \frac{1}{\sin^3 x}dx = \int \frac{(1+t^2)^3}{8t^3} \cdot \frac{2}{1+t^2}dt = \frac{1}{4}\int \frac{(1+t^2)^2}{t^3}dt$

$$= \frac{1}{4}\int \frac{t^4+2t^2+1}{t^3}dt = \frac{1}{4}\int \left(t + 2\cdot\frac{1}{t} + \frac{1}{t^3}\right)dt =$$

$$= \frac{1}{4}\left(\frac{1}{2}t^2 + 2\ln|t| - \frac{1}{2t^2}\right) + C =$$

$$= \frac{1}{8}\tan^2\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{8\tan^2\frac{x}{2}} + C =$$

$$= \frac{1}{8}\left(\tan^2\frac{x}{2} - \cot^2\frac{x}{2}\right) + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C.$$

12.43 | Bestäm alla primitiver till $\frac{\sqrt{x^2+2}-x}{\sqrt{x^2+2}+x}$

Lösning. Förläng med konjugatet (får bort rottecknet)

$$\int \frac{\sqrt{x^2+2}-x}{\sqrt{x^2+2}+x}dx = \int \frac{(\sqrt{x^2+2}-x)^2}{x^2+2-x^2}dx = \frac{1}{2}\int \left(x^2+2+x^2-2x\sqrt{x^2+2}\right)dx$$

$$= \int \left(x^2+1-x\sqrt{x^2+2}\right)dx = \frac{1}{3}x^3 + x - \frac{1}{2}\int \underbrace{2x}_{\text{inre derivatan}}\sqrt{x^2+2}\,dx$$

$$= \frac{1}{3}x^3 + x - \frac{1}{2}\cdot\frac{2}{3}(x^2+2)^{3/2} + C =$$

$$= \frac{1}{3}x^3 + x - \frac{1}{3}(x^2+2)^{3/2} + C.$$

Går även att lösa med substitutionen $x = \sqrt{2}\sinh t$, men det blir mycket jobbigare!

13.18a) $\displaystyle\int_1^2 \frac{x\ln x}{(1+x^2)^2}\,dx = \frac12\int_1^2 \underbrace{\frac{2x}{(1+x^2)^2}}\,\underbrace{\ln x}\,dx =$

$\displaystyle =\frac12\left[-\frac{1}{1+x^2}\ln x\right]_1^2 - \frac12\int_1^2\left(-\frac{1}{1+x^2}\cdot\frac1x\right)dx =$

$\displaystyle =\frac12\left(-\frac15\ln 2+\frac12\underbrace{\ln 1}_{=0}\right)+\frac12\int_1^2\frac{1}{x(1+x^2)}\,dx =$

$\displaystyle =-\frac1{10}\ln 2+\frac12\int_1^2\left(\frac1x-\frac{x}{1+x^2}\right)dx =$

$\displaystyle =-\frac1{10}\ln 2+\left[\frac12\ln|x|-\frac14\ln(1+x^2)\right]_1^2 =$

$\displaystyle =-\frac1{10}\ln 2+\frac12\ln 2-\frac14\ln 5-\frac12\underbrace{\ln 1}_{=0}+\frac14\ln 2$

$\displaystyle =\frac{13}{20}\ln 2-\frac14\ln 5.$

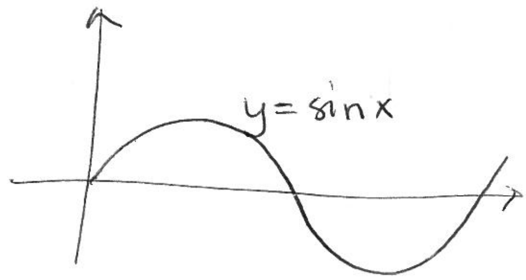
$\displaystyle \frac{1}{x(1+x^2)}=\frac{A}{x}+\frac{Bx+C}{1+x^2}$

$A=1$ (handpåläggning)

$\displaystyle \frac{1}{x(1+x^2)}-\frac1x=\frac{1-1-x^2}{x(1+x^2)}=$

$\displaystyle =-\frac{x}{1+x^2}=\frac{Bx+C}{1+x^2}\quad B=-1,\ C=0$

13.19b) $\displaystyle\int_0^{2\pi} e^{-x}|\sin x|\,dx = ?$



$\sin x\geq 0$ då $0\leq x\leq\pi$ och ≤ 0 då $\pi\leq x\leq 2\pi$.

$\displaystyle\int_0^{2\pi}e^{-x}|\sin x|\,dx=\int_0^{\pi}e^{-x}\underbrace{|\sin x|}_{=\sin x\text{ här}}dx+\int_{\pi}^{2\pi}e^{-x}\underbrace{|\sin x|}_{=-\sin x\text{ här}}dx =$

$\displaystyle =\int_0^{\pi}e^{-x}\sin x\,dx-\int_{\pi}^{2\pi}e^{-x}\sin x\,dx$

Bestäm först en primitiv till $e^{-x}\sin x$. (undvik dubbelarbete!)

$\displaystyle\int e^{-x}\sin x\,dx=-e^{-x}\sin x+\int e^{-x}\cos x\,dx=$

$\displaystyle =-e^{-x}\sin x-e^{-x}\cos x+\int e^{-x}(-\sin x)\,dx=$

$\displaystyle =-e^{-x}(\sin x+\cos x)-\int e^{-x}\sin x\,dx.$

$\displaystyle \Longrightarrow\int e^{-x}\sin x\,dx=-\frac12 e^{-x}(\sin x+\cos x)+C$

$$\Rightarrow \int_0^{2\pi} e^{-x}|\sin x|\,dx = \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^{\pi} -$$

$$\left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_{\pi}^{2\pi} =$$

$$= -\frac{1}{2}\left(e^{-\pi}\cdot(-1) - 1\right) + \frac{1}{2}\left(e^{-2\pi} - e^{-\pi}\cdot(-1)\right) =$$

$$= \frac{1}{2}\left(1 + 2e^{-\pi} + e^{-2\pi}\right) = \frac{1}{2}\left(1 + e^{-\pi}\right)^2$$

Om man vill