An Introduction to Nonlinear Waves

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The aim of these notes is to give an introduction to the mathematics of nonlinear waves. The waves are modelled by partial differential equations (PDE), in particular hyperbolic or dispersive equations. Some aspects of completely integrable systems and soliton theory are also discussed. While the goal is to discuss the nonlinear theory, this cannot be achieved without first discussing linear PDE. The prerequisites include a good background in real and complex analysis, vector calculus and ordinary differential equations. Some basic knowledge of Lebesgue integration and (the language of) functional analysis is also useful. There are two appendices providing the necessary background in these areas. I have not assumed any prior knowledge of PDE.

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Chapter 1
Introduction

You are probably familiar with many different kinds of wave phenomena such as surface waves on the ocean, sound waves in the air or other media, and electromagnetic waves, of which visible light is a special case. A common feature of these examples is that they all can be described by partial differential equations (PDE). The purpose of these lecture notes is to give an introduction to various kinds of PDE describing waves. In particular we will focus on nonlinear equations. In this chapter we introduce some basic concepts and give an overview of the contents of the lecture notes.

1.1 Linear waves

1.1.1 Sinusoidal waves

The first encounter with the mathematical theory of waves is usually with cosine (or sine) waves of the form

$$u(x,t) = a \cos(kx \pm \omega t).$$

Here $x, t \in \mathbb{R}$ denote space and time, respectively, and the parameters $a$, $k$ and $\omega$ are positive numbers. $a$ is the amplitude of the wave, $k$ the wave number and $\omega$ the angular frequency. Note that $u$ is periodic in both space and time with periods

$$\lambda = \frac{2\pi}{k}$$

and

$$T = \frac{2\pi}{\omega}.$$ 

$\lambda$ is usually called the wavelength and $T$ simply the period. The wave has crests (local maxima) when $kx \pm \omega t$ is an even multiple of $\pi$ and troughs (local minima) when it is an odd multiple of $\pi$.

Rewriting $u$ as

$$u(x,t) = a \cos(k(x \pm ct)).$$
where 
\[ c = \frac{\omega}{k}, \]  
we see that as \( t \) changes, the fixed shape 
\[ X \mapsto \cos(kX) \]
is simply being translated at constant speed \( c \) along the \( x \)-axis. The number \( c \) is called the wave speed or phase speed. The wave moves to the left if the sign is positive and to the right if the sign is negative. A wave of permanent shape moving at constant speed is called a travelling wave or sometimes a steady wave.

### 1.1.2 The wave equation

Assume that the phenomenon being studied is linear so that more complicated wave forms can be obtained by adding different cosine waves. In order to do this, we must be more specific about the relation between the parameters \( c \) and \( k \) (the parameter \( \omega \) is determined by \( c \) and \( k \) through (1.1)). One often makes the assumption that \( c \) is independent of \( k \), so that all cosine waves move at the same speed. Note that the waves then satisfy the PDE
\[ u_{tt}(x,t) = c^2 u_{xx}(x,t) \]  
(subscripts denote partial derivatives). This is the wave equation in one (spatial) dimension. The assumption that one can add the waves together agrees with the linearity of the wave equation; any linear combination of solutions of (1.2) is also a solution of (1.2).

A better way of deriving the wave equation is to start from physical principles. Here is an example.

**Example 1.1.** What we perceive as sound is really a pressure wave in the air. Sound waves are longitudinal waves, meaning that the oscillations take place in the same direction as the wave is moving. The equations governing acoustic waves are the equations of gas dynamics.
These form a system of nonlinear PDE for the velocity $u$ and the density $\rho$. Assuming that the oscillations are small, one can derive a set of linearized equations:

\[
\begin{align*}
\rho_0 u_t + c_0^2 \nabla \rho &= 0, \\
\rho_t + \rho_0 \nabla \cdot u &= 0,
\end{align*}
\]

where $\rho_0$ is the density and $c_0$ the speed of sound in still air. Here $u(\cdot,t) : \Omega \to \mathbb{R}^3$ while $\rho(\cdot,t) : \Omega \to \mathbb{R}$ for some open set $\Omega \subset \mathbb{R}^3$. If we consider a one-dimensional situation (e.g. a thin pipe), the system simplifies to

\[
\begin{align*}
\rho_0 u_t + c_0^2 \rho_x &= 0, \\
\rho_t + \rho_0 u_x &= 0.
\end{align*}
\]

Differentiating both equations with respect to $t$ and $x$, we obtain after some simplifications that both $u$ and $\rho$ satisfy the wave equation:

\[
\begin{align*}
u_{tt} &= c_0^2 u_{xx}
\end{align*}
\]

and

\[
\begin{align*}
\rho_{tt} &= c_0^2 \rho_{xx}.
\end{align*}
\]

Note that $\rho$ describes the fluctuation of the density around $\rho_0$. In the linear approximation, the pressure fluctuation $p$ around the constant atmospheric pressure is simply proportional to $\rho$. It follows that $p$ also satisfies the wave equation. The nonlinear equations of gas dynamics will be discussed in Chapter 4.

We have assumed that the waves are even in $x \pm ct$. To find the most general form of the wave we also have to include terms of the form $\sin(k(x \pm ct))$. In order to make the notation simpler it is convenient to use the complex exponential function. The original real-valued function can be recovered by separating into real and imaginary parts. It is convenient also to allow $k$ to be negative. A general linear combination of sinusoidal waves can be written

\[
u(x,t) = \sum_k (a_k e^{ik(x-ct)} + b_k e^{ik(x+ct)}),
\]

which is often rewritten (with other coefficients) in the form

\[
u(x,t) = \sum_k e^{ikx}(a_k \cos(kct) + b_k \sin(kct)).
\]

Depending on the physical situation there might be boundary conditions which limit the range of wave numbers $k$. We might e.g. only look for a wave in a bounded interval, say $[0, \pi]$, which vanishes at the end points. This forces us to look for solutions of the form

\[
u(x,t) = \sum_{k=1}^{\infty} \sin(kx)(a_k \cos(kct) + b_k \sin(kct)).
\]
In these notes we will not discuss boundaries, and hence there is no restriction on \( k \). The most general linear combination is then not a sum, but an integral

\[
u(x, t) = \int_{\mathbb{R}} e^{i\xi x} \left( a(\xi) \cos(\xi c t) + b(\xi) \sin(\xi c t) \right) d\xi
\]

(we have changed \( k \) to \( \xi \) to emphasize that it is a continuous variable). If you have studied Fourier analysis, you will recognize this as a Fourier integral. Suppose that we know the initial position and velocity of \( u \), that is, \( u(x, 0) \) and \( u_t(x, 0) \). Then \( a \) and \( b \) are determined by

\[
u(x, 0) = \int_{\mathbb{R}} e^{i\xi x} a(\xi) d\xi
\]

and

\[
u_t(x, 0) = \int_{\mathbb{R}} e^{i\xi x} c \xi b(\xi) d\xi.
\]

There are of course many question marks to be sorted out if one wants to make this into rigorous mathematics. In particular, one has to know that the functions \( u(x, 0) \) and \( u_t(x, 0) \) can be written as Fourier integrals and that the functions \( a \) and \( b \) are unique. This question is discussed in Chapters 2 and 6. In Chapter 3 we will discuss the application of Fourier methods to solving PDE such as the wave equation.

There is also another way of solving the wave equation discovered by d’Alembert. The idea is that the equation can be rewritten in the factorized form

\[(\partial_t - c \partial_x) (\partial_t + c \partial_x) u = 0.
\]

Introducing the new variable \( v = u_t + cu_x \), we therefore obtain the two equations

\[
u_t + cu_x = v,
\]

\[
v_t - cv_x = 0.
\]

The expression \( v_t - cv_x \) is the directional derivative of the function \( v \) in the direction \((-c, 1)\). The second equation therefore expresses that \( v \) is constant on the lines \( x + ct = \text{const} \). In other words, \( v \) is a function of \( x + ct \), \( v(x, t) = f(x + ct) \). The first equation now becomes

\[u_t + cu_x = f(x + ct)
\]

The left hand side of this equation can be interpreted as the directional derivative of \( u \) in the direction \((c, 1)\). Evaluating \( u \) along the straight line \( \gamma(s) = (y + cs, s) \), where \( y \in \mathbb{R} \) is a parameter, we find that

\[\frac{d}{ds} ((u \circ \gamma)(s)) = (v \circ \gamma)(s),\]

and hence

\[(u \circ \gamma)(s) = \int (v \circ \gamma)(r) \, dr
\]

\[= \int f(y + 2cr) \, dr
\]

\[= F(y + 2cs) + G(y)
\]
where \( F(s) = \frac{1}{2c} \int_0^s f(r) \, dr \) and \( G(y) \) is an integration constant (\( f \) is used to denote the indefinite integral). Setting \( s = t, y = x - ct \), we obtain that

\[
 u(x,t) = F(x + ct) + G(x - ct).
\]

This can be interpreted as saying that the general solution is a sum of a wave moving to the left, \( F(x + ct) \), and a wave moving to the right, \( G(x - ct) \). One usually derives this formula by making the change of variables \( y = x - ct, z = x + ct \), which results in the equation \( u_{yz} = 0 \). The approach used here is however useful to keep in mind when we discuss the method of characteristics in Chapter 4 (see also Section 1.2 below).

The functions \( F \) and \( G \) can again be uniquely determined once we know the initial data \( u(x,0) = u_0(x) \) and \( u_t(x,0) = u_1(x) \). Indeed, we obtain the system of equations

\[
\begin{align*}
F + G &= u_0, \\
F - G &= \frac{1}{c} U_1,
\end{align*}
\]

where \( U'_1 = u_1 \), which has the unique solution

\[
\begin{align*}
F &= \frac{1}{2} u_0 + \frac{1}{2c} U_1, \\
G &= \frac{1}{2} u_0 - \frac{1}{2c} U_1
\end{align*}
\]

Since \( U_1(s) = \int_0^s u_1(r) \, dr + C \), this results in d’Alembert’s solution formula

\[
 u(x,t) = \frac{1}{2} (u_0(x + ct) + u_0(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(s) \, ds.
\]

Since we have derived a formula for the solution, it follows that it is unique under suitable regularity assumptions.

**Theorem 1.2.** Assume that \( u_0 \in C^2(\mathbb{R}), u_1 \in C^1(\mathbb{R}) \) and \( c > 0 \). There exists a unique solution \( u \in C^2(\mathbb{R}^2) \) of the initial-value problem

\[
\begin{cases}
 u_{tt} = c^2 u_{xx} \\
 u(x,0) = u_0(x) \\
 u_t(x,0) = u_1(x).
\end{cases}
\]

The solution is given by d’Alembert’s formula

\[
 u(x,t) = \frac{1}{2} (u_0(x + ct) + u_0(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(s) \, ds.
\]
1.1.3 Dispersion

In many cases, the assumption that $c$ is constant as a function of $k$ is unrealistic. If $c$ is a non-constant function of $k$ one says that the wave (or rather the equation) is *dispersive*. This means that waves of different wavelengths travel with different speeds. An initially localized wave will disintegrate into separate components according to wavelength and disperse. The relation between $\omega$ and $k$ is called the *dispersion relation* (although some authors use this term for the relation between $c$ and $k$).

**Example 1.3.** The equation

$$ihu_t + \frac{\hbar^2}{2m} u_{xx} = 0$$

is the one-dimensional version of the famous *free Schrödinger equation* from quantum mechanics, describing a quantum system consisting of one particle. $\hbar$ is the reduced Planck’s constant, and $m$ the mass of the particle. The function $u$, called the *wave function*, is complex-valued and its modulus square, $|u|^2$, can be interpreted as a probability density. Assuming that $u$ is square-integrable and has been normalized so that

$$\int_{\mathbb{R}} |u(x,t)|^2 \, dx = 1,$$

the probability of finding the particle in the interval $[a, b]$ is given by

$$\int_a^b |u(x,t)|^2 \, dx.$$ 

After rescaling $t$ we can rewrite the Schrödinger equation in the form

$$iu_t + u_{xx} = 0.$$ 

Making the Ansatz $u(x,t) = e^{(ikx-\omega(k)t)}$, we obtain the equation

$$\omega(k) = k^2$$

and therefore

$$c(k) = \frac{\omega(k)}{k} = k.$$ 

Note that we allow $c$, $k$ and $\omega$ to be negative now, so that waves can travel either to the left ($k < 0$) or to the right ($k > 0$).

**Example 1.4.** The *linearized Korteweg-de Vries (KdV) equation*

$$u_t + u_{xxx} = 0$$

appears as a model for small-amplitude water waves with long wavelength. The equation is sometimes called *Airy’s equation*, although that name is also used for the closely related ordinary differential equation $y'' - xy = 0$. Substituting $u(x,t) = e^{(ikx-\omega(k)t)}$ into the equation, we find that

$$\omega(k) = -k^3.$$
and hence
\[ c(k) = -k^2. \]
The fact that \( c \) is negative means that the waves travel to the left.

Since \( c(k) \) depends on \( k \) for dispersive waves, the wave speed only gives the speed of a wave with a single wave number. A different speed can be found if one looks for a solution which consists of a narrow band of wave numbers. For simplicity, we suppose that the equation has cosine solutions \( \cos(kx - \omega(k)t) \) and consider the sum of two such waves with nearly the same wave numbers \( k_1 \) and \( k_2 \). Let \( \omega_1 \) and \( \omega_2 \) be the corresponding angular frequencies and suppose that the waves have the same amplitude \( a \). Using the trigonometric identity \( \cos(\alpha) + \cos(\beta) = 2\cos((\alpha - \beta)/2)\cos((\alpha + \beta)/2) \), the sum can be written
\[
2a \cos\left(\frac{(k_1 - k_2)x}{2} - \frac{(\omega_1 - \omega_2)t}{2}\right) \cos\left(\frac{(k_1 + k_2)x}{2} - \frac{(\omega_1 + \omega_2)t}{2}\right).
\]
This is a product of two cosine waves, one with wave speed \( (\omega_1 + \omega_2)/(k_1 + k_2) \) and the other with speed \( (\omega_1 - \omega_2)/(k_1 - k_2) \). Setting \( k_1 = k_0 + \Delta k \) and \( k_2 = k_0 - \Delta k \), we obtain as \( \Delta k \to 0 \) the approximate form
\[
2a \cos(\Delta k(x - \omega'(k_0)t)) \cos(k_0(x - c(k_0)t)).
\]
We thus find an approximate solution in the form of a carrier wave, \( \cos(k_0(x - c(k_0)t)) \), with speed \( c(k_0) \), multiplied by an envelope, \( \cos(\Delta k(x - \omega'(k_0)t)) \), with speed \( \omega'(k_0) \). This is an example of a wave packet. Notice that the carrier has wavelength \( 2\pi/k_0 \) whereas the envelope has the much longer wavelength \( 2\pi/\Delta k \). The speed
\[
c_g = \frac{d\omega}{dk}
\]
of the envelope is called the group speed. Notice the similarity with the definition of the phase speed
\[
c = \frac{\omega}{k}.
\]
In a wave packet, the group speed measures the speed of the packet as a whole, whereas the phase speed measures the speed of the individual components of the packet. See Figure 1.2 for an illustration. We will analyze this in more detail in Chapter 6.

**Example 1.5.**

- For the wave equation, we have \( \omega_{\pm}(k) = \pm ck \), so the group speed and the phase speed are equal.
- For the Schrödinger equation we have \( c_g(k) = \omega'(k) = 2k \) while \( c(k) = k \).
- For the linearized KdV equation, we have \( c_g(k) = \omega'(k) = -3k^2 \) and \( c(k) = -k^2 \).

We remark that for each \( k \neq 0 \), the wave equation has solutions travelling to the left and solutions travelling to the right, whereas the linearized KdV equation only have solutions travelling in one direction. We say that the wave equation is bidirectional while the linearized KdV equation is unidirectional. The Schrödinger equation is somewhere in between; it allows waves travelling in both directions, but the direction is determined by the sign of \( k \).
1.2 Nonlinear waves

The (inviscid) Burgers equation

\[ u_t + uu_x = 0 \]  \hspace{1cm} (1.3)

is a simplified model for fluid flow. It is named after the Dutch physicist Jan Burgers who studied the viscous version

\[ u_t + uu_x = \varepsilon u_{xx}, \quad \varepsilon > 0, \]

(viscosity is the internal friction of a fluid) as a model for turbulent flow. Recalling our discussion of the wave equation above, we can think of the Burgers equation as

\[ u_t + c(u)u_x = 0, \]

with ‘speed’ \( c(u) = u \). Again, the left hand side can be thought of as a directional derivative in the direction \((u, 1)\), which now depends on \( u \). When studying the equation \( u_t + cu_x = 0 \) with constant \( c \), we found that the solution was constant on straight lines in the direction \((c, 1)\). This suggests that the solution of Burgers’ equation should be constant on curves with tangent \((u(x,t), 1)\). Such curves are given by \((x(t), t)\), where \( x(t) \) is a solution of the differential equation

\[ \dot{x}(t) = u(x(t), t). \]

This doesn’t seem to help much since we don’t know \( u \). However, since \( u \) is constant on the curve, the equation actually simplifies to

\[ \dot{x}(t) = u_0(x_0), \]

where \( x_0 = x(0) \) and \( u_0(x) = u(x, 0) \). This means that the curves \((x(t), t)\) actually are straight lines,

\[ (x(t), t) = (x_0 + u_0(x_0)t, t), \]

and that we have

\[ u(x_0 + u_0(x_0)t, t) = u_0(x_0). \]  \hspace{1cm} (1.4)
This determines, at least formally, the solution in an implicit way. Note that this gives credence to our interpretation of \( u \) as a speed: considering a ‘particle’ situated originally on the graph of \( u \) at \((x_0, u_0(x_0))\), this particle will travel parallel to the \( x \)-axis with constant speed \( u_0(x_0) \) (the particle will travel backwards if \( u_0(x_0) < 0 \)). This suggests that (1.4) actually does define the solution implicitly for some time. However, suppose that the function \( u_0 \) has negative slope somewhere, so that there exist points \( x_1 < x_2 \) with \( u_0(x_1) > u_0(x_2) \). We suppose for simplicity that these values are positive. The point originally located at \((x_1, u_0(x_1))\) will then have greater speed than the point located at \((x_2, u_0(x_2))\) so that at some point \( T \) the solution will develop a vertical tangent. Continuing the ‘Lagrangian’\(^1\) solution beyond this point in time, the function \( u \) will become multivalued (see Figure 1.3). Clearly, \( u \) then ceases to be a solution of the original equation (1.3). In Chapter 4 we will study a different way of continuing the solution past the time \( T \), namely as a shock wave. The solution then continues to be the graph of a function, but now has a jump discontinuity. Clearly, this solution can’t satisfy (1.3) either, and so we have to develop a different way of interpreting solutions.

The above example shows that nonlinear waves may develop singularities. It turns out that dispersion can sometimes help, however. If we add to the Burgers equation the dispersion term \( u_{xxx} \), we obtain the famous Korteweg-de Vries (KdV) equation

\[
    u_t + 6uu_x + u_{xxx} = 0
\]

(1.5)

(the coefficient 6 is inessential). This equation is named after the Dutch mathematician Diederik Korteweg and his student Gustav de Vries, who derived it as an equation for long water waves in 1895. The equation had in fact been derived previously by Joseph Boussinesq [3]. As we will show in Chapter 6, the solutions of the KdV equation remain smooth for all times. Naïvely one can think of the nonlinear terms as trying to focus and steepen the wave. The dispersive term on the other hand tends to spread the wave apart and therefore might be able to counteract the nonlinearity.

\(^1\)In the Lagrangian approach to fluid mechanics, the coordinates follow the fluid particles. In the Eulerian approach the coordinates are fixed while the fluid moves.
1.3 Solitons

The KdV equation was derived in attempt to explain a phenomenon observed earlier by the Scottish naval engineer John Scott Russell. In 1834, on the Union canal between Edinburgh and Glasgow, Scott Russell made a remarkable discovery. This is his own account of the event [22].

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation; then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to foot and half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August, 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

As far as we know this is the first reported observation of a solitary water wave. Scott Russell was to spend a great deal of time reproducing the phenomenon in laboratory experiments. He built a small water channel in the end of which he dropped weights, producing solitary bump-like waves of the sort he had earlier observed. Figure 1.4 shows Scott Russell’s own illustrations of his experiments.

At that time the predominant mathematical theory for water waves was linear. In particular, the esteemed mathematicians and fluid dynamicists Sir George Gabriel Stokes and Sir George Biddell Airy—both with chairs at University of Cambridge—did not accept Russell’s theories, because they believed that it contradicted the mathematical/physical laws of fluid motion. In fact, if one considers only linear dispersive equations one finds that solitary travelling waves on the whole line are impossible. For the linearized KdV equation, any wave is the superposition of periodic travelling waves of different wave numbers which disperse with time.

To explain Scott Russell’s observations, we look for a travelling wave solution of the KdV equation which decays to 0 at infinity. Such a solution is called a solitary wave.² We thus make the Ansatz \( u(x,t) = U(x - ct) \). This yields the ODE

\[
-cU'(X) + 6U(X)U'(X) + U'''(X) = 0, \tag{1.6}
\]

where \( X = x - ct \). Since we are looking for a solitary wave we also require that

\[
\lim_{X \to \pm \infty} U(X) = 0. \tag{1.7}
\]

²A solitary wave could also converge to a non-zero constant at infinity.
Figure 1.4: John Scott Russell’s illustrations of his experiments from [22].
Figure 1.5: The phase portrait of (1.9) with $c > 0$. The thick curve represents the homoclinic orbit corresponding to the solitary wave. The filled circles represent the equilibria $(0,0)$ and $(0,c/3)$.

We may integrate equation (1.6) to

$$U'' = cU - 3U^2 + C_1, \quad C_1 \in \mathbb{R}.$$  

In view of (1.7), we set $C_1 = 0$. Otherwise $U''$ would converge to a non-zero number, yielding a contradiction. We thus arrive at the second order ODE

$$U'' = cU - 3U^2. \quad (1.8)$$

This can be written as the first-order system

$$\begin{cases}
U' = V \\
V' = cU - 3U^2
\end{cases} \quad (1.9)$$

and we find that the function

$$H(U,V) = V^2 - cU^2 - 2U^3$$

is constant on the orbits. We can thus find the phase portrait by drawing the level curves of $H$. Moreover, the orbit corresponding to a solitary wave must lie in the level set $H^{-1}(0)$. This reveals that we must take $c > 0$ in order to find a solitary wave. The phase portrait in this case is shown in Figure 1.5. We find that there is a homoclinic orbit connecting the origin with itself. Since the ODE is autonomous, this orbit corresponds to a family of solutions $X \mapsto U(X - X_0)$, $X_0 \in \mathbb{R}$, of (1.6)–(1.7). We also note from Figure 1.5 that $U(X) > 0$ for all $X$.

We can in fact find an explicit formula for $U(X)$. To make $U$ unique, we assume (without loss of generality) that $U(0) = c/2$ and hence $U'(0) = V(0) = 0$ (from the equation $H(U,V) = 0$). Note that

$$X \mapsto (U(-X), -V(-X))$$
also is a solution of (1.9) with the same initial data, so that by the Picard-Lindelöf theorem, 
\( U(X) = U(-X) \) \((U\) is even). It therefore suffices to find the solution for \( X > 0 \). Also, note that 
\( U(X) \in (0, c/2] \) for all \( X \). The equation \( H(U, U') = 0 \) can be written
\[
(U')^2 = cU^2 - 2U^3,
\]
which yields
\[
U' = -U\sqrt{c - 2U},
\]
for \( X \geq 0 \). This is a separable ODE. For \( X > 0 \) we know that \( U(X) < c/2 \) and hence
\[
-\int_{U(X)}^{U(X_0)} \frac{dU}{U\sqrt{c - 2U}} = X - X_0, \quad X_0 > 0, \quad X > 0.
\]
By continuity this also holds in the limit \( X_0 \to 0 \), so that
\[
-\int_{c/2}^{U(X)} \frac{dU}{U\sqrt{c - 2U}} = X, \quad X > 0.
\]
To calculate this integral, we introduce the new variable \( \theta \) defined by
\[
U = \frac{c}{2\cosh^2(\theta)}, \quad \frac{\partial U}{\partial \theta} = -\frac{c\sinh(\theta)}{\cosh^3(\theta)}, \quad \theta > 0.
\]
Note that \( \theta = 0 \) corresponds to \( U = c/2 \) and \( \theta = \infty \) to \( U = 0 \). The identity \( \cosh^2(\theta) - \sinh^2(\theta) = 1 \) implies that
\[
\sqrt{c - 2U} = \sqrt{c \left(1 - \frac{1}{\cosh^2(\theta)}\right)} = \sqrt{c \left(\cosh^2(\theta) - 1\right)\cosh^2(\theta)} = \sqrt{c} \frac{\sinh(\theta)}{\cosh(\theta)},
\]
and
\[
\frac{1}{U} \frac{1}{\sqrt{c - 2U}} \frac{\partial U}{\partial \theta} = -\frac{2\cosh^2(\theta)}{c} \frac{\cosh(\theta)}{\sqrt{c} \sinh(\theta)} \frac{c \sinh(\theta)}{\cosh^3(\theta)} = -\frac{2}{\sqrt{c}}.
\]
We therefore obtain that
\[
\frac{2}{\sqrt{c}} \theta(X) = \int_{0}^{\theta(X)} \frac{2}{\sqrt{c}} d\theta
= -\int_{c/2}^{U(X)} \frac{dU}{U\sqrt{c - 2U}}
= X,
\]
where
\[
U(X) = \frac{c}{2\cosh^2(\theta(X))}.
\]
It follows that
\[
U(X) = \frac{c}{2\cosh^2\left(\frac{\sqrt{c}}{2}X\right)} = \frac{c}{2} \text{sech}^2\left(\frac{\sqrt{c}}{2}X\right), \quad X \geq 0,
\]
where \( \text{sech} = 1/\cosh \) is the hyperbolic secant. Since \( U \) and \( \text{sech} \) are even, the formula also holds for \( X < 0 \). Going back to the original variables, we have proved the following result.
Figure 1.6: The great wave of translation. Three solitary-wave solutions of the KdV equation of the form (1.10). Notice that fast waves are tall and narrow.

**Theorem 1.6.** For each $c > 0$ there is a family of solitary-wave solutions

$$u(x, t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x - ct - x_0) \right), \quad x_0 \in \mathbb{R}, \quad (1.11)$$

of the KdV equation (1.5). For $c < 0$ there are no solitary wave solutions.

Note that the amplitude is proportional to the wave speed $c$, so that higher waves travel faster. The width of the wave\(^3\), on the other hand, is inversely proportional to $\sqrt{c}$. In other words, the waves become narrower the taller they are (see Figure 1.6). Another interesting aspect is that the solitary waves move in the opposite direction of the travelling waves of the linearized KdV equation (cf. Example 1.4). In fact, if one perturbs a solitary wave, one will generally see a small ‘dispersive tail’ travelling in the opposite direction.

The KdV equation was largely forgotten about by the mathematical community until the 1960’s. In a numerical study of the initial-value problem

$$u_t + uu_x + \delta^2 u_{xxx} = 0, \quad u(x, 0) = \cos(\pi x), \quad (1.12)$$

with periodic boundary conditions and $\delta \neq 0$ but small, Martin Kruskal and Norman Zabusky discovered that the solution splits up into finitely many distinct waves resembling the sech\(^2\)-wave, which interacted with each other almost as the equation was linear. They called the waves *solitons*. We will investigate this in much more detail in Chapter 7. Figure 1.7 shows some pictures from Kruskal and Zabusky’s original paper.

\(^3\)The width can e.g. be defined as the distance between to points where the elevation is half of the amplitude, $u = c/4$. 

```
Figure 1.7: The first reported solitons, from the pioneering paper [30] by Kruskal and Zabusky. The left plot depicts the temporal development of a solution of (1.12) with $\delta = 0.22$. The wave first seems to break, but instead eventually separates into eight solitary waves whose paths through space-time can be followed in the picture to the right. We see that the velocities are constant but for when the solitons interact (the paths are nearly straight lines interrupted by sudden shifts).
Chapter 2

The Fourier transform

2.1 The Schwartz space

We begin by studying the Fourier transform on a class of very nice functions. This means that we don’t have to worry much about technical details. In order to define this class it is practical to use the ‘multi-index’ notation due to Laurent Schwartz. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we define

$$\partial_\alpha x = \partial_{\alpha_1} x_1 \partial_{\alpha_2} x_2 \cdots \partial_{\alpha_d} x_d.$$ 

One can think of $\partial_x$ as the $d$-tuple $(\partial_{x_1}, \ldots, \partial_{x_d})$. We similarly define $x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $x \in \mathbb{R}^d$ (or even $\mathbb{C}^d$). The number $|\alpha| = \alpha_1 + \cdots + \alpha_d$ is called the order of the multi-index and we also write $\alpha! = \alpha_1! \cdots \alpha_d!$. All functions in this chapter are allowed to be complex-valued.

**Definition 2.1** (Schwartz space). The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the vector space of rapidly decreasing smooth functions,

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^{\alpha} \partial_\beta f(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^d \right\}.$$

$\mathcal{S}(\mathbb{R}^d)$ contains the class of $C^\infty$ functions with compact support, $C^\infty_0(\mathbb{R}^d)$. See Lemma B.24 for a non-trivial example of functions belonging to this class. The main reason for using $\mathcal{S}$ instead of $C^\infty_0$ is that, as we will see, the former class is invariant under the Fourier transform. This is not the case for $C^\infty_0$.

The topology on $\mathcal{S}(\mathbb{R}^d)$ is defined by the family of semi-norms (see Section A.1),

$$\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^d} |x^{\alpha} \partial_\beta f(x)|, \quad \alpha, \beta \in \mathbb{N}^d.$$ 

In other words, we say that

$$\varphi_n \to \varphi$$

in $\mathcal{S}(\mathbb{R}^d)$ if

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{\alpha, \beta} = 0.$$
for all $\alpha, \beta \in \mathbb{N}^d$. One can in fact show that
$$
\rho(f, g) := \sum_{\alpha, \beta} \frac{1}{2^{||\alpha|+|\beta||}} \frac{\|f-g\|_{\alpha, \beta}}{1+\|f-g\|_{\alpha, \beta}},
$$
defines a metric on $\mathcal{S}(\mathbb{R}^d)$. The topology defined by this metric is the same as the one defined by the semi-norms and the metric space $(\mathbb{R}^d, \rho)$ is complete. $\mathcal{S}(\mathbb{R}^d)$ is an example of a generalization of Banach spaces called Frechét spaces, but we will not use that in any substantial way. Notice, though, that the Schwartz space is not a Banach space, i.e. it cannot be equipped with a norm making it complete (in particular, $f \mapsto \rho(f, 0)$ does not define a norm).

**Proposition 2.2.**

1. For all $N \geq 0$ and $\alpha \in \mathbb{N}^d$ there exists a $C > 0$ such that $|\partial_\alpha^\alpha f(x)| \leq C(1 + |x|)^{-N}$, $x \in \mathbb{R}^d$.
2. $\mathcal{S}(\mathbb{R}^d)$ is closed under multiplication by polynomials.
3. $\mathcal{S}(\mathbb{R}^d)$ is closed under differentiation.
4. $\mathcal{S}(\mathbb{R}^d)$ is an algebra: $fg \in \mathcal{S}(\mathbb{R}^d)$ if $f, g \in \mathcal{S}(\mathbb{R}^d)$.
5. $C_0^\infty(\mathbb{R}^d)$ is a dense subspace of $\mathcal{S}(\mathbb{R}^d)$.
6. $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

**Proof.** The first property is basically a restatement of the definition of $\mathcal{S}(\mathbb{R}^d)$. Properties (2) and (3) follow from the definition of the Schwartz space. Property (4) is a consequence of Leibniz’ formula for derivatives of products.

It is clear that $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. To show that it is dense, let $f \in \mathcal{S}(\mathbb{R}^d)$ and define $f_n(x) = \varphi(x/n)f(x)$, where $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi(x) = 1$ for $|x| \leq 1$ (see Lemma B.24). Then
$$
f_n - f = (\varphi(x/n) - 1)f(x)
$$
with support in $\mathbb{R}^d \setminus B_n(0)$. From property (1) it follows that $|f(x)| \leq C_N(1 + |x|)^{-N}$ for any $N \geq 0$. Therefore
$$
\|f_n - f\|_{L^\infty(\mathbb{R}^d)} \leq C_N(1 + \|\varphi\|_{L^\infty(\mathbb{R}^d)})(1+n)^{-N} \to 0
$$
as $n \to \infty$. Since $N$ is arbitrary, the same is true if we multiply $f_n - f$ by a polynomial. Since the partial derivatives of $\varphi(x/n) - 1$ all have support in $\mathbb{R}^d \setminus B_n(0)$ and are bounded uniformly in $n$, the convergence remains true after differentiation. This proves (5).

By estimating
$$
|f(x)| \leq C(1 + |x|)^{-d-1}
$$
and using the fact that
$$
\int_{\mathbb{R}^d}(1 + |x|)^{-p(d+1)}\,dx = C \int_0^\infty \frac{r^{d-1}}{1 + rp(d+1)}\,dr < \infty,
$$
it follows that \( f \in L^p(\mathbb{R}^d) \) for \( 1 \leq p < \infty \), if \( f \in \mathcal{S}(\mathbb{R}^d) \). Since \( C^\infty_0(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d) \) (cf. Corollary B.26), property (6) now follows from property (5) (we only use the part that \( C^\infty_0(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \)).

\[ \]

**Example 2.3.** The Gaussian function \( e^{-|x|^2} \) is an example of a function in \( \mathcal{S}(\mathbb{R}^d) \) which is not in \( C^\infty_0(\mathbb{R}^d) \).

### 2.2 The Fourier transform on \( \mathcal{S} \)

**Definition 2.4** (Fourier transform). The Fourier transform, \( \mathcal{F}(f) \), of a function \( f \in \mathcal{S}(\mathbb{R}^d) \) is defined by

\[
\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx.
\]

We also use the notation \( \hat{f} = \mathcal{F}(f) \).

It is clear from the inequality

\[
\left| \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx \right| \leq \int_{\mathbb{R}^d} |f(x)| \, dx
\]

and Proposition 2.2 (6) that \( \hat{f}(\xi) \) is defined for each \( \xi \in \mathbb{R}^d \).

**Lemma 2.5.** Let \( f \in \mathcal{S}(\mathbb{R}^d) \). Then \( \hat{f} \) is a bounded continuous function with

\[
\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\|f\|_{L^1(\mathbb{R}^d)}
\]

**Proof.** The boundedness follows directly from the inequality (2.2). The continuity is a consequence of the dominated convergence theorem (see Theorem B.15).

**Proposition 2.6.** For \( f \in \mathcal{S}(\mathbb{R}^d) \), we have that \( \hat{f} \in C^\infty(\mathbb{R}^d) \) and

\[
\begin{align*}
f(x + y) & \mapsto e^{iy \cdot \xi} \hat{f}(\xi), \\
e^{iy \cdot x} f(x) & \mapsto \hat{f}(\xi - y), \\
f(Ax) & \mapsto |\text{det}A|^{-1} \hat{f}(A^{-T} \xi), \\
f(\lambda x) & \mapsto |\lambda|^{-d} \hat{f}(\lambda^{-1} \xi), \\
\partial_\xi^\alpha f(x) & \mapsto (i\xi)^\alpha \hat{f}(\xi), \\
x^\alpha f(x) & \mapsto (i\partial_\xi)^\alpha \hat{f}(\xi),
\end{align*}
\]

where \( y \in \mathbb{R}^d, \lambda \in \mathbb{R} \setminus \{0\} \) and \( A \) is an invertible real \( d \times d \) matrix.
The Fourier transform

Proof. The first four properties follow by changing variables in the integral in (2.1). The fifth property follows by partial integration and the sixth by differentiation under the integral sign. This also proves that \( \hat{f} \in C^\infty(\mathbb{R}^d) \).

Example 2.7. Let us compute the Fourier transform of the Gaussian function \( e^{-|x|^2/2}, x \in \mathbb{R}^d \). We begin by considering the case \( d = 1 \). The function \( u(x) = e^{-x^2/2} \) satisfies the differential equation \( u'(x) = -xu(x) \). Taking Fourier transforms we therefore find that \( i\xi \hat{u}(\xi) = -i\hat{u}'(\xi) \), or equivalently \( \hat{u}'(\xi) = -\xi \hat{u}(\xi) \). But this means that

\[
\hat{u}(\xi) = Ce^{-\xi^2/2},
\]

where

\[
C = \hat{u}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} \, dx.
\]

This integral can be computed using the standard trick

\[
C^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy = \int_{0}^{\infty} e^{-\tau/2} \tau \, d\tau = 1.
\]

Since \( C > 0 \) it follows that \( C = 1 \). Hence,

\[
\mathcal{F}(e^{-x^2/2})(\xi) = e^{-\xi^2/2},
\]

that is, \( u \) is invariant under the Fourier transform!

This can be generalized to \( \mathbb{R}^d \), since

\[
e^{-|x|^2/2} = \prod_{j=1}^{d} e^{-x_j^2/2}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.
\]

From this it follows that

\[
\mathcal{F}(e^{-|x|^2/2})(\xi) = \prod_{j=1}^{d} \mathcal{F}(e^{-x_j^2/2})(\xi_j) = \prod_{j=1}^{d} e^{-\xi_j^2/2} = e^{-|\xi|^2/2}.
\]

Note in particular that the Fourier transform of a Gaussian function is in \( \mathcal{S}(\mathbb{R}^d) \). This is no coincidence.

Theorem 2.8. For any \( f \in \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^d) \), and \( \mathcal{F} \) is a continuous transformation of \( \mathcal{S}(\mathbb{R}^d) \) to itself.
Proof. We estimate the $\mathcal{S}(\mathbb{R}^d)$ semi-norms of $\hat{f}$ as follows:

$$
\|\hat{f}\|_{\alpha, \beta} = \|\xi^\alpha \partial_\xi^\beta \hat{f}\|_{L^\infty(\mathbb{R}^d)}
= \|\mathcal{F}(\partial_x^\alpha (x^\beta f))\|_{L^\infty(\mathbb{R}^d)}
\leq (2\pi)^{-d/2} \|\partial_x^\alpha (x^\beta f)\|_{L^1(\mathbb{R}^d)}
= (2\pi)^{-d/2} \|(1 + |x|)^{-(d+1)} (1 + |x|)^d \partial_x^\alpha (x^\beta f)\|_{L^1(\mathbb{R}^d)}
\leq C\|(1 + |x|)^{d+1} \partial_x^\alpha (x^\beta f)\|_{L^\infty(\mathbb{R}^d)}.
$$

The last line can be estimated by a finite number of semi-norms of $f$ using Leibniz’ formula. It follows that $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^d)$ to itself continuously. \hfill \Box

Recall that the convolution of two functions $f$ and $g$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y) \, dy, \quad (2.3)$$

whenever this makes sense (see Section B.5).

Theorem 2.9 (Convolution theorem). Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then $\hat{f} \hat{g} = (2\pi)^{d/2} \hat{f} \hat{g}$.

Proof. Using Fubini’s theorem, we have that

$$(2\pi)^{d/2} \mathcal{F}(f * g)(\xi) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x-y)g(y) \, dy \right) e^{-ix \cdot \xi} \, dx
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x-y) e^{-ix \cdot \xi} \, dx \right) g(y) \, dy
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(z) e^{-i(z+y) \cdot \xi} \, dz \right) g(y) \, dy
= \left( \int_{\mathbb{R}^d} f(z) e^{-iz \cdot \xi} \, dz \right) \left( \int_{\mathbb{R}^d} g(y) e^{-iy \cdot \xi} \, dy \right)
= (2\pi)^{d} \hat{f}(\xi) \hat{g}(\xi). \quad \Box$$

The inverse Fourier transform is defined by

$$
\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{ix \cdot \xi} \, dx. \quad (2.4)
$$

It is clearly also continuous on $\mathcal{S}(\mathbb{R}^d)$. The name is explained by the following theorem.

Theorem 2.10 (Fourier’s inversion formula). Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then $f(x) = \mathcal{F}^{-1}(\hat{f})(x)$.

Proof. Let $K(x) = (2\pi)^{-d/2} e^{-|x|^2 / 2}$ and recall from Example 2.7 that $\hat{K} = K$. In particular, this implies that $\int_{\mathbb{R}^d} K(x) \, dx = (2\pi)^{d/2} K(0) = 1$. By assumption, the iterated integral

$$
\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(y) e^{-iy \cdot \xi} \, dy \right) e^{ix \cdot \xi} \, d\xi
$$
exists for all $x \in \mathbb{R}^d$. Squeezing a Gaussian in makes it absolutely convergent, and

$$ f_\varepsilon(x) := \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i x \cdot \xi} K(\varepsilon \xi) \, d\xi $$

$$ \to \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i x \cdot \xi} \, d\xi \quad \text{as} \quad \varepsilon \to 0, \quad (2.5) $$

by the dominated convergence theorem. Changing order of integration, we also see that

$$ f_\varepsilon(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) e^{i (x-y) \cdot \xi} K(\varepsilon \xi) \, dy \, d\xi $$

$$ = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y) \left( \int_{\mathbb{R}^d} e^{-i (y-x) \cdot \xi} K(\varepsilon \xi) \, d\xi \right) \, dy $$

$$ = \int_{\mathbb{R}^d} f(y) \mathcal{F}(K(\varepsilon \xi))(y-x) \, dy $$

$$ = \varepsilon^{-d} \int_{\mathbb{R}^d} f(y) K(\varepsilon^{-1}(y-x)) \, dy $$

$$ = (K_\varepsilon * f)(x), $$

where $K_\varepsilon(x) = \varepsilon^{-d} K(\varepsilon^{-1} x)$. Theorem B.27 implies that

$$ \lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x). \quad (2.6) $$

The result follows by combining (2.5) and (2.6). \hfill \Box

**Corollary 2.11.** The Fourier transform is a linear bijection of $\mathcal{S}(\mathbb{R}^d)$ to itself with inverse equal to $\mathcal{F}^{-1}$. That is,

$$ \mathcal{F}^{-1} \mathcal{F} = \mathcal{F} \mathcal{F}^{-1} = \text{Id on } \mathcal{S}(\mathbb{R}^d). $$

**Proof.** The above results implies that $\mathcal{F}^{-1} \mathcal{F} = \text{Id on } \mathcal{S}(\mathbb{R}^d)$. To prove the corollary, we must also show that $\mathcal{F} \mathcal{F}^{-1} = \text{Id}$. Define the reflection operator $\mathcal{R}: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ by $\mathcal{R}(f)(x) = f(-x)$. $\mathcal{R}$ commutes with $\mathcal{F}$ and $\mathcal{F}^{-1}$ (Proposition 2.6), is its own inverse and satisfies $\mathcal{F}^{-1} = \mathcal{R} \mathcal{F}$. It follows that

$$ \mathcal{F} \mathcal{F}^{-1} = \mathcal{F} \mathcal{R} \mathcal{F} = \mathcal{F}^{-1} \mathcal{F} = \text{Id}. \hfill \Box $$

**Corollary 2.12.** Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then $f \ast g \in \mathcal{S}(\mathbb{R}^d)$.

**Proof.** This follows from the convolution theorem, Fourier’s inversion formula and the fact that $\mathcal{S}(\mathbb{R}^d)$ is an algebra, since $f \ast g = (2\pi)^{d/2} \mathcal{F}^{-1}(\hat{f} \hat{g})$. \hfill \Box
2.3 Extension of the Fourier transform to $L^p$

Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ (Proposition 2.2), it is natural to try to extend the Fourier transform to $L^p$ by continuity. We have the following abstract result.

**Proposition 2.13.** Suppose that $X$ is a normed vector space, $E \subset X$ is a dense linear subspace, and $Y$ is a Banach space. If $T : E \to Y$ is a bounded linear operator in the sense that there exists $C > 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X$$

for each $x \in E$, then there is a unique bounded extension $\tilde{T} : X \to Y$ of $T$.

**Proof.** Let $S$ be another bounded extension. If $x \in X$ and $\{x_n\}$ is a sequence in $E$ which converges to $X$, we find that

$$\lim_{n \to \infty} (\tilde{T} - S)x_n = (\tilde{T} - S)x,$$

while at the same time

$$(\tilde{T} - S)x_n = (T - T)x_n = 0.$$ 

It follows that $\tilde{T}x = Sx$.

To prove the existence of $\tilde{T}$, let $x$ and $\{x_n\}$ be as above. Then $\{x_n\}$ is a Cauchy sequence in $X$. Since $T$ is bounded it follows that $\{Tx_n\}$ is a Cauchy sequence in $Y$. By completeness of $Y$ there exists a limit, which we define to be $\tilde{T}x$. The limit is clearly independent of the sequence $x_n$ since $T$ applied to the difference of two such sequence will converge to $0$. Linearity and boundedness follow by a limiting procedure.

We begin by extending the Fourier transform to $L^1$. This extension can in fact be defined without using the above proposition. In order to simplify the notation, we also write $\hat{f} = \mathcal{F}(f)$ for the extension. Let $C_b(\mathbb{R}^d)$ be the space of bounded continuous functions on $\mathbb{R}^d$, equipped with the supremum norm.

**Theorem 2.14 (Riemann-Lebesgue lemma).** $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ extends uniquely to a bounded linear operator $L^1(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$, defined as the absolutely convergent integral (2.1). Moreover,

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\|f\|_{L^1(\mathbb{R}^d)}$$

(2.7)

and $\lim_{x \to \infty} \hat{f}(x) = 0$.

**Proof.** Repeating the proof of Lemma 2.5, we find that the extension exists and defines a bounded operator satisfying the estimate (2.7). Proposition 2.13 shows that it is unique. To prove that $\hat{f}(x) \to 0$ as $x \to \infty$, we take $\epsilon > 0$ and $g \in \mathcal{S}(\mathbb{R}^d)$ with $\|\hat{f} - \hat{g}\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\|f - g\|_{L^1(\mathbb{R}^d)} < \epsilon/2$. Choosing $R$ large enough, we find that $|\hat{g}(x)| < \epsilon/2$ for $|x| \geq R$. Hence,

$$|\hat{f}(x)| \leq \|\hat{f} - \hat{g}\|_{L^\infty(\mathbb{R}^d)} + |\hat{g}(x)| < \epsilon$$

for $|x| \geq R$. 

In order to extend the Fourier transform to $L^2(\mathbb{R}^d)$, we need an estimate similar to (2.7). In fact, we can do even better.
Lemma 2.15. For \( f, g \in \mathcal{S}(\mathbb{R}^d) \), we have that
\[
(\mathcal{F}(f), g)_{L^2(\mathbb{R}^d)} = (f, \mathcal{F}^{-1}(g))_{L^2(\mathbb{R}^d)},
\]
where
\[
(f,g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx
\]
is the inner product on \( L^2(\mathbb{R}^d) \).

Proof. Just as in the proof of Theorem 2.9, we use Fubini’s theorem to change the order of integration:
\[
\int_{\mathbb{R}^d} \mathcal{F}(f)(x) \overline{g(x)} \, dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(y) e^{-ix \cdot y} \, dy \right) \overline{g(x)} \, dx
\]
\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} g(x) e^{ix \cdot y} \, dx \right) f(y) \, dy
\]
\[
= \int_{\mathbb{R}^d} f(y) \overline{\mathcal{F}^{-1}(g)(y)} \, dy.
\]

In particular, if \( f, g \in \mathcal{S}(\mathbb{R}^d) \), one finds that
\[
(\mathcal{F}(f), \mathcal{F}(g))_{L^2(\mathbb{R}^d)} = (f, \mathcal{F}^{-1}(g))_{L^2(\mathbb{R}^d)} = (f, g)_{L^2(\mathbb{R}^d)}
\]
(2.8)
by Fourier’s inversion theorem. Taking \( g = f \), we find that
\[
\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{L^2(\mathbb{R}^d)}^2 = \|\mathcal{F}^{-1}(f)\|_{L^2(\mathbb{R}^d)}^2,
\]
where the last equality follows from the first by replacing \( f \) with \( \mathcal{F}^{-1}(f) \). This is sometimes called Parseval’s formula. A bijective linear operator which preserves the inner product is called a unitary operator. Note that a unitary operator automatically is bounded.

Theorem 2.16 (Plancherel’s theorem). The Fourier transform \( \mathcal{F} \) and the inverse Fourier transform \( \mathcal{F}^{-1} \) on \( \mathcal{S}(\mathbb{R}^d) \) extend uniquely to unitary operators on \( L^2(\mathbb{R}^d) \) satisfying \( \mathcal{F}^{-1} \mathcal{F} = \mathcal{F} \mathcal{F}^{-1} = \text{Id} \).

Proof. The existence and uniqueness of the extensions follow from Proposition 2.13 and (2.9). Since \( \mathcal{F} \mathcal{F}^{-1} = \mathcal{F}^{-1} \mathcal{F} = \text{Id} \) on a dense subspace, it follows that the same is true on \( L^2(\mathbb{R}^d) \). By continuity, we have that \( (\mathcal{F}(f), g)_{L^2(\mathbb{R}^d)} = (f, \mathcal{F}^{-1}(g))_{L^2(\mathbb{R}^d)} \) for \( f, g \in L^2(\mathbb{R}^d) \), so \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are unitary.

Since the Fourier transform is defined for \( L^1(\mathbb{R}^d) \) and \( L^2(\mathbb{R}^d) \), one can in fact extend it to \( L^p(\mathbb{R}^d) \), \( 1 \leq p \leq 2 \), by interpolation. The following result is a direct consequence of Theorems 2.14, 2.16 and B.28.

Theorem 2.17 (Hausdorff-Young inequality). Let \( p \in [1, 2] \) and \( q \in [2, \infty] \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). The Fourier transform extends uniquely to a bounded linear operator \( \mathcal{F} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \), with
\[
\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^d)} \leq (2\pi)^{\frac{d}{2}} (1 - \frac{2}{p})^{\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d).
\]
The extension to $L^p(\mathbb{R}^d)$ for $p > 2$ is a completely different matter. We will return to this in Chapter 5.

Note that many of the properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ automatically hold on $L^p(\mathbb{R}^d)$, $1 \leq p \leq 2$, by continuity. This is e.g. true for the four first properties in Proposition 2.6. One also obtains the following extension of Fourier’s inversion formula.

**Proposition 2.18.** Let $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, 2]$ and $\hat{f} \in L^1(\mathbb{R}^d)$. Then $f$ is continuous\footnote{Meaning that there is an element of the equivalence class which is continuous.} and $f(x) = \mathcal{F}^{-1}(\hat{f})(x)$ for all $x \in \mathbb{R}^d$.

Note that we have defined the extension in two different ways if e.g. $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. However, approximating $f$ by a sequence in $\{f_n\} \subset C_0^\infty(\mathbb{R}^d)$ with $f_n \to f$ in $L^1$ and $L^2$, we find that $\hat{f}_n \to \mathcal{F}_{L^1}(f)$ in $C_b$ and $\hat{f}_n \to \mathcal{F}_{L^2}(f)$ in $L^2$, where the subscripts on the Fourier transforms are used to differentiate the two extensions. But this implies that $\mathcal{F}_{L^2}(f) = \mathcal{F}_{L^1}(f)$ a.e.

**Example 2.19.** Let

$$f(x) = \begin{cases} 1, & x \in [-1, 1], \\ 0, & x \notin [-1, 1] \end{cases}$$

be the characteristic function of the interval $[-1, 1]$. $f \in L^1(\mathbb{R})$, so

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\xi} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ e^{-ix\xi} \right]_{x=-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left( e^{i\xi} - e^{-i\xi} \right)$$

$$= \frac{\sqrt{2}}{\pi} \frac{\sin(\xi)}{\xi}.$$  

$\hat{f}$ is not in $L^1(\mathbb{R})$, so we can’t use the pointwise inversion formula in Proposition 2.18. We do however have $f = \mathcal{F}^{-1}(\hat{f})$ in the sense of Theorem 2.16 since $f \in L^2(\mathbb{R})$. Moreover, we have

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = \|\hat{f}\|_{L^2(\mathbb{R}^d)}^2,$$

by Parseval’s formula. When written out, this means that

$$\int_{\mathbb{R}} \frac{\sin^2 \xi}{\xi^2} \, d\xi = \pi.$$
Chapter 3

Linear PDE with constant coefficients

In this chapter we will study the simplest kind of PDE, namely linear PDE with constant coefficients. This means in particular that the superposition principle holds, so that new solutions can be constructed by forming linear combinations of known solutions. There are two main reasons for why this chapter is included in the lecture notes. The first reason is that some important equations describing waves are linear. The second is that many nonlinear wave equations can be treated as perturbations of linear ones. A knowledge of linear equations is therefore essential for understanding nonlinear problems.

We will concentrate on the Cauchy problem (the initial-value problem), in which, in addition to the PDE, the values of the unknown function and sufficiently many time derivatives are prescribed at $t = 0$. This presupposes that there is some distinguished variable $t$ which can be interpreted as time. Usually this is clear from the physical background of the equation in question. If you are familiar with the Cauchy problem for ordinary differential equations you know that there always exists a unique solution under sufficient regularity hypotheses. We will look for conditions under which the same property is true for PDE. Our goal is in fact to convert the PDE to ODE by using the Fourier transform with respect to the spatial variables $x$. From now on, Fourier transforms will only be applied with respect to the spatial variables. We will consider functions $u(x,t)$ defined for all $x \in \mathbb{R}^d$. This is of course a simplification; in the ‘physical world’, $u(x,t)$ will only be defined for $x$ in some bounded domain and there will be additional boundary conditions imposed on the boundary of this domain. Our study will not be completely restricted to equations describing waves, but in the end we will arrive at a condition which guarantees that the solutions behave like waves in the sense that there is a well-defined speed of propagation. Equations having this property are called hyperbolic. Not all equations which describe waves are hyperbolic. In particular, dispersive equations such as the Schrödinger equation and the linearized KdV equation are not. Nevertheless, we will prove that the Cauchy problem is uniquely solvable for these equations as well. Further properties of dispersive equations will be studied in Chapter 6.

We begin by considering three classical second order equations, namely the heat equation, the wave equation and the Laplace equation. These examples are instructive in that they illustrate that the properties of the solutions of PDE depend heavily on the form of the equation. They also give us some intuition of what to expect in general.
36 Linear PDE with constant coefficients

The presentation of the material in this chapter is close that in Rauch [20]. The treatment of Kirchhoff’s solution formula for the wave equation in \( \mathbb{R}^3 \) follows Stein and Shakarchi [24]. A more detailed account of the theory of linear PDE with constant coefficients can be found in Hörmander [13].

3.1 The heat equation

3.1.1 Derivation

The heat equation in \( \mathbb{R}^3 \) is the equation

\[
    u_t = k \Delta u, \tag{3.1}
\]

where \( \Delta u = \partial_{x_1}^2 u + \partial_{x_2}^2 u + \partial_{x_3}^2 u \) and \( k > 0 \) is a constant. As the name suggest, the equation describes the flow of heat. Recall that heat is a form of energy transferred between two bodies, or regions of space, at different temperatures. Heat transfer to a region increases or decreases the internal energy in that region. The internal energy of a system consists of all the different forms of energy of the molecules in the system, but excludes the energy associated with the movement of the system as a whole. We look at this from a macroscopic perspective and suppose that the internal energy can be described by a density \( e(x,t) \). If \( E(t) \) is the total internal energy in the domain \( \Omega \subset \mathbb{R}^3 \) we therefore have that

\[
    E(t) = \int_{\Omega} e(x,t) \, dx, 
\]

and the rate of change is

\[
    E'(t) = \int_{\Omega} e_t(x,t) \, dx. 
\]

The heat flux \( J \) is a vector-valued function which describes the flow of heat. The total transfer of heat across an oriented surface \( \Sigma \) is given by

\[
    \int_{\Sigma} J(x,t) \cdot n(x) \, dS(x),
\]

where \( n \) denotes the unit normal and \( dS \) the element of surface area on \( \Sigma \). The rate at which heat leaves \( \Omega \) is therefore \( \int_{\partial \Omega} J \cdot n \, dS \), where \( n \) is the outer unit normal. Using the divergence theorem, this can be rewritten as \( \int_{\Omega} \nabla \cdot J \, dx \). Assuming that the only change in internal energy is due to conduction of heat across the boundary, we find that

\[
    \int_{\Omega} (e_t + \nabla \cdot J) \, dx = 0.
\]

Since this holds for all sufficiently nice domains \( \Omega \) we obtain the continuity equation

\[
    e_t + \nabla \cdot J = 0
\]
if all the involved functions are continuous. To derive the heat equation we use two constitutive laws. First of all, we assume that over a sufficiently small range of temperatures, $e$ depends linearly on the temperature $T$, that is,

$$e = e_0 + \rho c (T - T_0),$$

where $\rho$ is the density, $c$ is the specific heat capacity and $T_0$ a reference temperature. We assume that $\rho$ and $c$ are independent of $T$. We can simply think of the temperature as what we measure with a thermometer. On a microscopic level, the temperature is a measure of the average kinetic energy of the molecules. The above assumption means that an increase in internal energy results in an increase in temperature. This is not always true, as an increase in internal energy may instead result in an increase in potential energy. This is e.g. what happens in phase transitions. The second constitutive law is Fourier’s law, which says that heat flows from hot to cold regions proportionally to the temperature gradient. In other words,

$$J = -\kappa \nabla T,$$

where $\kappa$ is a scalar quantity called the thermal conductivity. Assuming that $c$ and $\rho$ are independent of $t$, we therefore obtain the equation

$$c \rho T_t = \nabla \cdot (\kappa \nabla T).$$

If in addition $c$, $\rho$ and $\kappa$ are constant, we find that the temperature satisfies the heat equation with $k = \kappa / (c \rho)$.

The heat equation is sometimes also called the diffusion equation. The physical interpretation is then that $u$ describes the concentration of a substance in some medium (e.g. a chemical substance in some liquid). Diffusion means that the substance tends to move from regions of higher concentration to regions of lower concentration. The heat equations is obtained by assuming that the rate of motion is proportional to the concentration gradient (Fick’s law of diffusion).

### 3.1.2 Solution by Fourier’s method

The Cauchy problem for the heat equation in $\mathbb{R}^d$ reads

$$
\begin{cases}
    u_t(x,t) = \Delta u(x,t), & t > 0, \\
    u(x,0) = u_0(x),
\end{cases}
$$

where $u: \mathbb{R}^d \times [0,\infty) \to \mathbb{C}$ and $u_0: \mathbb{R}^d \to \mathbb{C}$. Note that one can assume that $k = 1$ in (3.1) by using the transformation $t \mapsto kt$. It is useful to first consider a solution in the form of an individual Fourier mode $\hat{u}(t)e^{i\xi \cdot x}$. Substituting this in the equation yields

$$\hat{u}'(t) = -|\xi|^2 \hat{u}(t).$$

This is an ODE with solution $\hat{u}(t) = e^{-|\xi|^2 t} \hat{u}(0)$. The corresponding solution is therefore

$$u(x,t) = \hat{u}(0)e^{-|\xi|^2 t} e^{i\xi \cdot x} = \hat{u}(0)e^{i(x \cdot \xi + i|\xi|^2 t)}.$$
This is written in the form of a complex sinusoidal wave $ae^{i(x \xi - \omega(t) \mu)}$ considered in Chapter 1, except that the angular frequency $\omega(t) = -i|\xi|^2$ now is imaginary. This implies that the absolute value of the solution is strictly decreasing in $t$. Since there are no temporal oscillations, we don’t normally think of the heat equation as an equation describing waves.

Our strategy for solving the Cauchy problem in general is to find a solution which is a ‘continuous linear combination’ of solutions of the form (3.3), that is, a Fourier integral in $x$. We work formally and assume that $u$ and $u_0$ are sufficiently regular, so that their Fourier transforms are well-defined. We also assume that we can differentiate under the integral sign, so that $\partial_t \mathcal{F}(u) = \mathcal{F}(\partial_t u)$. We then obtain the transformed problem

$$
\begin{cases}
\hat{u}_t(t, \xi) = -|\xi|^2 \hat{u}(\xi, t), & t > 0, \\
\hat{u}(\xi, 0) = \hat{u}_0(\xi),
\end{cases}
$$

with solution $\hat{u}(\xi, t) = e^{-t|\xi|^2} \hat{u}_0(\xi)$. Applying the inverse Fourier transform, we obtain that

$$
u(x, t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\xi} e^{-t|\xi|^2} \hat{u}_0(\xi) d\xi, \quad t \geq 0. \quad (3.4)$$

The integral will in general not converge for $t < 0$ due to the exponentially growing factor $e^{-t|\xi|^2}$. Under the assumption that $u_0 \in \mathcal{S}(\mathbb{R}^d)$, one can differentiate under the integral sign as many times as one likes for $t > 0$ by Theorem B.15. This still is true if we only allow one-sided time derivatives at $t = 0$. It follows that $u$ defined by this formula belongs to $C^\infty(\mathbb{R}^d \times [0, \infty))$.

Carrying out the differentiation one also verifies that $u$ satisfies the heat equation and the initial condition. Since we have derived an explicit formula, it follows that the solution is unique whenever the steps involved in deriving the formula can be justified. This holds e.g. if $u(\cdot, t), u_t(\cdot, t) \in \mathcal{S}(\mathbb{R}^d)$ for each $t \geq 0$ and $\hat{u}(\cdot, t)$ and $\hat{u}_t(\cdot, t)$ can be dominated by some $L^1$ functions which are independent of $t$, so that Theorem B.15 applies. There is however a more natural condition which guarantees that the steps can be justified. The idea is to take the analogy with ordinary differential equations one step further and think of the solution $u$ as a function of $t$ with ‘values’ in the function space $\mathcal{S}(\mathbb{R}^d)$.

**Definition 3.1.** Let $I$ be an open interval in $\mathbb{R}$.

1. A function $u: I \to \mathcal{S}(\mathbb{R}^d)$ is said to be continuous if $\|u(t + h) - u(t)\|_{\alpha, \beta} \to 0$ as $h \to 0$ for every semi-norm $\|\cdot\|_{\alpha, \beta}$ and every $t \in I$. $C(I; \mathcal{S}(\mathbb{R}^d))$ is the vector space of continuous functions $I \to \mathcal{S}(\mathbb{R}^d)$.

2. A function $u: I \to \mathcal{S}(\mathbb{R}^d)$ is said to be differentiable with derivative $u': I \to \mathcal{S}(\mathbb{R}^d)$ if

$$
\lim_{h \to 0} \left\| \frac{u(t + h) - u(t)}{h} - u'(t) \right\|_{\alpha, \beta} = 0
$$

for all $\alpha, \beta \in \mathbb{N}^d$ and $t \in I$. The space $C^1(I; \mathcal{S}(\mathbb{R}^d))$ consists of differentiable functions $u: I \to \mathcal{S}(\mathbb{R}^d)$ such that $u, u' \in C(I; \mathcal{S}(\mathbb{R}^d))$.

\(^{1}\)Meaning that at $t = 0$ derivatives with respect to $t$ are only taken from the right.
The space $C^k(I; \mathcal{S}(\mathbb{R}^d))$, $k \geq 2$, is defined inductively by requiring that $u \in C(I; \mathcal{S}(\mathbb{R}^d))$ and $u' \in C^{k-1}(I; \mathcal{S}(\mathbb{R}^d))$, and $C^\infty(I; \mathcal{S}(\mathbb{R}^d)) = \bigcap_{k=0}^\infty C^k(I; \mathcal{S}(\mathbb{R}^d))$.

If $I$ is any interval (not necessarily open), we say that $u \in C^k(I; \mathcal{S}(\mathbb{R}^d))$ if it is $k$ times continuously differentiable from the interior of $I$ to $\mathcal{S}(\mathbb{R}^d)$ and all the derivatives extend continuously in $\mathcal{S}(\mathbb{R}^d)$ to the endpoints.

A function $u \in C(I; \mathcal{S}(\mathbb{R}^d))$ also defines a function $v \in \mathbb{R}^d \times I \to \mathbb{C}$, $v(x,t) = u(t)(x)$. The definition implies that $v \in C(\mathbb{R}^d \times I)$ and that $v(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ for each $t \in I$. When there is no risk of confusion we shall write $u$ for $v$. Also, to clarify the notation, we will write $u_t$ or $\partial_t u$ for $u'$. If $u \in C^1(I; \mathcal{S}(\mathbb{R}^d))$ it follows in particular that $t \mapsto \partial_t^\alpha u(x,t)$ is continuously differentiable on $I$ for each $\alpha \in \mathbb{N}^d$ and each fixed $x \in \mathbb{R}^d$.

**Lemma 3.2.** Assume that $u \in C^k(I; \mathcal{S}(\mathbb{R}^d))$. Then $\hat{u} \in C^k(I; \mathcal{S}(\mathbb{R}^d))$ with $\partial_t^\alpha \mathcal{F}(u) = \mathcal{F}(\partial_t^\alpha u)$.

**Proof.** The result follows immediately from Lemma 2.8. \qed

We therefore obtain uniqueness in the class $C^\infty([0,\infty); \mathcal{S}(\mathbb{R}^d))$ for instance. The fact that the solution $u$ given by (3.4) belongs to this class follows immediately from the fact that $t \mapsto \hat{u}_0(\xi)e^{-t|\xi|^2}$ does so and Lemma 3.2.

There is a final important property to discuss. In real life it is impossible to measure something with infinite accuracy. It is therefore essential that the solution depends continuously on the data in some sense, so that small measuring errors don’t grow large (at least in finite time). In the case of the heat equation, we e.g. have that the map

$$\mathcal{S}(\mathbb{R}^d) \ni u_0 \mapsto u \in C^\infty([0,\infty); \mathcal{S}(\mathbb{R}^d))$$

is continuous if the space $C^\infty([0,\infty); \mathcal{S}(\mathbb{R}^d))$ is equipped with the countable family of semi-norms

$$\sup_{0 \leq t \leq n} \|\partial_t^\alpha u(\cdot,t)\|_{\alpha,\beta}, \quad n, k \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^d.$$ 

Note that $t \mapsto \|\partial_t^\alpha u(\cdot,t)\|_{\alpha,\beta}$ is a continuous function so that the supremum is finite. The continuity of the map $u_0 \mapsto u$ follows from the formula $\hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{u}_0(\xi)$.

In summary, we have proved the following result.

**Theorem 3.3.** There exists a unique solution $u \in C^\infty([0,\infty); \mathcal{S}(\mathbb{R}^d))$ of the Cauchy problem (3.2). Moreover, the solution map $\mathcal{S}(\mathbb{R}^d) \ni u_0 \mapsto u \in C^\infty([0,\infty); \mathcal{S}(\mathbb{R}^d))$ is continuous.

A convenient way of describing the solution is by looking at the map $S(t) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ from the data $u_0$ to the solution $u(\cdot,t)$ at time $t \geq 0$. $S$ is called the propagator for the heat equation. The continuous dependence of the solution on the initial data means that $S(\cdot) \in C(\mathcal{S}(\mathbb{R}^d); C^\infty([0,\infty); \mathcal{S}(\mathbb{R}^d)))$. The map $S$ has two important properties. First of all,

$$S(0) = \text{Id},$$

(3.5)
where $\text{Id}$ is the identity map on $\mathcal{S}(\mathbb{R}^d)$. Secondly, note that the heat equation is invariant under translations in $t$. This means that the solution with initial data $u_0$ at $t = t_0$ is given by $S(t - t_0)u_0$, $t \geq t_0$. Consider the solution $u$ with initial data $u_0$ at time 0. If $s, t \geq 0$ we can express $u(\cdot, s + t)$ in two different ways. We can of course express it as $S(s + t)u_0$. On the other hand, we can solve the equation up to time $t$, which gives us $S(t)u_0$. We can then use this as our new initial data and solve up to time $s + t$. In other words

$$S(s + t) = S(s)S(t), \quad s, t \geq 0. \quad (3.6)$$

**Definition 3.4.** Let $X$ be a vector space. A family of linear operators $\{S(t)\}_{t \geq 0}$ on $X$ with the properties (3.5) and (3.6) is called a **semigroup**. If the family is defined for all $t \in \mathbb{R}$ and (3.6) holds for all $s, t \in \mathbb{R}$ it is called a **group**.

### 3.1.3 The heat kernel

Define the **heat kernel**

$$K_t(x) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0.$$  

Using the convolution theorem and the fact that

$$\mathcal{F}^{-1}(e^{-|\cdot|^2})(x) = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}} = (2\pi)^{d/2}K_t(x), \quad t > 0$$

(see Example 2.7), we can express the solution of the heat equation without using the Fourier transform.

**Theorem 3.5.** The unique solution of (3.2) with $u_0 \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$u(x, t) = (K_t * u_0)(x), \quad t > 0, \quad (3.7)$$

where $K_t$ is the heat kernel.

Note that $(x, t) \mapsto K_t(x)$ is itself a solution of the heat equation in the half space $\mathbb{R}^d \times (0, \infty)$ with $(t \mapsto K_t) \in C^\infty((0, \infty); \mathcal{S}(\mathbb{R}^d))$. Solving the heat equation with initial data $K_t$ we therefore find that

$$K_s * K_t = K_{s+t}, \quad s, t > 0.$$  

This is another way of stating the semigroup property of the propagator $S(t)$. We also note that

$$\|K_t\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} K_t(x) \, dx = (2\pi)^{d/2} \mathcal{F}(K_t)(0) = 1.$$
3.1.4 Qualitative properties

Decay in time

The solutions of the heat equation decay as $t \to \infty$. Physically, this is related to the fact that heat flows from hot regions to cold regions, so that the temperature evens out. Since we are working in the whole space $\mathbb{R}^d$ and assume that the solutions converge to 0 at spatial infinity, the temperature will converge to 0 everywhere as $t \to \infty$. The total internal energy is however conserved. Mathematically this is reflected in the conservation law

$$\int_{\mathbb{R}^d} u(x,t) \, dx = (2\pi)^{d/2} \hat{u}(0,t) = (2\pi)^{d/2} \hat{u}_0(0).$$

The $L^2$ norm decays to zero however.

**Proposition 3.6.** Assume that $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and let $u$ be the corresponding solution of (3.2). Then

$$\lim_{t \to \infty} \|u(\cdot,t)\|_{L^2} = 0.$$  

**Proof.** Using Parseval’s formula it follows that

$$\|u(\cdot,t)\|_{L^2}^2 = \int_{\mathbb{R}^d} e^{-2|\xi|^2} |\hat{u}_0(\xi)|^2 \, d\xi.$$  

Since $e^{-2|\xi|^2} \to 0$ monotonically as $t \to \infty$, the proposition follows. \(\square\)

The $L^2$ norm of the solution or its derivative can often be interpreted as some form of energy. Due to the above result, one often thinks of the heat equation as a dissipative equation, meaning that energy is lost. This is not true in the physical sense however, since the physical energy is given by the conserved quantity $\int_{\mathbb{R}^d} u(x,t) \, dx$. From a mathematical point of view, it is however useful to think of the heat equation as dissipative, in contrast to e.g. the wave equation (see Section 3.2.4).

One can also use the heat kernel to prove the decay of the solution as $t \to \infty$. Using (3.7) and Young’s inequality one can prove a number of estimates.

**Proposition 3.7.** Assume that $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and let $u$ be the corresponding solution of (3.2). All the norms $\|u(\cdot,t)\|_{L^p}$, $p \geq 1$, are decreasing functions of $t$. For $p > 1$, we have that

$$\lim_{t \to \infty} \|u(\cdot,t)\|_{L^p} = 0.$$  

**Proof.** Due to the translation invariance in $t$, it suffices to show that $\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}$ to prove the monotonicity of the $L^p$ norm. This bound follows from Young’s inequality (Theorem B.29):

$$\|u(\cdot,t)\|_{L^p} \leq \|K_t\|_{L^1} \|u_0\|_{L^p} = \|u_0\|_{L^p}.$$  

The second part also follows from Young’s inequality:

$$\|u(\cdot,t)\|_{L^p} \leq \|K_t\|_{L^p} \|u_0\|_{L^1} = C t^{-d/2} (1-\frac{p}{d}) \|u_0\|_{L^1} \to 0$$  

if $p > 1$, for some constant $C > 0$. \(\square\)
We mention an alternative way of proving the monotonicity. This is especially useful if one wants to consider the heat equation in a domain other than $\mathbb{R}^d$, where the Fourier transform is not available. For simplicity we assume that $u_0$ (and hence $u$) is real-valued. Multiplying the equation $u_t = \Delta u$ by $u$ and integrating we find that
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^d} u^2 \, dx \right) = 2 \int_{\mathbb{R}^d} uu_t \, dx = 2 \int_{\mathbb{R}^d} u \Delta u \, dx = -2 \int_{\mathbb{R}^d} \nabla u \cdot \nabla u \, dx \leq 0
\]
where we used integration by parts in the last step. The monotonicity of the other $L^p$ norms, $1 \leq p < \infty$ can be proved in a similar way, although special care has to be taken in the case $1 \leq p < 2$ due to regularity problems. The method of multiplying the equation by some quantity and integrating to obtain a bound is called the energy method. One can sometimes guess the form of the integral directly.

Smoothing

The heat equation also has a smoothing property. Again this makes sense physically. Since the heat tends to spread out evenly, we expect discontinuities to be ‘evened out’. To formalize this smoothing property we need to consider non-smooth initial data.

**Proposition 3.8.** The propagator $S(t) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, $t \geq 0$, extends uniquely to a continuous map $S(t) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$, given by
\[
S(t)u_0 = K_t * u_0, \quad t > 0
\]
and $S(0)u_0 = u_0$.

**Proof.** It follows from Young’s inequality that $K_t * u_0$ is defined for $t > 0$ and defines a continuous extension of $S(t) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. The uniqueness follows from Propositions 2.13 and 3.7.

**Proposition 3.9.** Let $u_0 \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Then $S(\cdot)u_0 \in C^\infty(\mathbb{R}^d \times (0, \infty))$ and $S(t)u_0$ solves the heat equation for $t > 0$. Moreover, $S(t)u_0 \to u_0$ in $L^p(\mathbb{R}^d)$.

**Proof.** This follows by using the formula $S(t)u_0 = K_t * u_0$. Since $K_t(x)$ is exponentially decaying in $x$ for $t > 0$, we can apply Theorem B.15 to differentiate under the integral sign as many times as we like. It follows that $S(t)u_0$ is smooth in $\mathbb{R}^d \times (0, \infty)$. The convergence of $S(t)u_0 = K_t * u_0$ to $u_0$ in $L^p$ is a consequence of Theorem B.27.

Infinite speed of propagation

Suppose that $u_0 \geq 0$. Formula (3.7) then shows that $u(x,t) \geq 0$ for $t > 0$ and all $x \in \mathbb{R}$. In fact it also shows that $u(x,t) > 0$ unless $u_0 \equiv 0$. This holds even if $u_0$ has compact support. We say that heat equation has infinite speed of propagation. In fact, a much stronger property holds. One can show that $u$ is a real analytic function for $t > 0$, that is, it can be expanded in a convergent power series in a neighbourhood of each point $(x,t) \in \mathbb{R}^d \times (0, \infty)$. If $u(\cdot,t)$ vanishes on an open subset $\Omega \subset \mathbb{R}^d$ for some $t > 0$, then $\partial_t^k \partial_\alpha u(x,t) = \Delta^k \partial_\alpha u(x,t) = 0$ for $(x,t) \in \Omega$ for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$. It then follows that $u(x,t) \equiv 0$ on $\mathbb{R}^d \times (0, \infty)$ by the unique continuation property for real analytic functions.
3.2 The wave equation

3.2.1 Derivation

We consider next the wave equation in $\mathbb{R}^d$

$$u_{tt}(x,t) = c^2 \Delta u(x,t).$$

The equation describes many different physical phenomena. We discuss two examples.

**Example 3.10** (Acoustic waves). Consider again the linearized equations of gas dynamics

$$\rho_0 u_t + c_0^2 \nabla \rho = 0,$$

$$\rho_t + \rho_0 \nabla \cdot u = 0,$$

from Example 1.1. Taking the divergence of the first equation with respect to $x$ and differentiating the second equation with respect to $t$, one finds that

$$\rho_{tt} = c_0^2 \Delta \rho.$$

Suppose next that the vector field $u$ is *irrotational*, $\nabla \times u = 0$, for some $t$. It is an easy matter show that it is then irrotational for all $t$. Differentiating the first equation with respect to $t$ and the second with respect to $x$ and using the identity

$$\nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \Delta F; \tag{3.8}$$

one finds that $u$ satisfies the wave equation

$$u_{tt} = c_0^2 \Delta u$$

(meaning that each component of $u_j$ satisfies the wave equation).

**Example 3.11** (Electromagnetic waves). Maxwell’s equations couple electric and magnetic fields. In vacuum they are

$$\nabla \cdot E = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t};$$

$$\nabla \cdot B = 0, \quad \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t},$$

where $E$ is the *electric field*, $B$ the *magnetic field*, $\mu_0$ the *electric constant* and $\varepsilon_0$ the *magnetic constant*. Uppercase letters are used to indicate vector quantities and lowercase letters to indicate scalar quantities. Taking the curl of the last two equations we obtain that

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t} \nabla \times B = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2},$$

$$\nabla \times (\nabla \times B) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times E = -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}.$$

Using (3.8) it follows that

$$E_{tt} = c_0^2 \Delta E,$$

$$B_{tt} = c_0^2 \Delta B,$$

where $c_0 = 1/\sqrt{\mu_0 \varepsilon_0}$ is the speed of light in vacuum.
3.2.2 Solution by Fourier’s method

To simplify the notation, we shall assume that \( c = 1 \). In fact, this can always be achieved by the change of variables \( t \mapsto ct \). We thus consider the Cauchy problem

\[
\begin{align*}
  u_{tt}(x,t) &= \Delta u(x,t), \quad t > 0, \\
  u(x,0) &= u_0(x), \\
  u_t(x,0) &= u_1(x).
\end{align*}
\]  

(3.9)

In contrast to the heat equation, there are two initial conditions. This is due to the fact that the equation is of second order in time. The transformed problem is

\[
\begin{align*}
  \hat{u}_{tt}(\xi,t) &= -|\xi|^2 \hat{u}(\xi,t), \quad t > 0, \\
  \hat{u}(\xi,0) &= \hat{u}_0(\xi), \\
  \hat{u}_t(\xi,0) &= \hat{u}_1(\xi),
\end{align*}
\]

with solution

\[
\hat{u}(\xi,t) = \cos(t|\xi|)\hat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{u}_1(\xi)
\]

and hence

\[
u(x,t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \left( \cos(t|\xi|)\hat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{u}_1(\xi) \right) d\xi.
\]

Just as for the heat equation, one obtains an existence and uniqueness result from these formulas. In contrast to the heat equation, the solution is now also defined for negative times. This is related to the fact that, contrary to the heat equation, the wave equation is invariant under the transformation \( t \mapsto -t \).

**Theorem 3.12.** There exists a unique solution \( u \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d)) \) of the Cauchy problem (3.9). Moreover, the solution map \( \mathcal{S}(\mathbb{R}^d)^2 \ni (u_0, u_1) \mapsto u \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d)) \) is continuous.

To define a propagator with the semigroup property, one has to consider both \( u \) and \( u_t \). We leave it as an exercise to prove that the map \( S(t)(u_0, u_1) := (u(\cdot, t), u_t(\cdot, t)) \) in fact is a group in \( \mathcal{S}(\mathbb{R}^d)^2 \).

3.2.3 Solution formulas

One can express the solution \( u \) without using the Fourier transform, although it requires more work than for the heat equation. The problem is that the functions \( \cos(t|\xi|) \) and \( \sin(t|\xi|)/|\xi| \) don’t belong to \( \mathcal{S}(\mathbb{R}^d) \) and that we therefore can’t use the convolution theorem and Fourier’s inversion formula directly. Note that

\[
\hat{u}_t \left( \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} \hat{f}(\xi) d\xi \right) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \cos(t|\xi|) \hat{f}(\xi) d\xi,
\]

which implies that the solution corresponding to the case \( u_1 = 0 \) can be obtained by differentiating the solution corresponding to the case \( u_0 = 0 \) with respect to \( t \). It turns out that the form of the solution depends on the dimension. We discuss the cases \( d = 1, 2 \) and 3 separately.
One dimension

In this case we already know from Theorem 1.2 that the solution is given by d’Alembert’s formula

\[ u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} u_1(y) \, dy. \]  

(3.10)

We can recover this formula by calculating the inverse Fourier transform of \( \sin(t|\xi|)/|\xi| \). Using d’Alembert’s formula we can in fact guess what the solution must be. For \( t > 0 \) we find that

\[ \mathcal{F}(\chi_{(-t,t)}(\xi)) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-ix\xi} \, dx = \frac{2}{\sqrt{2\pi}} \frac{\sin(t|\xi|)}{|\xi|} = \frac{2}{\sqrt{2\pi}} \frac{\sin(t|\xi|)}{|\xi|}. \]

It follows that

\[ \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) = \frac{\sqrt{2\pi}}{2} \chi_{(-t,t)}, \quad t > 0. \]

Using the convolution theorem (which extends to \( L^1 \) by continuity) we now find that

\[ \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) \hat{u}_1(\xi) (x) = \frac{1}{2} \left( \chi_{(-t,t)} * g \right)(x) \]

\[ = \frac{1}{2} \int_{\mathbb{R}} \chi_{(-t,t)}(x-y) u_1(y) \, dy \]

\[ = \frac{1}{2} \int_{x-t}^{x+t} u_1(y) \, dy. \]

The formula extends directly to negative \( t \). Finally, the solution corresponding to \( u_0 \) is obtained by differentiating with respect to \( t \) and replacing \( u_1 \) by \( u_0 \).

Three dimensions

In three dimensions, it is not possible to make sense of the identity

\[ \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) \hat{u}_1(\xi) = (2\pi)^{-3/2} \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) * u_1 \]

directly. As we shall see, one can make sense of it by thinking of \( \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) \) not as a function, but as a linear functional. This is our first encounter with distributions. They will be discussed more in Chapter 5. As in one dimension, we shall compute the ‘inverse transform’ by making an educated guess. Note that d’Alembert’s formula involves the average of the function \( u_1 \) over the interval \([x-t, x+t]\) and of \( u_0 \) over the boundary of the interval (that is, its endpoints). It turns out that the correct idea in the three-dimensional case is to consider the spherical average

\[ M_t(f)(x) := \frac{1}{|\partial B_t(x)|} \int_{\partial B_t(x)} f(y) \, dS(y), \]
of a function $f$ over the sphere $\partial B_t(x)$ of radius $t$ around $x$, where $|\partial B_t(x)| = 4\pi t^2$ is the area of the sphere. It is useful to rewrite this as

$$M_t(f)(x) = \frac{1}{4\pi} \int_{\partial B_t(0)} f(x-ty) \, dS(y).$$

Note that $M_t(f)$ extends to an even function of $t$ defined by taking the average over the ball of radius $|t|$ for $t < 0$.

**Lemma 3.13.** Let $f \in \mathscr{S} (\mathbb{R}^3)$. Then the map $t \mapsto M_t(f)$ belongs to $C^\infty (\mathbb{R}; \mathscr{S}(\mathbb{R}^3))$.

**Proof.** To prove that $M_t(f)$ is rapidly decreasing for fixed $t$, we simply estimate $|f(x-ty)| \leq C_N(1+|x-ty|)^{-N} \leq C'_N(1+|x|)^{-N}$ when $|y| = 1$ (separate into the cases $|x| \leq 2t|y|$ and $|x| \geq 2t|y|$ and use the reverse triangle inequality in the latter case) and hence $|M_t(f)(x)| \leq C''_N(1+|x|)^{-N}$ by integration. Differentiating under the integral sign, we find that $M_t(f) \in \mathscr{S}(\mathbb{R}^3)$ for all $t \in \mathbb{R}$. The difference quotient

$$\frac{f(x-(t+h)y) - f(x-ty)}{h} = -\nabla f(x) \cdot (-y)$$

can be estimated using Taylor’s theorem by $(\max_{|\alpha|=2} \sup_{z \in B_{|\alpha|+|h|}(x)} |\partial^\alpha f(z)|)/h$ if $h$ is small. Again, this is rapidly decreasing in $x$ and therefore converges to zero as $h \to 0$ when multiplied by any polynomial. Similar estimates for the derivatives $\partial^\alpha_x f(x-ty)$ are easily obtained. It follows that $t \mapsto M_t(f) \in C^\infty (\mathbb{R}; \mathscr{S}(\mathbb{R}^3))$. □

**Lemma 3.14.**

$$\frac{1}{4\pi} \int_{\partial B_1(0)} e^{-iv\xi} \, dS(x) = \frac{\sin |\xi|}{|\xi|}.$$

**Proof.** We begin by noting that the integral is radial in $\xi$. Indeed, if $R$ is a rotation matrix, then $R^T R = I$, so

$$\int_{\partial B_1(0)} e^{-iv\xi} \, dS(x) = \int_{\partial B_1(0)} e^{-iRx \cdot \xi} \, dS(x) = \int_{\partial B_1(0)} e^{-iv\xi} \, dS(y),$$

where $y = R^Tx$. It therefore suffices to take $\xi = (0,0,|\xi|)$, in which case the integral equals

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi e^{-i|\xi| \cos \theta} \sin \theta \, d\theta \, d\phi = \frac{1}{2} \left[ \frac{e^{-i|\xi| \cos \theta}}{i|\xi|} \right]_0^\pi = \frac{\sin |\xi|}{|\xi|}. □$$

**Theorem 3.15** (Kirchhoff’s formula). The solution of (3.9) when $d = 3$ is given by

$$u(x,t) = \frac{1}{4\pi t} \int_{|x-y|=|t|} u_1(y) \, dS(y) + \frac{\partial}{\partial t} \left( \frac{1}{4\pi t} \int_{|x-y|=|t|} u_0(y) \, dS(y) \right). \quad (3.11)$$
Proof. It suffices to take \( u_0 = 0 \). We find that
\[
(2\pi)^{3/2}\mathcal{F}(M_l(u_1))(\xi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} \int_{\partial B_1(0)} u_1(x - ty) \, dS(y) \, dx
\]
\[
= \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\partial B_1(0)} u_1(z)e^{-i(z+ty)\cdot\xi} \, dS(y) \, dz
\]
\[
= \left( \int_{\mathbb{R}^3} u_1(z)e^{-i\zeta \cdot \xi} \, dz \right) \left( \frac{1}{4\pi} \int_{\partial B_1(0)} e^{-iy\cdot\xi} \, dS(y) \right)
\]
\[
= (2\pi)^{3/2} \frac{\sin(t|x|)}{t|x|} \hat{u}_1(\xi).
\]

Two dimensions

It’s more difficult to directly calculate a solution formula in the two-dimensional case, but there is a trick which works well. The idea is to look at the solution \( u(x_1,x_2,t) \) as a solution of the three-dimensional wave equation which is independent of \( x_3 \). This is called the method of descent.

**Theorem 3.16.** The solution of (3.9) when \( d = 2 \) is given by
\[
u(x,t) = \frac{\text{sgn} t}{2\pi} \int_{|x-y| \leq |t|} \frac{u_1(y)}{(t^2 - |x-y|^2)^{1/2}} \, dy + \frac{\partial}{\partial t} \left( \frac{\text{sgn} t}{2\pi} \int_{|x-y| \leq |t|} \frac{u_0(y)}{(t^2 - |x-y|^2)^{1/2}} \, dy \right).
\]

(3.12)

*Proof.* As usual it suffices to take \( u_0 = 0 \). Let \( \tilde{u} \) be the solution of the wave equation for \( d = 3 \) with initial data \( \tilde{u}_0(x) = 0 \) and \( \tilde{u}_1(x) = u_1(x) \), where \( \tilde{x} = (x_1, x_2), x = (x_1, x_2) \in \mathbb{R}^2 \). According to Theorem 3.15, we then have that
\[
\tilde{u}(\tilde{x}, t) = \frac{1}{4\pi t} \int_{|\tilde{x} - \tilde{y}| = |t|} \tilde{u}_1(\tilde{y}) \, dS(\tilde{y}) = \frac{1}{2\pi t} \int_{|\tilde{x} - \tilde{z}| = |t|} \tilde{u}_1(\tilde{z}) \, dS(\tilde{z}),
\]

since \( \tilde{u}_1 \) is independent of \( y_3 \). The upper hemisphere \( \{ \tilde{y}: |\tilde{x} - \tilde{y}| = |t|, y_3 \geq 0 \} \) can be parametrized by setting \( y_3 = x_3 + (t^2 - |x-y|^2)^{1/2}, |x-y| \leq |t| \). The element of surface area is then \( dS = |t| (t^2 - |x-y|^2)^{-1/2} \, dy \), and
\[
u(x,t) = \frac{\text{sgn} t}{2\pi} \int_{|x-y| \leq |t|} \frac{u_1(y)}{(t^2 - |x-y|^2)^{1/2}} \, dy
\]

There is a small problem in that \( \tilde{u}_1 \notin \mathcal{S}(\mathbb{R}^3) \). This can e.g. be taken care of by replacing it with \( u_1(x)\phi(x_3) \) where \( \phi \) is in \( C_0^\infty(\mathbb{R}) \) with \( \phi(x) = 1 \) for \( |x| \leq R \). As long as \( |t| \leq R \) this will not affect \( \nu \). One therefore finds that \( \nu \) defined by (3.12) is the unique solution of the wave equation. \( \square \)
3.2.4 Qualitative properties

Finite speed of propagation

The solution formulas (3.10), (3.11) and (3.12) show that the wave equation has finite speed of propagation, in the sense that the initial data at $x_0$ only influences the solution at $(x,t)$ if $|x-x_0| \leq |t|$. There are various ways of making this more precise.

Definition 3.17.

1. We say that a point $P = (x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}$ influences a future point $Q = (x,t) \in \mathbb{R}^d \times \mathbb{R}$, $t \geq t_0$, if for every spatial neighbourhood $U \subset \mathbb{R}^d$ of $x_0$ there exist two solutions $u$ and $v$ of the wave equation, such that $(u,u_t) = (v,v_t)$ outside of $U$ at time $t_0$ but $(u,u_t) \neq (v,v_t)$ at $Q$.

2. The (future) domain of influence of a point $P$ is the set of future points which are influenced by $P$. If $U \subset \mathbb{R}^d \times \mathbb{R}$, the domain of influence of $U$ is the union of the domains of influence of all $P \in U$.

3. The domain of dependence of a point $Q \in \mathbb{R}^d \times \mathbb{R}$ is the set of past points which influences it.

Remark 3.18. The reader should be aware that the terminology isn’t standardized. In the definition of the domain of influence one e.g. sometimes considers both future and past points. Note also that since the equation is linear one can replace $v$ by the zero function in the definition.

Proposition 3.19. Consider the wave equation in $\mathbb{R}^d$. If $d = 1,2$, the domain of influence of a point $(x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}$ is the set $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \geq t_0, |x-x_0| \leq t-t_0\}$. If $d = 3$, it is the set $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \geq t_0, |x-x_0| = t-t_0\}$. Similarly, the domain of dependence of $(x_0,t_0)$ is $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \leq t_0, |x-x_0| \leq t_0-t\}$ for $d = 1,2$ and $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \leq t_0, |x-x_0| = t_0-t\}$ for $d = 3$.

Proof. It suffices to consider the domain of influence. The domain of dependence is easily obtained by considering the domains of influence of points in the past. Since the wave equation is invariant under translations in $t$, we can assume that $t_0 = 0$.

Consider first the cases $d = 1,2$. The proof follows by inspecting the solution formulas (3.10) and (3.12). If $u_0$ and $u_1$ have supports in $\overline{B}_R(x_0)$, the supports of $u(\cdot,t)$ and $u_1(\cdot,t)$ for $t \geq 0$ are contained in $\{x \in \mathbb{R}^d : \exists y \in B_R(x_0), |x-y| \leq t\}$. Picking $u_0(x) \equiv 0$ and $u_1(x) \geq 0$ with $\text{supp} u_1 = \overline{B}_R(x_0)$, we find that $\text{supp} u(\cdot,t) = \{x \in \mathbb{R}^d : \exists y \in \overline{B}_R(x_0), |x-y| \leq t\}$. Choosing $R$ arbitrarily small, the result follows.

In the three-dimensional case, the solution formula (3.11) only involves integration over spheres rather than balls. Repeating the above argument shows that the domain of influence is now $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \geq 0, |x-x_0| = t\}$. □

The cones $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \geq t_0, |x-x_0| = t-t_0\}$ and $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \leq t_0, |x-x_0| = t_0-t\}$ are called the future and past light cones of $(x_0,t_0)$.
We can summarize this phenomenon by saying that information travels with speed less than or equal to 1 for the wave equation. This is in fact true in all dimensions. In three-dimensional space, a localized disturbance will cause a wave which is only observed for a finite amount of time by an observer who is standing still. In two dimensions, the observer will notice the wave for all future times, although the amplitude decays in time. A more useful fact is that Huygens’ principle holds in all dimensions if one considers the speed of which singularities travel. This is known as the generalized Huygens’ principle.

Extension of the solution formulas and conservation of energy

The solution formulas (3.10), (3.11) and (3.12) extend in two different ways. First of all, it is not necessary to require that \( u_0 \) and \( u_1 \) are rapidly decreasing. Because of the finite speed of propagation, it suffices to assume that \( u_0, u_1 \in C^\infty(\mathbb{R}^d) \). Indeed, given a non-negative function \( \varphi \in C^0_0(\mathbb{R}^d) \), we construct a family \( \{ \psi_\alpha \}_{\alpha \in \mathbb{Z}^d} \) of non-negative functions in \( C^\infty_0(\mathbb{R}^d) \) such that \( \text{supp } \psi_\alpha = \text{supp } \varphi + \alpha \) and \( \sum_{\alpha \in \mathbb{Z}^d} \psi_\alpha(x) = 1 \), by setting \( \varphi_\alpha(x) = \varphi(x-\alpha) \) and

\[
\psi_\alpha(x) = \frac{\varphi_\alpha(x)}{\sum_{\alpha \in \mathbb{Z}^d} \varphi_\alpha(x)}.
\]

Note that the sum in the denominator only ranges over finitely many \( \alpha \) for every fixed \( x \). The sequence \( \{ \psi_\alpha \}_{\alpha \in \mathbb{Z}^d} \) is an example of a locally finite partition of unity. Writing \( u_0 = \sum_\alpha u_0 \psi_\alpha \) and \( u_1 = \sum_\alpha u_1 \psi_\alpha \) and solving the wave equation with initial data \( u_0 \psi_\alpha, u_1 \psi_\alpha \in C^\infty_0(\mathbb{R}^d) \) for every \( \alpha \), we obtain a solution for the initial data \( u_0, u_1 \in C^\infty(\mathbb{R}^d) \) by adding all the solutions. Due to the finite speed of propagation, the sum will only range over finitely many indices at each point.

Secondly, the initial data don’t have to be infinitely differentiable. We saw already in Theorem 1.2 that it suffices that \( u_0 \in C^2 \) and \( u_1 \in C^1 \) in one dimension. The amount of regularity depends on the dimension however. In three dimensions we e.g. have the following result.

**Proposition 3.20.** Let \( u_0 \in C^k(\mathbb{R}^3) \), \( u_1 \in C^{k-1}(\mathbb{R}^3) \) with \( k \geq 3 \). Formula (3.11) defines a solution \( u \in C^{k-1}(\mathbb{R}^3 \times \mathbb{R}) \) of the wave equation with initial data \( u(x,0) = u_0(x) \) and \( u_t(x,0) = u_1(x) \).
The proof simply relies on changing variables to fix the domain of integration and differentiation under the integral sign. The difference with respect to the one-dimensional case has to do with the fact that the integral in (3.11) is over a two-dimensional surface in \( \mathbb{R}^3 \), whereas the integral in d’Alembert’s formula is over an open interval in \( \mathbb{R} \). One can show that the ‘loss of derivatives’ gets worse the higher the dimension. It is however possible to avoid this by instead measuring the regularity in terms of Sobolev norms. We will return to this in Chapter 5. All of this is in stark contrast to the heat equation, for which initial data which may not even be continuous evolve into smooth solutions for \( t > 0 \) (cf. Proposition 3.9).

Proposition 3.20 only provides an existence result. Uniqueness can be also be proved using the energy method.

**Theorem 3.21.** Let \( B_R(0) \) be the ball of radius \( R \) around the origin in \( \mathbb{R}^d \) and let \( \Gamma = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |x| \leq R - |t|\} \). Assume that \( u \in C^2(\Gamma) \) is a solution of the wave equation and that \( u \) and \( u_t \) vanish in \( B_R(0) \times \{0\} \). Then \( u(x,t) \equiv 0 \) in \( \Gamma \).

**Proof.** For simplicity we consider the case \( t \geq 0 \). The other case follows by the change of variables \( t \mapsto -t \). Since the real and imaginary parts each satisfy the wave equation, we may also assume that \( u \) is real. Let \( n(x) = x/(R-t) \) be the exterior unit normal of the ball \( B_{R-t}(0) \) in \( \mathbb{R}^d \) and \( dS(x) \) the surface measure on \( \partial B_{R-t}(0) \). Introduce

\[
e(t) = \frac{1}{2} \int_{|x| \leq R-t} (u_t^2 + |\nabla u|^2) \, dx, \quad 0 \leq t \leq R/c.
\]

\[
e'(t) = \int_{|x| \leq R-t} (u_t u_{tt} + \nabla u \cdot \nabla u_t) \, dx - \frac{1}{2} \int_{|x|=R-t} (u_t^2 + |\nabla u|^2) \, dS(x)
\]

\[
= \int_{|x| \leq R-t} u_t (u_{tt} - \Delta u) \, dx - \frac{1}{2} \int_{|x|=R-t} (u_t^2 + |\nabla u|^2 - 2u_t \nabla u \cdot n(x)) \, dS(x)
\]

\[
\leq 0,
\]

since \( u \) satisfies the wave equation and \( 2|u_t \nabla u \cdot n(x)| \leq u_t^2 + |\nabla u|^2 \). This shows that \( e(t) \) is decreasing. Since \( e(0) = 0 \) and \( e(t) \geq 0 \) it follows that \( e(t) \equiv 0 \). \( \square \)

Since the wave equation is linear, this guarantees that any \( C^2 \) solution is uniquely determined by its initial data. If we assume that \( \nabla u \) and \( u_t \) are decaying at infinity and instead integrate of \( \mathbb{R}^d \), the boundary terms disappear when we integrate by parts. This proves the following result.

**Proposition 3.22.** Let \( u_0, u_1 \in \mathcal{S}(\mathbb{R}^d) \). Then,

\[
E(t) = \frac{1}{2} \int_{\mathbb{R}^d} (|u_t|^2 + |\nabla u|^2) \, dx
\]

is independent of \( t \).

The quantity \( E \) can be interpreted as the total energy of the wave. This explains the name ‘energy method’ used above. Alternatively, the conservation of energy can be derived by using the Fourier representation of the solution. We leave this as an exercises.
3.3 The Laplace equation

Laplace’s equation
\[ \Delta u = 0 \]
arises e.g. when considering time-independent solutions of the heat or wave equation. A solution is called a harmonic function. Consider Laplace’s equation in \( \mathbb{R}^{d+1} \). Though one usually thinks of this as an equation for steady states and not as an evolution equation, we can of course pick one of the variables, say \( x_{d+1} \), and think of that as our time-variable. We thus consider Laplace’s equation in the half-space \( x_{d+1} > 0 \) with data given at the hyperplane \( x_{d+1} = 0 \). To emphasize this, we set \( t = x_{d+1} \) and end up with the problem

\[
\begin{cases}
  u_{tt}(x,t) = -\Delta u(x,t), & t > 0, \\
  u(x,0) = u_0(x), \\
  u_t(x,0) = u_1(x).
\end{cases}
\]

The corresponding problem on the Fourier side is

\[
\begin{cases}
  \hat{u}_{tt}(\xi,t) = |\xi|^2 \hat{u}(\xi,t), & t > 0, \\
  \hat{u}(\xi,0) = \hat{u}_0(\xi), \\
  \hat{u}_t(\xi,0) = \hat{u}_1(\xi),
\end{cases}
\]

yielding
\[
\hat{u}(\xi,t) = \cosh(t|\xi|)\hat{u}_0(\xi) + \frac{\sinh(t|\xi|)}{|\xi|} \hat{u}_1(\xi).
\]

Here we run into problems since \( \hat{u}(\cdot,t) \) will not even be bounded for \( t > 0 \) unless \( \hat{u}_0 \) and \( \hat{u}_1 \) are exponentially decaying (note that \( \cosh(t|\xi|) \) and \( \sinh(t|\xi|) \) grow like \( e^{|\xi|} \) at infinity). Thus, there is in general no solution of the form discussed above, unless we assume e.g. that \( \hat{u}_0 \) and \( \hat{u}_1 \) have compact support. One could of course look for a solution with less regularity, but using distribution theory one can in fact show under very mild conditions that there is no solution. There is however the possibility that \( \hat{u}_0 \) and \( \hat{u}_1 \) fit together in such a way that the exponentially large terms cancel. This will be the case precisely if \( \hat{u}_1(\xi) = -|\xi|\hat{u}_0(\xi) \). If we are satisfied with only prescribing \( u(x,0) \) we can therefore find a solution, namely \( u(x,t) = e^{-t|\xi|} \hat{u}_0(\xi) \).

This gives the solution formula
\[
u(x,t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|} \hat{u}_0(\xi) \, d\xi,
\]

defining a harmonic function in the half plane \( t > 0 \). As for the heat equation, it is possible to obtain a solution formula directly in terms of \( u_0 \) by calculating \( \mathcal{F}^{-1}(e^{-t|\xi|}) \). The result is
\[
u(x,t) = (P_t * u_0)(x),
\]

where
\[
P_t(x) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{t}{(t^2 + |x|^2)^{(d+1)/2}},
\]
is the Poisson kernel for the half-space \( t > 0 \), \( \Gamma \) denoting the Gamma function.
3.4 General PDE and the classification of second order equations

In general, a homogeneous linear second order PDE with real constant coefficients can be written

$$\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} a_i \frac{\partial u}{\partial x_i} + a_0 u = 0. \quad (3.13)$$

The coefficients $a_{ij}$, $a_i$ are real numbers. We assume that not all $a_{ij} = 0$. Without loss of generality we can assume that $a_{ij} = a_{ji}$. Consider a linear change of variables $y = Bx$, where $B$ is a $d \times d$ matrix. That is,

$$y_k = \sum_l b_{kl} x_l.$$

Using the chain rule, we obtain that

$$\frac{\partial u}{\partial x_i} = \sum_k \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i} = \sum_k b_{ki} \frac{\partial u}{\partial y_k}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \left( \sum_k b_{ki} \frac{\partial}{\partial y_k} \right) \left( \sum_l b_{lj} \frac{\partial}{\partial y_l} \right) u = \sum_{k,l} b_{ki} b_{lj} \frac{\partial^2 u}{\partial y_k \partial y_l}.$$

The second order part of the PDE are therefore converted as follows:

$$\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l} \left( \sum_{i,j} b_{ki} a_{ij} b_{lj} \right) \frac{\partial^2 u}{\partial y_k \partial y_l}.$$

We associate to the second order terms the quadratic form

$$p(\xi) = \sum_{i,j} a_{ij} \xi_i \xi_j = \xi^T A \xi,$$

where $\xi \in \mathbb{R}^d$ is thought of as a column vector and $\xi^T$ is the transpose of $\xi$. Let $q(\eta)$ be the form corresponding to the transformed second order part. Then

$$q(\eta) = \eta^T BAB^T \eta.$$

Note that this is just the change of variables formula associated with the transformation $\xi = B^T \eta$. Recall that the number of positive ($m_+$) and negative ($m_-$) eigenvalues of the symmetric matrix associated with a quadratic form is invariant under linear transformations and that we can find a transformation $B$ so that

$$q(\eta) = p(\sigma \eta) = \sum_{j=1}^{d} \sigma_j \eta_j^2,$$

in which $\sigma_j \in \{-1, 0, 1\}$ with $m_+$ positive and $m_-$ negative squares. We can make a classification of second order PDE by considering the quadratic form $q$ (or $p$). The names correspond to the character of the quadratic surface $q(\eta) = c$. 

**Definition 3.23.**

- We say that the PDE (3.13) is **elliptic** if the quadratic form $p$ is positive or negative definite, that is, if all the eigenvalues are positive or if all are negative. In other words this means that after linear change of variables, we can transform the PDE to the Laplace equation plus terms of lower order.

- If one eigenvalue is positive and the others negative (or vice versa) we say that the equation is **hyperbolic**. This means that we can transform the equation to the wave equation plus lower order terms.

- If the quadratic form is non-degenerate, but not hyperbolic or elliptic, we say that the equation is **ultrahyperbolic**. This means that both $m_+$ and $m_-$ are $\geq 2$ and that $d = m_+ + m_-$. Note in particular that we must have $d \geq 4$ for this to be possible.

- Finally, if one eigenvalue is zero and the others have the same sign, we say that the equation is **parabolic**. Note that this is a degenerate case and that one needs information on the first order terms to somehow compare the equation with the heat equation.

Roughly speaking, we expect elliptic equations to behave like Laplace’s equation, hyperbolic equations to like the wave equation and parabolic equations like the heat equation. A large portion of the history of PDE has been devoted to finding suitable generalizations of these definitions for operators of arbitrary order. Note that this classification by no means is exhaustive, not even for second order equations.

Using multi-index notation, we can write a general linear partial differential operator of order $m$ with constant coefficients as

$$
\sum_{|\alpha| \leq m} a_\alpha \partial^\alpha,
$$

where $a_\alpha \in \mathbb{C}$. There is no need to assume that the coefficients are real, but note that a complex equation always can be written as a system of two real equations. It turns out to be more convenient to work with the partial derivatives $D_j = \partial_j/i$ and $D = (\partial_1, \ldots, \partial_d)/i$. We therefore consider operators of the form

$$
P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha,
$$

and associate with such an operator the **symbol**

$$
P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha
$$

obtained by replacing $D_j$ by $\xi_j$. The reason for this choice is that $D_j$ corresponds to $\xi_j$ on the Fourier side, and hence

$$
\mathcal{F}(P(D)u)(\xi) = P(\xi)\hat{u}(\xi).
$$

(3.14)
The principal symbol \( P_m(\xi) \) is defined by
\[
P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha,
\]
and the principal part is \( P_m(D) \). Just as for second order operators, one can relate properties of the PDE to those of the polynomials \( P \) and \( P_m \). This is e.g. evident by considering relation (3.14). Our aim will be to find a suitable general definition of hyperbolicity in terms of \( P \) and \( P_m \). The idea is to consider two particular characteristics of the wave equation, namely the well-posedness of the initial-value problem and the finite speed of propagation. We shall therefore devote the next section to a study of the initial-value problem for partial differential operators with constant coefficients.

For completeness, we also mention the general definition of an elliptic operator.

**Definition 3.24 (Ellipticity).** We say that the operator \( P(D) \) is elliptic if \( P_m(\xi) \neq 0 \) for \( \xi \in \mathbb{R}^d \setminus \{0\} \).

Note that this agrees with the definition for second order equations. The idea is essentially that one can solve the equation \( P(D)u = f \) by dividing with \( P(\xi) \) on the Fourier side. This idea doesn’t quite work since \( P(\xi) \) might vanish for small \( |\xi| \). However, ellipticity implies that \( |P(\xi)| \geq c|\xi|^m \) for large \( |\xi| \) and this implies e.g. that \( u \) is smooth if \( f \) is smooth, since the regularity of \( u \) has to do with the decay of \( \hat{u} \) at infinity. The equation
\[
P_m(\xi) = 0
\]
is called the characteristic equation. We will return to this later on. On the other hand, one can show that the initial-value problem is always ill-posed in the sense described in Section 3.3 for elliptic equations if \( d \geq 2 \). We will not study elliptic operators more in these notes since they’re not directly relevant to waves. Note however that elliptic equations appear in the study of special types of waves, such as standing or travelling waves.

### 3.5 Well-posedness of the Cauchy problem

We consider the Cauchy problem
\[
\begin{cases}
  u_t = A(D_x)u, & t > 0, \\
  u = u_0, & t = 0,
\end{cases}
\]
(3.15)

where \( A(\xi) \) is an \( n \times n \) matrix with polynomial entries and \( u \) is defined on \( \mathbb{R}^d \times [0, \infty) \) with values in \( \mathbb{R}^n \). Note that scalar equations of order \( n \) in \( t \) can be reduced to this form by introducing the time derivatives of order \( \leq n - 1 \) as new variables. The corresponding partial differential operator is now the matrix-valued operator with symbol
\[
P(\xi, \tau) = i\tau I - A(\xi),
\]
where \( I \) the unit \( n \times n \) matrix and we use \( \tau \) to denote the Fourier variable corresponding to \( t \).
Example 3.25. The wave equation \( u_{tt} = \Delta u \) can be written
\[
\begin{cases}
u_t = v, \\
v_t = \Delta u,
\end{cases}
\]
which is of the above form with
\[
A(\xi) = \begin{pmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{pmatrix}.
\]

Definition 3.26 (Well-posedness in the sense of Hadamard). We say that the Cauchy problem is well-posed if:

1. there exists a solution for each \( u_0 \);
2. the solution is unique;
3. the solution depends continuously on the data \( u_0 \).

Of course, this has to be made more precise in the sense that one has to specify the class of initial data, the class of solutions and the topology used to measure the continuity of the map \( u_0 \mapsto u \). We will consider data \( u_0 \in \mathcal{S}(\mathbb{R}^d)^n \), meaning that each component of \( u_0 \) belongs to \( \mathcal{S}(\mathbb{R}^d) \). Note that Definition 3.1 extends in a natural way to functions with values in \( \mathcal{S}(\mathbb{R}^d)^n \) by imposing the conditions on each component.

Definition 3.27. The Cauchy problem (3.15) is said to be well-posed in \( \mathcal{S} \) if

1. for every \( u_0 \in \mathcal{S}(\mathbb{R}^d)^n \) there exists a unique solution
\[
u = S(\cdot)u_0 \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d)^n),
\]
2. the map \( S: \mathcal{S}(\mathbb{R}^d)^n \to C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d)^n), S(t)u_0 = u(t), \) is continuous.

If the Cauchy problem is not well-posed it is said to be ill-posed (in \( \mathcal{S} \)).

Remark 3.28. The assumption that \( u \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d)^n) \) in the uniqueness result can be relaxed. It suffices e.g. to assume that \( u \in C([0, \infty); \mathcal{S}(\mathbb{R}^d)^n) \cap C^1((0, \infty); \mathcal{S}(\mathbb{R}^d)^n) \). From the equation \( u_t = A(D_x)u \) it is then easy to prove that \( u \in C^2((0, \infty); \mathcal{S}(\mathbb{R}^d)^n) \) with \( \partial_t u_t = A(D_x)u_t \) and \( u_t(0) = A(D_x)u_0 \). Using induction, we find that \( u \in C^\infty((0, \infty); \mathcal{S}(\mathbb{R}^d)^n) \). Moreover, all the derivatives extend continuously to \( t = 0 \).

We look for a general condition under which the initial-value problem is well-posed in \( \mathcal{S} \). Taking the Fourier transform of (3.15) we obtain the problem
\[
\begin{cases}
\hat{u}_t = A(\xi)\hat{u}, \\ \hat{u} = \hat{u}_0,
\end{cases} \quad t > 0, \quad t = 0.
\]

(3.16)
Note that (3.16) is a linear system of ordinary differential equations. We therefore know that for each $\xi$ and $u_0(\xi)$, (3.16) has a unique solution

$$\hat{u}(\xi, t) = e^{tA(\xi)}\hat{u}_0(\xi).$$

Moreover, since $A(\xi)$ is a polynomial in $\xi$, $\hat{u} \in C^\infty(\mathbb{R}^d \times (0, \infty))^n$. In order for $\hat{u}(\cdot, t)$ to be of Schwartz class, we need to control the growth of $e^{tA(\xi)}\hat{u}_0(\xi)$ for large $\xi$. The idea is that this can be done by prescribing a condition on the eigenvalues of $A$.

**Definition 3.29** (Petrowsky’s condition). We say that $A(D_x)$ satisfies Petrowsky’s condition if there exists a number $C_A \in \mathbb{R}$ such that the eigenvalues $\lambda(\xi)$ of $A(\xi)$ satisfy $\text{Re} \lambda(\xi) \leq C_A$ for all $\xi \in \mathbb{R}^d$.

**Remark 3.30.**

1. Consider the symbol $P(\xi, \tau) = i\tau I - A(\xi)$ of the operator $\partial_t I - A(D_x)$. An equivalent formulation of Petrowsky’s condition is that $\det P(\xi, \tau) \neq 0$ if $\text{Im} \tau \leq -C_A$.

2. Petrowsky’s original condition was in fact a logarithmic bound on $\text{Re} \lambda(\xi)$. It was shown by Gårding in the special case of hyperbolic operators that this condition is equivalent to the one in the definition. The equivalence in general follows from the Tarski-Seidenberg theorem discussed below. This was first observed by Hörmander.

**Theorem 3.31.** The Cauchy problem (3.15) is well-posed in $\mathcal{S}$ if $A(D_x)$ satisfies Petrowsky’s condition.

To clarify the proof, we first treat the scalar case $n = 1$. $A(\xi) = a(\xi)$ is then a complex number for each $\xi$ and the condition means that $\text{Re} a(\xi) \leq C_A$ for all $\xi \in \mathbb{R}^d$. It follows that

$$|e^{ta(\xi)}| = e^{t\text{Re} a(\xi)} \leq e^{C_A t}, \quad t \geq 0.$$ 

Hence

$$\hat{u}(\xi, t) = e^{ta(\xi)}\hat{u}_0(\xi) \in \mathcal{S}(\mathbb{R}^d)$$

for each $t \geq 0$. A familiar argument shows that

$$u(x, t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{ta(\xi)}\hat{u}_0(\xi) \, d\xi,$$

defines a solution of the equation. A straight-forward proof shows that the function $t \mapsto u(\cdot, t)$ satisfies the conditions of Definition 3.27.

For the general case one needs a property of the matrix exponential. This property is best proved using complex analysis.

**Lemma 3.32.** There exists a constant $C > 0$ such that

$$\|e^A\| \leq C(1 + \|A\|^{n-1})e^{\text{max Re} \sigma(A)}$$

for $A \in \mathbb{C}^{n \times n}$, where $\sigma(A)$ denotes the spectrum (set of eigenvalues) of $A$ and $\| \cdot \|$ denotes the matrix norm.
Proof. The proof relies on the following representation of the matrix exponential:

\[ e^{tA} = \frac{1}{2\pi i} \oint_{|z| = r} e^{iz} (zI - A)^{-1} \, dz, \]  

where \( r \) is chosen so that all the eigenvalues of \( A \) are contained in the ball \( B_r(0) = \{ z \in \mathbb{C} : |z| < r \} \) and the contour is traversed counterclockwise. The path integral of a matrix-valued function is just the matrix formed by taking the path integral of each element of he matrix.

To prove formula (3.17), let \( B(t) \) be the function defined by the path integral. Notice first that the function \( z \mapsto e^{iz} (zI - A)^{-1} \) is analytic away from the eigenvalues of \( A \) (again defined by considering each element). It follows that \( B(t) \) is independent of \( r \). By the properties of the matrix exponential, it suffices to check that

\[ B'(t) = AB(t) \]

and \( B(0) = I \). Using the identity

\[ (zI - A)^{-1} = z^{-1} (I + A(zI - A)^{-1}), \]

and Cauchy’s theorem, we find that

\[
B(0) = \frac{1}{2\pi i} \oint_{|z| = r} (zI - A)^{-1} \, dz
= \frac{1}{2\pi i} \oint_{|z| = r} z^{-1} (I + A(zI - A)^{-1}) \, dz
= I + \frac{1}{2\pi i} \oint_{|z| = r} z^{-1} A(zI - A)^{-1} \, dz.
\]

Recall now Cramer’s rule, which says that

\[ A^{-1}_{j,k} = (-1)^{j+k} \frac{\det A^{k,j}}{\det A}, \]

where \( A^{k,j} \) is the matrix obtained if one removes row number \( k \) and column number \( j \) from \( A \). From this it follows that \( \|(zI - A)^{-1}\| \leq C|z|^{-1} \) as \( |z| \to \infty \). Consequently,

\[
\left\| \frac{1}{2\pi i} \oint_{|z| = r} z^{-1} A(zI - A)^{-1} \, dz \right\| \leq \frac{C}{r} \to 0
\]

as \( r \to \infty \), so that \( B(0) = I \). Finally, we have that

\[ B'(t) - AB(t) = \frac{1}{2\pi i} \oint_{|z| = r} e^{iz} I \, dz = 0. \]

This proves formula (3.17).

Define now a path \( \Gamma \) in \( \mathbb{C} \) as follows. Let \( \lambda_j, j = 1, \ldots, k \) be the distinct eigenvalues of \( A \), in order of increasing real part. For each \( j \), let \( B_1(\lambda_j) \) be the open disc of radius 1 around \( \lambda_j \).
Form the union \( U = \cup_{j=1}^{k} B_{1}(\lambda_{j}(A)) \) (note that some of the discs may intersect). Let \( \Gamma \) be the boundary of \( U \), traversed counterclockwise. Deforming the contour \( |z| = r \) into \( \Gamma \), we obtain from (3.17) that
\[
e^{A} = \frac{1}{2\pi i} \oint_{\Gamma} e^{z}(zI - A)^{-1} \, dz.
\]
On \( \Gamma \) we have that \( |z - \lambda_{j}| \geq 1 \) for each \( j \). Writing,
\[
det(zI - A) = \prod_{j=1}^{k} (z - \lambda_{j})^{m_j},
\]
where \( m_j \) is the algebraic multiplicity of \( \lambda_j \), it follows that
\[
|\det(zI - A)| \geq 1.
\]
Using Cramer’s rule, we find that
\[
(zI - A)_{j,k}^{-1} = (-1)^{j+k} \frac{\det((zI - A)^{k-j})}{\det(zI - A)^{-1}},
\]
where \( \det((zI - A)^{k-j}) \) is a polynomial of degree at most \( n - 1 \) in \( z \), the coefficients of which are products of the elements of \( A \). We can therefore estimate
\[
\|(zI - A)^{-1}\| \leq C_1(|z|^{n-1} + ||A||^{n-1}).
\]
On the other hand, when \( z \) belongs to \( \Gamma \) it follows that \( |z - \lambda_j| = 1 \) for some \( j \). Using the trivial estimate \( |\lambda_j| \leq ||A|| \), which follows from the definition of an eigenvalue, we find that \( |z| \leq ||A|| + 1 \) on \( \Gamma \). We conclude that \( \|(zI - A)^{-1}\| \leq C_2(1 + ||A||^{n-1}) \). Finally, \( |e^{z}| = e^{Re z} \leq e^{Re \lambda_k + 1} \) on \( \Gamma \) and the total length of \( \Gamma \) is at most \( 2\pi n \). It follows that
\[
\|e^{A}\| \leq \frac{1}{2\pi i} \oint_{\Gamma} e^{z}(zI - A)^{-1} \, dz \leq C(1 + ||A||^{n-1})e^{Re \lambda_k},
\]
where \( C = nC_2e \).

Proof of Theorem 3.31. Given the assumptions in the theorem and the previous lemma it is not difficult to see that
\[
\|e^{A(\xi)}\| \leq C(1 + |t|^{n-1}|\xi|^{m})e^{C|t|},
\]
where \( m \) is the maximum degree of the elements of \( A(\xi) \) times \( n - 1 \). We therefore find that
\[
\|\xi^{\alpha} \hat{u}(\xi, t)\|_{L^{\infty}(\mathbb{R}^d)} \leq C \sum_{|\beta| \leq |\alpha| + m} \|\xi^{\beta} \hat{u}_{0}(\xi)\|_{L^{\infty}(\mathbb{R}^d)}
\]
uniformly for \( t \) in a compact interval, for some other constant \( C \). The solution of a system of ordinary differential equations whose coefficients are \( C^{\omega} \) functions of some parameter depends smoothly on that parameter. It follows that \( \hat{u} \) is \( C^{\omega} \). Differentiating (3.16) with respect to \( \xi_k \), we find that
\[
\begin{align*}
\partial_{\xi_k} \hat{u}_t &= A(\xi)\partial_{\xi_k} \hat{u} + (\partial_{\xi_k} A(\xi)) \hat{u}, \quad t > 0, \\
\partial_{\xi_k} \hat{u} &= \partial_{\xi_k} \hat{u}_{0}, \quad t = 0,
\end{align*}
\]
and hence
\[ \partial_{\xi} u(\xi, t) = e^{iA(\xi)} \partial_{\xi} u_0(\xi) + \int_0^t e^{(t-s)A(\xi)}(\partial_{\xi} A(\xi)) \hat{u}(t, \xi) \, d\xi. \]

We also have that
\[ \partial_t u(\xi, t) = e^{iA(\xi)} A(\xi) \hat{u}_0(\xi). \]

This will at most increase $|\hat{u}(\xi, t)|$ by a polynomial factor. The same holds whenever we take a finite number of derivatives. Since $u_0 \in \mathcal{S}$, we therefore find that $\hat{u}(\cdot, t) \in \mathcal{S}$ and hence $u(\cdot, t) \in \mathcal{S}$.

To prove that $t \mapsto u(\cdot, t)$ is continuous we use the formula
\[ u(x, t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{iA(\xi)} \hat{u}_0(\xi) \, d\xi. \]

It follows that
\[ \|u(\cdot, t+h) - u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} \|e^{(t+h)A(\xi)} - e^{tA(\xi)}\| \|\hat{u}_0(\xi)\| \, d\xi. \]

For any $\varepsilon$ we can choose $R$ so large that
\[ \int_{|\xi| \geq R} \|e^{(t+h)A(\xi)} - e^{tA(\xi)}\| \|\hat{u}_0(\xi)\| \, d\xi \leq C \int_{|\xi| \geq R} (1 + |\xi|^{m'}) \|\hat{u}_0(\xi)\| \, d\xi \leq \varepsilon, \]

while
\[ \int_{|\xi| \leq R} \|e^{(t+h)A(\xi)} - e^{tA(\xi)}\| \|\hat{u}_0(\xi)\| \, d\xi \leq Ch \int_{|\xi| \leq R} \|\hat{u}_0(\xi)\| \, d\xi \to 0 \]

as $h \to 0$. It follows that $\|u(\cdot, t+h) - u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $h \to 0$. Similar estimates are easily obtained for all the other semi-norms. To prove that $u$ is $C^1$ in $t$ one needs in addition the identity
\[ e^{(t+h)A(\xi)} - e^{tA(\xi)} = \int_t^{t+h} A(\xi) e^{sA(\xi)} \, ds, \]

from which it follows that
\[ \frac{\|e^{(t+h)A(\xi)} - e^{tA(\xi)}\|}{h} = (1 + |\xi|^{m'}) e^{(t+h)CA}, \]

for some $m'$. Splitting the integral as above it then follows that $\|u(\cdot, t+h) - u(\cdot, t)\|_{\alpha, \beta} \to 0$ as $h \to 0$ for every semi-norm, where $u_t$ is the formal time-derivative of $u$,
\[ u_t(x, t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} A(\xi) e^{iA(\xi)} \hat{u}_0(\xi) \, d\xi. \]

The continuity of $u_t$ is shown as above. It follows that $t \mapsto u(\cdot, t)$ belongs to $C^1([0, \infty); \mathcal{S}'(\mathbb{R}^d))$ and hence to $C^\infty([0, \infty); \mathcal{S}'(\mathbb{R}^d))$ using the equation (see Remark 3.28). The continuity of the map $u_0 \mapsto u$ follows from the estimates
\[ \|u(\cdot, t)\|_{\alpha, \beta} \leq C \sum_{|\beta'| \leq |\beta|+m} \|u_0\|_{\alpha, \beta'}, \]

and the linearity of the equation (again using Remark 3.28).
One can show that Petrowsky’s condition is also necessary for well-posedness in $\mathcal{S}$. For this it is necessary to use a result from algebraic geometry known as the Tarski-Seidenberg theorem. This result implies that the maximum real part of the eigenvalues of $A(\xi)$ either is bounded or grows as a power in $|\xi|$ as $\xi \to \infty$. More precisely, one has that
\[
\max\{\Re \lambda : \det(\lambda I - A(\xi)) = 0 \text{ for some } \xi \in \mathbb{R}^d \text{ with } |\xi| = r\} = cr^\alpha(1 + o(1))
\]
as $r \to \infty$ for some $\alpha \in \mathbb{Q}$ and $c \in \mathbb{R}$. Suppose that Petrowsky’s condition is violated. We can then find a sequence $\xi_j \to \infty$ and corresponding eigenvalues $\lambda_j$ with $\Re \lambda_j \sim c|\xi_j|^\alpha$, $c, \alpha > 0$. Let $e_j$ be a sequence of corresponding unit eigenvectors. Choosing
\[
\hat{u}_j(\xi) = |\xi_j|^{-1}\varphi(\xi - \xi_j)e_j,
\]
for some fixed function $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi(0) = 1$, we obtain that
\[
|e^{tA(\xi)}\hat{u}_j(\xi_j)| = |\xi_j|^{-1}e^{(\Re \lambda_j)t}|e_j| \sim |\xi_j|^{-1}e^{c|\xi_j|^\alpha t}|e_j|
\]
so that $u_j(\cdot, t) = \mathcal{F}^{-1}(\hat{u}_j(\cdot, t))$ is unbounded in $\mathcal{S}(\mathbb{R}^d)$ for any fixed $t > 0$, while $u_{0,j} = \mathcal{F}^{-1}(\hat{u}_j) \to 0$ in $\mathcal{S}(\mathbb{R}^d)$. This proves that (2) in Definition 3.27 is violated. In fact, one can show that for ‘most’ initial data, there is not even a solution in the sense of Definition 3.27.

**Example 3.33.** Consider the scalar equation $u_t = a(D_x)u$, where
\[
a(\xi) = a_m \xi^m,
\]
with $a_m \neq 0$. To avoid trivialities, we suppose that $m \geq 1$. If $m$ is odd, the equation is well-posed if and only if $a_m$ is purely imaginary. If $m$ is even, the equation is well-posed if and only if $\Re a_m \leq 0$.

**Example 3.34.** The linearized KdV equation $u_t + u_{xxx} = 0$ can be written $u_t = a(D_x)u$ with $a(\xi) = i\xi^3$. Thus, it is well-posed. So is the Schrödinger equation $iu_t + u_{xx} = 0$, which can be written $u_t = a(D_x)u$ with $a(\xi) = -i\xi^2$.

**Example 3.35.** The heat equation $u_t = \Delta u$ can be written $u_t = a(D_x)u$ with $a(\xi) = -|\xi|^2$. Thus, it is well-posed. However, the *backwards* heat equation $u_t = -\Delta u$ is ill-posed. This equation is obtained by considering the evolution of the heat equation for $t < 0$.

Since we have assumed that $A(D_x)$ is independent of $t$, the semigroup property follows at once if Petrowsky’s condition is satisfied. On the Fourier side this corresponds to the identity
\[
e^{(s+t)A(\xi)} = e^{sA(\xi)}e^{tA(\xi)}.
\]
One sometimes uses the notation
\[
S(t) = e^{tA(D_x)}
\]
due to the similarity with the matrix exponential.

Finally, we consider the *inhomogeneous* problem
\[
\begin{align*}
\begin{cases}
u_t &= A(D_x)u + f, &t > 0, \\
u &= u_0, &t = 0,
\end{cases}
\end{align*}
\tag{3.18}
\]
which will play a big role when we consider nonlinear equations.
Theorem 3.36 (Duhamel’s formula). Let \( f \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d)) \) and \( u_0 \in \mathcal{S}(\mathbb{R}^d) \) and assume that \( A \) satisfies Petrowsky’s condition. The inhomogeneous Cauchy problem (3.18) has a unique solution \( u \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d)) \) given by

\[
u(t) = e^{tA(D_x)}u_0 + \int_0^t e^{(t-s)A(D_x)}f(s) \, ds, \quad t \geq 0.
\]

The solution formula should be interpreted pointwise for each \( x \in \mathbb{R}^d \).

Proof. Working on the Fourier side, we find that any solution must satisfy

\[
\hat{u}(\xi, t) = e^{tA(\xi)}\hat{u}_0(\xi) + \int_0^t e^{(t-s)A(\xi)}\hat{f}(\xi, s) \, ds, \quad t \geq 0
\]

from the theory of ordinary differential equations. On the other hand, \( \hat{u}(\xi, t) \) defined by this formula is a solution of the the transformed version of the problem. Using the same approach as in the proof of Theorem 3.31 one shows that \( e^{(t-s)A(\xi)}\hat{f}(\xi, s) \) is a smooth function of \( s \) and \( t \) with values in \( \mathcal{S}(\mathbb{R}^d) \) when \( 0 \leq s \leq t \). It is then an easy matter to prove that \( \hat{u} \) belongs to \( C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d)) \). It follows from Lemma 3.2 that the same is true for \( u = \mathcal{F}^{-1}(\hat{u}) \). Since \((\xi, s) \mapsto e^{(t-s)A(\xi)}\hat{f}(\xi) \) is in \( L^1(\mathbb{R}^d \times [0, t]) \) we can change the order of integration to deduce that

\[
\mathcal{F}^{-1}\left(\int_0^t e^{(t-s)A(\xi)}\hat{f}(\xi, s) \, ds\right) = \int_0^t \mathcal{F}^{-1}(e^{(t-s)A(\xi)}\hat{f}(\xi, s)) \, ds = \int_0^t e^{(t-s)A(D_x)}f(s) \, ds.
\]

\[\square\]

3.6 Hyperbolic operators

As mentioned in Section 3.4, hyperbolic operators are those for which the Cauchy problem is well-posed and which have the property of finite speed of propagation. By finite speed of propagation we mean that the solution \( u(x, t) \) has compact support for a fixed \( t \) if \( u_0 \) has compact support. One can show that a necessary and sufficient condition for this to hold is that

\[
\det(\lambda I - A(\xi)) \text{ has degree at most } n \text{ in } \xi.
\]

(3.19)

The idea is that if \( u(x, t) \) has compact support, then its Fourier transform

\[
\hat{u}(\xi, t) = (2\pi)^{-d/2} \int_{\text{supp} u(\cdot, t)} u(x, t)e^{-ix \cdot \xi} \, dx
\]

can be extended to the whole complex plane. Moreover, we have an estimate of the form

\[
|\hat{u}(\xi, t)| \leq Ce^{K|\text{Im} \xi|}, \quad \xi \in \mathbb{C}.
\]

(3.20)

On the other hand, we know that \( \hat{u}(\xi, t) = e^{tA(\xi)}u_0(\xi) \) and one can show that this together with (3.20) implies a growth condition of the form \( |\lambda(\xi) - C(1 + |\xi|) \) on the eigenvalues of \( A(\xi) \). This leads to the condition (3.19). For the converse, one needs a characterization of functions of compact support in terms of their Fourier transform. One can show that if \( \hat{u}(\cdot, t) \) is an entire
analytic function satisfying (3.20), then \( u(\cdot, t) \) has compact support. Results of this type are called Paley-Wiener theorems. A proof of these facts is beyond the scope of these notes.

An equation which is well-posed in \( \mathscr{S}(\mathbb{R}^d) \) and has finite speed of propagation is in fact well-posed in \( C^\infty \) (see the discussion before Proposition 3.20). The hypothesis of finite speed of propagation is crucial for this to hold. It can be shown that if the condition (3.19) is violated, there exists a non-zero solution \( u \in C^\infty(\mathbb{R}^d \times \mathbb{R}) \) of \( u_t = A(D_x)u \) which vanishes in the half-space \( t \leq 1 \). Thus, uniqueness is lost for the Cauchy problem.

We are now ready for the definition of a hyperbolic operator with constant coefficients. The operator \( \partial_t I - A(D_x) \) is said to be hyperbolic if it satisfies condition (3.19) and Petrowsky’s condition (Definition 3.29). For a general partial differential operator \( P(D) \) in \( \mathbb{R}^d \), it may not be clear what to consider as the time-variable. Moreover, one may also wish to solve the Cauchy problem with data on a hypersurface in \( \mathbb{R}^d \). It is therefore useful to have a more general definition of hyperbolicity.

**Definition 3.37 (Hyperbolicity).**

1. A scalar partial differential operator \( P(D) \) of order \( m \) is said to be hyperbolic in the direction \( \nu \in \mathbb{R}^d \) if its principal part satisfies

   \[
P_m(\nu) \neq 0,
   \]

   and there exists \( \tau_0 \in \mathbb{R} \) such that

   \[
P(\xi + \tau \nu) \neq 0, \quad \xi \in \mathbb{R}^d, \quad \text{Im} \tau \leq \tau_0.
   \]

2. \( P(D) \) is said to be strictly hyperbolic in the direction \( \nu \) if, in addition, for every \( \xi \in \mathbb{R}^d \) not parallel with \( \nu \), the equation

   \[
P_m(\xi + \tau \nu) = 0,
   \]

   has only simple real roots \( \tau \).

3. A matrix-valued operator \( P(D) \) is said to be (strictly) hyperbolic if \( \det P(D) \) is (strictly) hyperbolic.

Note that we recover the definition of hyperbolicity for \( P(D) = \partial_t I - A(D_x) \) by taking \( \nu = (0, 1) \in \mathbb{R}^d \times \mathbb{R} \).

In the following discussion we consider a scalar operator \( P(D) \). The condition \( P_m(\nu) \neq 0 \) is the analogue of (3.19). We say that the direction \( \nu \) is non-characteristic for \( P \) if this condition holds. A smooth \((d - 1)\)-dimensional hypersurface \( \Sigma \) in \( \mathbb{R}^d \) is said to be non-characteristic if at each point of \( \Sigma \), the normal is a non-characteristic direction for \( P \). The Cauchy problem with data on \( \Sigma \) consists of prescribing the values of the function \( u \) and all normal derivatives of \( u \) of order strictly less than \( m \) on \( \Sigma \). One can show that the Cauchy problem for a hyperbolic operator with \( C^\infty \) data on a non-characteristic hypersurface is well-posed. If one only assumes
that the surface is non-characteristic, one can still prove the (local) existence and uniqueness of solutions with real-analytic data if $\Sigma$ is real-analytic. This is the Cauchy-Kovalevskaya theorem. Intuitively, the assumption means that the normal derivative of order $m$ can be expressed in terms of the normal derivatives of order $\leq (m - 1)$ (the data) and their tangential derivatives. This can be used to construct a formal power series solution, and the proof consists of showing that the series converges. In fact, the theorem also holds for nonlinear equations. The assumption of real-analyticity is often too strong for the theorem to be of any practical relevance. Moreover, the solution may not depend continuously on the data.

Let us briefly mention something about operators

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha,$$

with (smooth) variable coefficients. We say that $P$ is (strictly) hyperbolic in the direction $\nu$ at $x_0$ if $P(x_0, D)$ is (strictly) hyperbolic in the direction $\nu$. It turns out that strict hyperbolicity is sufficient for well-posedness, whereas the weaker notion is not. However, there are certain notions in between which are useful. Consider e.g. the first order system

$$A_0(x, t) \partial_t u - \sum_{k=1}^d A_k(x, t) \partial_k u - B(x, t) u = 0, \quad (3.21)$$

where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, and $B$ and $A_k, k = 0, \ldots, d$, are $n \times n$ real matrices.

**Definition 3.38.** The system (3.21) is called a symmetric hyperbolic system if all the matrices $A_k(x, t)$ are symmetric and $A_0(x, t)$ is positive definite for all $(x, t)$.

One can prove that the Cauchy problem with initial data on $t = 0$ is well-posed also for such systems.
Linear PDE with constant coefficients
Chapter 4

First order quasilinear equations

After our study of linear equations in the previous chapter, we now turn to nonlinear hyperbolic equations. We will focus on real first order equations since these can be solved by the method of characteristics. The main emphasis is on showing that initially smooth solutions can develop singularities. This is something one encounters already in a course on ordinary differential equations, where solutions of the nonlinear equation $\dot{x} = x^2$ blow up in finite time. We will also briefly discuss how one can continue a solution after the development of a singularity. To do this we introduce the concept of weak solutions. Since weak solutions are in general not differentiable, they cannot satisfy the PDE in the classical sense. To interpret the solution one has to go back to the original physical problem. We will do this for a class of equations called conservation laws, for which weak solutions have a natural interpretation.

More details on the method of characteristics can be found in Evans [7] or John [14]. For more on hyperbolic conservation laws, see e.g. Bressan [4], Hörmander [11], Holden and Risebro [10] and Lax [17].

4.1 Some definitions

A general partial differential equation of order $m$ can be written

$$F(x, \partial^\alpha u(x); |\alpha| \leq m) = 0, \quad x \in \Omega,$$

(4.1)

where $\Omega \subset \mathbb{R}^d$ and the notation means that the real-valued function $F$ depends on $u$ and all of its derivatives of order less than or equal to $m$. Note that we now assume that $u$ is real-valued. This will be the case throughout the entire chapter. In the previous chapter we discussed linear PDE,

$$\sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u(x) = f(x), \quad x \in \Omega,$$

for given functions $a_\alpha, f: \Omega \to \mathbb{R}$. We mostly discussed the homogeneous case $f = 0$. Nonlinear equations are divided into different classes depending on ‘how nonlinear’ they are.
Definition 4.1.

(1) A PDE is said to be semilinear if it has the form

\[ \sum_{|\alpha|=m} a_\alpha(x) \partial^\alpha u + b(x, \partial^\beta u; |\beta| \leq m-1) = 0. \]

(4.2)

(2) It is said to be quasilinear if it has the form

\[ \sum_{|\alpha|=m} a_\alpha(x, \partial^\beta u; |\beta| \leq m-1) \partial^\alpha u + b(x, \partial^\beta u; |\beta| \leq m-1) = 0. \]

(4.3)

(3) It is said to be fully nonlinear if it depends nonlinearly on the highest derivatives.

In other words, the highest derivatives appear linearly in a quasilinear equation. If the coefficients of the highest derivatives are also independent of the function \( u \) and its derivatives, the equation is semilinear. A rule of thumb is that a PDE is more difficult to handle the more the nonlinearity affects the highest derivatives.

Example 4.2.

- The KdV equation

\[ u_t + uu_x + u_{xxx} = 0 \]

is an example of a semilinear equation. So is the nonlinear wave equation

\[ u_{tt} - \Delta u + f(u) = 0, \]

where \( f: \mathbb{R} \to \mathbb{R} \) is a given function.

- Burgers’ equation

\[ u_t + uu_x = 0 \]

is an example of a quasilinear equation, which is not semilinear.

- An important example of a fully nonlinear equation is obtained as follows. Consider a hypersurface \( \Sigma \) in \( \mathbb{R}^d \times \mathbb{R} \), given implicitly by \( S(x, t) = 0 \), where \( S \) is a \( C^1 \)-function. \( \Sigma \) is characteristic for the wave equation \( u_{tt} = \Delta u \) if the normal is a characteristic direction for the wave equation at each point of \( \Sigma \). Since the normal at \( (x, t) \) is given by \( \nabla_{(x, t)} S(x, t) \), this condition can be written

\[ S_t^2 - |
\nabla S|^2 = 0. \]

This is the eikonal\(^1\) equation. Assuming that \( S(x, t) = t - u(x) \), the eikonal equation is sometimes rewritten

\[ |
\nabla u|^2 = 1. \]

\(^1\)From the Greek word for image.
One can define characteristic directions and hyperbolicity also for nonlinear equations. For a semilinear equation it is natural to define \( \xi \in \mathbb{R}^d \) to be a characteristic direction at the point \( x \in \Omega \) if \( \xi \) is a characteristic direction of the linear operator \( \sum_{|\alpha|=m} a_\alpha(x) \partial^\alpha \). The definition is similar in the quasilinear case: \( \xi \in \mathbb{R}^d \) is a characteristic direction if \( \xi \) is characteristic for \( \sum_{|\alpha|=m} a_\alpha(x, u(x), \ldots, \partial^{m-1} u(x)) \partial^\alpha \). This condition depends on both \( x \) and the function \( u \).

Strict hyperbolicity is defined in an analogous manner. In the fully nonlinear case, one can define strict hyperbolicity in terms of the linearization at \( u \). The linearization at \( u \) is obtained by calculating
\[
\left. \frac{d}{d\epsilon} F(x, \partial^\alpha (u + \epsilon v)) \right|_{\epsilon=0} = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha v(x),
\]
where
\[
a_\alpha(x) = \frac{\partial F}{\partial (\partial^\alpha u)}(x, \partial^\beta u(x)).
\]

We say that the equation (4.1) is strictly hyperbolic if its linearization is strictly hyperbolic. We leave it as an exercise to prove that this agrees with the previous definitions in the case that the equation is semilinear or quasilinear.

**Example 4.3.** Consider the first order quasilinear equation
\[
a_1(x, u) \partial_1 u + \cdots + a_d(x, u) \partial_d u = f(x, u)
\]
in \( \mathbb{R}^d \). The direction \( v \in \mathbb{R}^d \) is non-characteristic at \( x \) and \( u \) if
\[
a(x, u) \cdot v \neq 0,
\]
where \( a(x, u) = (a_1(x, u), \ldots, a_d(x, u)) \). The equation is then automatically strictly hyperbolic, since the equation
\[
a(x, u) \cdot (\xi + \tau v) = 0
\]
has the real root \( \tau = -(a(x, u) \cdot \xi) / (a(x, u) \cdot v) \) for each \( \xi \in \mathbb{R}^d \). The equation also shares the finite speed of propagation property with the linear hyperbolic equations with constant coefficients discussed in the previous chapter. This follows from the method of characteristics discussed below.

Many of the models which appear in physics are in fact systems of PDE. The only difference is that \( u \) and \( F \) are then vector valued. One usually assumes that the number of equations equals the number of unknown functions. One can make a similar division into linear, semilinear, quasilinear and fully nonlinear equations.

### 4.2 The method of characteristics

We consider now first order quasilinear equations
\[
a_1(x, u) \partial_1 u + \cdots + a_d(x, u) \partial_d u = f(x, u).
\]
The coefficients $a_j$ are assumed to be $C^1$ functions. Our goal is to solve (4.4) by turning it into an appropriate system of ODE. Given $u$, we can define a curve $x: I \to \mathbb{R}^d$, $I = (s_0, s_1) \subset \mathbb{R}$, by $\dot{x}(s) = a(x(s), u(x(s)))$. The left hand side of (4.4) is the derivative of $u$ along the curve $x(s)$. In other words,\[\dot{z}(s) = f(x(s), z(s)), \quad \text{where } z(s) = u(x(s)).\] This closes the system and we have obtained the characteristic equations

\[\begin{align*}
\dot{x}(s) &= a(x(s), z(s)), \\
\dot{z}(s) &= f(x(s), z(s)).
\end{align*}\] (4.5)

The curves $x(s)$ are called characteristics, although this name is sometimes used for the curves $(x(s), z(s))$ in $\mathbb{R}^{d+1}$. If equation (4.4) is semilinear, the first equation doesn’t involve $z(s)$. We can then first find the characteristic $x(s)$ and then solve the second equation. If the original equation is linear this simply entails integrating $f(x(s))$.

**Example 4.4.** Consider the linear equation $x_1 \partial_2 u - x_2 \partial_1 u = f(x)$ in $\mathbb{R}^2$. The characteristic equations are

\[\begin{align*}
\dot{x}_1(s) &= -x_2(s), \\
\dot{x}_2(s) &= x_1(s), \\
\dot{z}(s) &= f(x(s)).
\end{align*}\]

The characteristics are circles, $x(s) = r(\cos s, \sin s)$, $r > 0$ fixed. Once we know this, we obtain that

\[u(r \cos s, r \sin s) = \int_0^s f(r \cos \theta, r \sin \theta) \, d\theta + g(r),\]

where $g$ is arbitrary. Note that a necessary condition in order to obtain a global solution is that

\[\int_0^{2\pi} f(r \cos \theta, r \sin \theta) \, d\theta = 0\]

for every $r$, since otherwise we will not obtain the same value of $u$ when we go one time around the origin.

We now illustrate how the characteristics can be used to solve the Cauchy problem in the neighbourhood of a point. For simplicity we assume that the initial data $u_0$ is given on the hyperplane $x_d = 0$.

**Theorem 4.5** (Local solvability). *Let $u_0 \in C^1(U)$ for some open set $U \subset \mathbb{R}^{d-1}$ and let $y_0 \in U$. Assume that moreover that $a$ and $f$ are $C^1$ in $\mathbb{R}^d$ and that the equation is non-characteristic in the direction $e_d$ at $(y_0, 0)$ and $u_0$. There exists an open subset $W \subset \mathbb{R}^d$ with $W \cap (\mathbb{R}^{d-1} \times \{0\}) \subset U$ such that equation (4.4) has unique solution $u \in C^1(W)$ with $u(y, 0) = u_0(y)$.*

*Proof.* The solution is obtained by solving the characteristic equations (4.5) with initial data $x(0; y) = (y, 0)$ and $z(0; y) = u_0(y)$, where $y \in U$. By the theory of ordinary differential equations, there exists for each $y$ a maximal solution $x(s; y)$ defined for $s$ in some open interval $I(y)$ containing the origin. Moreover, the function $(y, s) \mapsto x(s; y)$ is $C^1$ on the open set.
\( \cup_{y \in U} \{ y \} \times I(y) \). We wish to define the solution for \( s \neq 0 \) as \( u(x(s;y)) = z(s;y) \). This requires that the map \( y \mapsto x(s;y) \) is injective, in other words, that the characteristics don’t cross. Moreover, as \( y \) and \( s \) vary, the curves \( x(s;y) \) should trace out an open set \( W \subset \mathbb{R}^d \). Both of these conditions follow from the inverse function theorem if we can prove that the Jacobian of the map \( (y,s) \mapsto x(s;y) \) is non-zero at \( s = 0 \) and \( y = y_0 \). Since \( x(0;y) = (y,0) \) it follows that \( \partial_y x_i(0;y) = \delta_{ij} \). We also have that \( \partial_s x_i(0;y) = a((y,0),u_0(y)) \) from (4.5). Hence, the Jacobian is
\[
\begin{vmatrix}
1 & 0 & \ldots & a_1((y_0,0),u_0(y_0)) \\
0 & 1 & \ldots & a_2((y_0,0),u_0(y_0)) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_d((y_0,0),u_0(y_0))
\end{vmatrix}
= a_d((y_0,0),u_0(y_0)) \neq 0
\]
by the non-characteristic assumption. The fact that \( u \) defined in this way is a solution of the problem follows from the inverse function theorem and the definition of the characteristics. The uniqueness of the solution follows from the construction.

For details on the generalization of the method of characteristics to fully nonlinear first order equations and the solution of the Cauchy problem with data on a noncharacteristic hypersurface in \( \mathbb{R}^d \) we refer to Evans [7, Chapter 3.2].

### 4.3 Conservation laws

We used the conservation of energy in the derivation of the heat equation in Section 3.1.1. Suppose more generally that we have a time-dependent real-valued function \( u \) on \( \mathbb{R}^d \) which describes the ‘density’ of some quantity, e.g. the concentration of some substance in a fluid. The total amount of substance in the region \( \Omega \subset \mathbb{R}^d \) is then given by
\[
\int_{\Omega} u(x,t) \, dx.
\]
Assume that the production of the substance at a point \( (x,t) \) in space and time is given by \( g(x,t) \) (substance is destroyed if \( g(x,t) < 0 \)) and that the flow of the substance is described by the vector-valued function \( f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d \). The same considerations as in Section 3.1.1 lead to the equation
\[
\frac{d}{dt} \int_{\Omega} u(x,t) \, dx = -\int_{\partial \Omega} f(x,t) \cdot n \, dS + \int_{\Omega} g(x,t).
\]
Assuming again that all the involved functions are continuous, we obtain the continuity equation
\[
u_t + \nabla \cdot f = g.
\]
To get any further one has to know the form of \( f \) and \( g \), which depends on the specific problem. In many cases, \( f \) and \( g \) depend on \( u \) and possibly its derivatives. The relation between these quantities is called a constitutive law. We shall assume throughout the rest of the chapter that
First order quasilinear equations

\( f(x,t) = f(u(x,t)) \), that is, \( f \) depends solely on \( u \). We will also mostly consider the case \( g \equiv 0 \). The equation of interest is therefore

\[ u_t + \nabla \cdot f(u) = 0. \]

This is called a scalar conservation law. The function \( f \) is sometimes called the flux function in this context. Carrying out the differentiation, we find that

\[ u_t + a_1(u) \partial_1 u + \cdots + a_d(u) \partial_d u = 0, \]

where \( a_i(u) = f'_i(u) \), \( i = 1, \ldots, d \). In particular, we notice that a scalar conservation law always is hyperbolic in the \( t \)-direction (cf. Example 4.3). Many of the interesting problems from physics are in fact systems of conservation laws.

\[ \partial_t u_j + \nabla \cdot f_j(u) = 0, \quad (4.6) \]

where \( u = (u_1, \ldots, u_n) : \mathbb{R}^d \to \mathbb{R}^n \) and \( f_j : \mathbb{R}^n \to \mathbb{R}^d \), \( j = 1, \ldots, n \).

**Example 4.6** (Burgers’ equation). Burgers’ equation \( u_t + uu_x = 0 \) can be rewritten as the scalar one-dimensional conservation law

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0 \]

with flux \( f(u) = u^2/2 \).

**Example 4.7** (Equations of gas dynamics). The macroscopic description of a gas (or any other fluid) involves the fluid velocity \( u \), the density \( \rho \), the pressure \( p \) and the energy \( E \). The first is a vector quantity while the others are scalar quantities. At a given moment in time, the functions are defined in some open subset of \( \mathbb{R}^3 \). The flow of substance across a surface is given by \( \rho u \cdot n \), where \( n \) is the unit normal. The conservation of mass can therefore be expressed in the form

\[ \rho_t + \nabla \cdot (\rho u) = 0. \quad (4.7) \]

Basic principles of physics require balance of momentum. This is expressed as the conservation law\(^2\)

\[ \partial_t (\rho u_j) + \nabla \cdot (\rho u_j u) + \partial_j p = \rho F_j, \quad (4.8) \]

where \( \rho u_j \) are the components of the momentum density, \( \rho u_j u \cdot n \) the components of the momentum flow across a surface, \( p \) is the pressure and \( F_j \) the components of the body forces (e.g. gravity). Here we have neglected viscosity. Equation (4.8) can be written in vector notation as

\[ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = \rho F, \]

\(^2\)We’re abusing the terminology here, since the equations are not of the form (4.6). Introducing new variables \( w_j = \rho u_j \) we would however obtain (4.6) with source terms.
where \( u \otimes u = (u_j u_k)_{j,k} \in \mathbb{R}^d \) is the outer product of the vector \( u \) with itself. In addition, one needs an equation for the conservation of energy. Ignoring heat conduction and production, this takes the form

\[
\partial_t E + \nabla \cdot (u(E + p)) = \rho F \cdot u, \tag{4.9}
\]

where \( E \) is the total energy per unit volume. \( E \) is the sum of internal energy and kinetic energy associated with the macroscopic motion, \( E = \rho e + \frac{1}{2} \rho |u|^2 \), where \( e \) denotes the specific internal energy (internal energy per unit mass). In general one should add a a heat flux term as in Section 3.1.1 and possibly a heat production term. The considerations so far are not specific for gases, and would work equally well for a liquid. Note that we have six unknowns (counting all components of \( u \)) but only five equations (4.7)–(4.9). This means that we have an underdetermined system. Liquids are assumed to be incompressible, mathematically meaning that introduces the extra equation \( \nabla \cdot u \). For a gas one adds instead a constitutive relation of the form \( e = e(p, \rho) \). Assuming that we are dealing with an ideal gas with constant entropy (an isentropic gas) the equations can be reduced to (4.7)–(4.8) together with a constitutive relation

\[
p = f(\rho),
\]

where \( f(\rho) = \kappa \rho^\gamma \) for some constants \( \kappa > 0 \) and \( \gamma > 1 \). Here we have also assumed that the gas is polytropic, that is, that it has constant specific heat capacities. Equation (4.8) can be simplified using (4.7). Carrying out the differentiation in (4.8), one obtains that

\[
(\partial_t + \nabla \cdot (\rho u))u_j + \rho (\partial_t u_j + u \cdot \nabla u_j) + \partial_j p = \rho F_j,
\]

which simplifies to

\[
u_t + (u \cdot \nabla)u + \frac{1}{\rho} \nabla p = F \tag{4.10}
\]
due to the identity (4.7). Equation (4.10) is the usual form of Euler’s equations. Note that (4.8) can be written in the similar form

\[
\rho_t + u \cdot \nabla \rho + \rho \nabla \cdot u = 0.
\]

The differential operator

\[
\frac{D}{Dt} = \partial_t + u \cdot \nabla,
\]

called the material derivative, has a very natural interpretation. Suppose that we follow a very small parcel of gas. The parcel is approximated by a single mass at the point \( x(t) \) (a ‘fluid particle’). The evolution of this particle is governed by the equation

\[
x(t) = u(x(t), t).
\]

The material derivative of some physical quantity, such as the density, is then simply the derivative of the quantity along the path of a fluid particle, that is,

\[
\frac{d}{dt} \left\{ \rho(x(t), t) \right\} = \rho_t(x(t), t) + u(x(t), t) \cdot \nabla \rho(x(t), t) = \frac{D\rho}{Dt}(x(t), t).
\]

See Whitham [28, Chapter 6] for a much more extensive discussion of gas dynamics.
Example 4.8 (The linearized equations of gas dynamics). We now show how the linearized equations of gas dynamics discussed in Example 1.1 can be derived from (4.7) and (4.10) under the assumption (4.7). We assume in addition that there are no external forces ($F = 0$). Note that a constant state $u \equiv 0$, $\rho \equiv \rho_0$ and $p \equiv p_0 := f(\rho_0)$ is a solution of the equations. Consider now a small perturbation of the constant state: $\rho = \rho_0 + \varepsilon \tilde{\rho}$ and $u = \varepsilon \tilde{u}$, where $\varepsilon$ is a small parameter. Substituting this into the governing equations, expanding in powers of $\varepsilon$ and dividing by $\varepsilon$, we obtain

$$\tilde{\rho}_t + \rho_0 \nabla \cdot \tilde{u} = O(\varepsilon)$$

and

$$\tilde{u}_t + \frac{f'(\rho_0)}{\rho_0} \rho_0 \nabla \tilde{\rho} = O(\varepsilon).$$

Neglecting terms of order $\varepsilon$ and dropping the tildes for notational convenience, we arrive at the linearized equations

$$\rho_0 u_t + c_0^2 \nabla \rho = 0,$$

$$\rho_t + \rho_0 \nabla \cdot u = 0,$$

from Example 1.1, where $c_0 = \sqrt{f''(\rho_0)}$.

Example 4.9 (Traffic model). We discuss a simple model for traffic flow introduced independently by Lighthill and Whitham [18] and Richards [21]. Consider a road consisting of a single lane. The traffic is described by the density $\rho(x,t)$ of cars per unit length. The number of cars in the stretch $[a,b]$ is given by $\int_a^b \rho(x,t) \, dx$. The function $\rho$ is in reality a piecewise continuous function of $x$ (if we count fractions of cars), but if we replace it by the local average over a an interval which is sufficiently large compared with the average length of the cars and sufficiently small compared with the total length of the road, we can assume that it is continuous. We also assume that it is sufficiently differentiable for the following considerations to go through. We suppose that there are no exists or entries, so that the only way a car can enter or leave a stretch of road is through the endpoints. Let $v(x,t)$ be the velocity of an individual vehicle at $(x,t)$. The rate of cars passing through the point $x$ at time $t$ is then given by $v(x,t)\rho(x,t)$ and we have the relation

$$\frac{d}{dt} \int_a^b \rho(x,t) \, dx = v(a,t)\rho(a,t) - v(b,t)\rho(b,t).$$

We therefore obtain the conservation law

$$\rho_t + (\rho v)_x = 0.$$ 

The model has to be completed with a constitutive relation between $v$ and $\rho$ (and possibly $(x,t)$). Different relations can be used to describe different traffic situations. Assuming that the road looks the same everywhere and at all times, we postulate that $v$ is a function only of $\rho$. A natural assumption is that $v$ is a decreasing function of $\rho$ (cars slow down when the traffic becomes denser). We assume that when the traffic is light, a car will drive at maximum speed...
and that when the traffic reaches a critical density $\rho_{\text{max}}$ the cars will stop. The simplest possible constitutive relation which satisfies these conditions is given by

$$v = v_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right),$$

meaning that $\rho$ satisfies the conservation law

$$\rho_t + f(\rho)_x = 0,$$

with $f(\rho) = \rho v_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$. According to this model, the maximum flow rate $v_{\rho}$ is attained when $\rho = \rho_{\text{max}}/2$. Note that the ‘wave velocity’ $a(\rho) = f'(\rho) = v_{\text{max}} \left(1 - \frac{2\rho}{\rho_{\text{max}}}\right)$ is less than the individual velocity of the cars $v(\rho)$. The interpretation is that information travels backward through the traffic and that the drivers are influenced by what happens ahead. Setting $u = \rho/\rho_{\text{max}}$ and $x = v_{\text{max}}\tilde{x}$ turns the equation into

$$u_t + (u(1-u))_x = 0,$$

where we have dropped the tilde for notational convenience. Setting $w = 1 - 2u$ one recovers Burgers’ equation. See Whitham [28, Chapter 3] for more on traffic models.

There are well-developed theories for scalar conservation laws in any dimension ($n = 1$) and for systems of conservation laws in one spatial dimension ($d = 1$). Very little is known when both $d > 1$ and $n > 1$. For simplicity, we will consider scalar conservation laws in one space-dimension ($d = n = 1$). The Cauchy problem is

$$u_t + f(u)_x = 0, \quad t > 0, \quad u(x, 0) = u_0(x).$$

(4.11) (4.12)

The integrated form of (4.11) is

$$\frac{d}{dt} \int_{a}^{b} u(x, t) \, dx + \left[f(u(x, t))\right]_{x=a}^{b} = 0.$$

(4.13)

Carrying out the differentiation in (4.11), we obtain

$$u_t + a(u)u_x = 0,$$

(4.14)

where $a(u) = f'(u)$. Equation (4.14) can be thought of as a generalization of the ‘transport equation’ $u_t + au_x = 0$, a constant, in which the velocity $a(u)$ depends on the solution $u$. As discussed in Section 1.2 of the introduction, the crossing of the characteristics can be interpreted as saying that the portion of the graph of $u$ initially located at $x_1$ eventually overtakes the portion of the graph initially located at $x_2 > x_1$, given that $a(u_0(x_1)) > a(u_0(x_2))$. This causes the function $u$ to become multivalued.

We now summarize the local theory for the problem (4.11)–(4.12).
Theorem 4.10. Let $u_0 \in C^1_c(\mathbb{R})$ and $f \in C^2(\mathbb{R})$. There exists a maximal time $T > 0$ such that (4.11)–(4.12) has a unique solution $u \in C^1([0,T] \times \mathbb{R})$. The corresponding characteristics are straight lines. If $a(u_0)$ is increasing, the maximal existence time is infinite, $T = \infty$. Otherwise,

$$T = -\frac{1}{\inf_{x \in \mathbb{R}} \partial_x(a(u_0(x)))}.$$ 

Proof. The proof essentially follows as for Theorem 4.5, the difference being that we are no considering the global picture. We begin by proving the uniqueness. Due to the form of the equation we can parametrize the characteristics using $t$ and write them in the form $(x(t;y),t)$. The characteristic equations simplify to

$$\begin{cases}
\dot{x} = a(z), \\
\dot{z} = 0,
\end{cases}$$

meaning that $u$ is constant along the characteristics and that the characteristics are straight lines $t \mapsto (y + a(u_0(y))t, t)$. The map $y \mapsto x(t;y)$ defines a change of variables as long as it is strictly increasing with $\lim_{y \to \pm \infty} x(t;y) = \pm \infty$. The latter property will hold for all $t$ since $u_0$ is bounded. To verify the former property, we calculate

$$\partial_y x(t;y) = 1 + \partial_y(a(u_0(y)))t,$$

which is positive for all $y$ precisely when $t < T$ as defined above. This shows that the solution is uniquely determined by $u_0$. Note that the map $y \mapsto x(t;y)$ is not monotone for $t > T$. This means that there cannot exist a $C^1$ solution beyond $T$ since we would then have $x(t;x_1) = x(t;x_2) := x$ and $u(x,t) = u_0(x_1) = u_0(x_2)$ for some numbers $x_1 < x_2$. This means that the characteristics starting at $x_1$ and $x_2$ intersect at $(x,t)$. On the other hand, the slope of these characteristics is $a(u(x_1)) = a(u(x_2))$, which shows that they can never intersect. Clearly this is preposterous. The existence of the solution simply follows by defining $u(x,t) = u_0(g(x,t))$, where $x \mapsto g(x,t)$ is the inverse of the map $y \mapsto x(t;y)$. The inverse function theorem guarantees that $g$, and hence $u$, is a $C^1$ function. From the definition of the characteristics it follows that $u$ satisfies the differential equation, and that $u(x,0) = u_0(g(x,0)) = u_0(x)$.

\[\square\]
4.3.1 Discontinuous solutions

The above shows that we must allow discontinuous solutions in order to have any hope of developing a global theory. Equation (4.11) does not make sense in this case, but the integrated form (4.13) might still be sensible. Suppose that \( u \) is a classical solution of (4.11) on either side of a curve \( x = x(t) \) and that \( u \) and its derivatives extend continuously to the curve from the left and the right. Let \( u_l(t) \) be the limit from the left and \( u_r(t) \) the limit from the right of \( u \). Taking \( a < x(t) < b \), we obtain

\[
\frac{d}{dt} \int_a^b u \, dx = \frac{d}{dt} \left( \int_a^{x(t)} u \, dx + \int_{x(t)}^b u \, dx \right)
\]

\[
= \int_a^b u_t \, dx + (u_l(t) - u_r(t)) \dot{x}(t)
\]

\[
= - \int_a^b f(u)_x \, dx + (u_l(t) - u_r(t)) \dot{x}(t)
\]

\[
= (u_l(t) - u_r(t)) \dot{x}(t) + f(u_r(t)) - f(u_l(t)) - \left[ f(u(x,t)) \right]_{x=a}^{x=b}.
\]

We therefore obtain the necessary condition

\[
f(u_r(t)) - f(u_l(t)) = \dot{x}(t)(u_r(t) - u_l(t))
\]

for \( u \) to be a solution of the integrated version of the conservation law. This is called the Rankine-Hugoniot condition. It is often written

\[
[[f(u)] = \dot{x}(t)[[u]],
\]

where \([[v]] = v_r(t) - v_l(t)\) for a function \( v \) having limits \( v_l(t) \) from the left and \( v_r(t) \) from the right at the curve \( x = x(t) \).

Equation (4.13) still requires that we can make sense of the derivative of \( \int_a^b u(x,t) \, dx \). One could obtain an even weaker notion of solutions by integrating this relation over a time interval. There is a related concept which we now describe. It has the advantage that it can be used to define discontinuous solutions of other types of differential equations than conservation laws. Let \( u \) be a \( C^1 \) solution of (4.11) and multiply the equation by a function \( \phi \in C_0^\infty(\mathbb{R}^2) \). If we now integrate by parts, we obtain that

\[
0 = \iiint_{\mathbb{R}^3_+} (u_t(x,t) + f_u(x,t)) \phi_x(x,t) \, dx \, dt
\]

\[
= - \iiint_{\mathbb{R}^3_+} (u(x,t) \phi_t(x,t) + f_u(x,t) \phi_x(x,t)) \, dx \, dt - \int_{\mathbb{R}} u(x,0) \phi(x,0) \, dx,
\]

where \( \mathbb{R}^3_+ = \mathbb{R} \times (0,\infty) \). Hence

\[
\iiint_{\mathbb{R}^3_+} (u(x,t) \phi_t(x,t) + f_u(x,t) \phi_x(x,t)) \, dx \, dt + \int_{\mathbb{R}} u_0(x) \phi(x,0) \, dx = 0.
\] (4.15)
Definition 4.11. Let $u_0 \in L^\infty_{\text{loc}}(\mathbb{R})$. A function $u \in L^\infty_{\text{loc}}(\mathbb{R}^2_+)$ is called a weak solution of (4.11) if (4.15) holds for every $\varphi \in C^0_0(\mathbb{R}^2)$. By $u \in L^\infty_{\text{loc}}(\mathbb{R}^2_+)$ we mean that $u \in L^\infty(K)$ for every compact subset $K$ of $\mathbb{R}^2_+ := \mathbb{R} \times [0, \infty)$. 

We have already seen that a classical solution is a weak solution. If a weak solution $u$ is $C^1$ it is easy to see that it must in fact be classical solution. Indeed, integrating by parts one finds that

$$\int_{\mathbb{R}^2_+} u_t(x,t) \varphi(x,t) \, dx \, dt + \int_{\mathbb{R}} (u(x,0) - u_0(x)) \varphi(x,0) \, dx = 0.$$ 

It now follows that that $u_t(x,t) + f(u(x,t)) = 0$ for $t > 0$. Otherwise one could find an open set $\Omega$ where this is non-zero and of constant sign and then choose $\varphi \geq 0$ and $\neq 0$ with support in $\Omega$ to obtain a contradiction. Having established this, the initial condition follows from a similar argument.

Proposition 4.12. Assume that $x : [0, \infty) \to \mathbb{R}^2_+$ is a $C^1$ function with $x(0) = 0$. Assume that $u$ is a classical $C^1$ solution on each side of the curve $x = x(t)$ and that $u$ and its derivatives extend continuously to $x = 0$ along the curve $x = x(t)$ and $u_t = \lim_{x \to x(t)^-} u(x,t)$ and $u_r = \lim_{x \to x(t)^+} u(x,t)$. Then $u$ is a weak solution if and only if it satisfies the Rankine-Hugoniot condition along the curve $x = x(t)$.

Proof. Let $\Omega_l = \{(x,t) \in \mathbb{R}^2_+ : x < x(t)\}$ and $\Omega_r = \{(x,t) \in \mathbb{R}^2_+ : x > x(t)\}$. We can write

$$\int_{\mathbb{R}^2_+} (u\varphi_t + f(u)\varphi_x) \, dx \, dt = \int_{\Omega_l} (u\varphi_t + f(u)\varphi_x) \, dx \, dt + \int_{\Omega_r} (u\varphi_t + f(u)\varphi_x) \, dx \, dt.$$ 

Since $\nabla_{(x,t)} \cdot (f(u)\varphi, u\varphi) = (u_t + f(u)\varphi_t + u\varphi_x + f(u)\varphi_x = u\varphi_t + f(u)\varphi_x)$ in $\Omega_l$, we obtain that

$$\int_{\Omega_l} (u\varphi_t + f(u)\varphi_x) \, dx \, dt = \int_{\Omega_l} \nabla_{(x,t)} \cdot (f(u)\varphi, u\varphi) \, dx \, dt = \int_{\partial \Omega_l} (f(u)\varphi, u\varphi) \cdot n \, ds$$

$$= - \int_{-\infty}^{x(0)} u_0(x) \varphi(x,0) \, dx + \int_{0}^{\infty} (f(u_l(t)) - \dot{x}(t)u_l(t))\varphi(x(t),t) \, dt$$

by the divergence theorem, where $n$ denotes the outer unit normal. In order to use the divergence theorem, we should work in a bounded domain. Note however that $\varphi(x,t) = 0$ outside a compact set, so that we could do the calculations in $\Omega_l \cap R$ where $R = (-A,A) \times (-B,B)$ is a large rectangle in $\mathbb{R}^2$ containing supp $\varphi$ and $\{(x(t),t) : 0 \leq t < B\}$. The result would be the same since $\varphi$ and its derivatives vanish on $\partial R$. Performing the same calculation for $\Omega_r$, we find that

$$\int_{\Omega_r} (u\varphi_t + f(u)\varphi_x) \, dx \, dt = - \int_{x(0)}^{\infty} u_0(x) \varphi(x,0) \, dx + \int_{0}^{\infty} (f(u_r(t)) - \dot{x}(t)u_r(t))\varphi(x(t),t) \, dt$$
Adding the two parts and comparing with (4.15), we find that $u$ is a weak solution if and only if
\[
\int_{0}^{\infty} ([f(u))] - \dot{x}(t) [u]) \phi(x(t), t) \, dt = 0
\]
for all $\phi$. This is equivalent to the Rankine-Hugoniot condition $[f(u)] = \dot{x}(t) [u]$.

Note that the condition is automatically satisfied if $u_l = u_r$, so that a piecewise $C^1$ function of the above form always is a weak solution.

### 4.3.2 The Riemann problem

Having obtained a necessary condition on a solution with a jump discontinuity, we are left with the question of whether such solutions exist. We consider initial data of the form

\[
u_0(x) = \begin{cases} u_l, & x \leq 0, \\ u_r, & x > 0, \end{cases}
\]

where $u_l$ and $u_r$ are constants with $u_l \neq u_r$. We suppose in addition that $a(u_l) \neq a(u_r)$.

Consider first the case $a(u_l) > a(u_r)$. If we try to solve the problem using the method of characteristics, we see that the region $x < a(u_r)t$ only contains characteristics emanating from the negative half-axis, while the region $x > a(u_l)t$ only contains characteristics emanating from the positive half-axis. We should therefore set

\[
u(x, t) = \begin{cases} u_l, & x < a(u_r)t, \\ u_r, & x > a(u_l)t. \end{cases}
\]

However, in the region $a(u_r)t \leq x \leq a(u_l)t$ the characteristics intersect and it is unclear how to define the solution (see Figure 4.2). We propose the solution

\[
u(x, t) = \begin{cases} u_l, & x \leq ct, \\ u_r, & x > ct. \end{cases}
\]
First order quasilinear equations

Clearly $u$ solves (4.11) on either side of the discontinuity. From the previous discussion, we see that $u$ is a weak solution if and only if the Rankine-Hugoniot condition

$$c = \frac{[f(u_l)]}{[u]}$$

(4.17)

is satisfied. The solution defined in (4.16)–(4.17) is called a shock wave.

Consider next the case $a(u_l) < a(u_r)$ and assume for simplicity that $a$ is strictly monotone (see e.g. Holden and Risebro [10] for the general case). We no longer have the problem that characteristics cross. Instead we have the problem that no characteristics emanating from the $x$-axis enter the region $a(u_l)t \leq x \leq a(u_r)t$. Thus, it is again unclear how to define the solution in this region. The shock solution (4.16)–(4.17) is still a weak solution even if $u_l < u_r$. However, we can also define a solution by setting

$$u(x,t) = \begin{cases} 
  u_l, & x \leq a(u_l)t, \\
  a^{-1}(x/t), & a(u_l)t \leq x \leq a(u_r)t, \\
  u_r, & x \geq a(u_r)t.
\end{cases}$$
Indeed, it is straightforward to show that this defines a piecewise $C^1$ function, which satisfies the differential equation in each of the three different regions. It follows that it is a weak solution. We thus have the unhappy situation that weak solutions are not unique. The function $u$ is called a rarefaction wave. Figure 4.5 shows a rarefaction wave corresponding to Burgers’ equation, for which $a^{-1}(x/t) = x/t$ and solution is given by a straight line in the interval $a(u_l)t \leq x \leq a(u_r)t$.

### 4.3.3 Entropy conditions

There is now way to single out the ‘correct’ solution by simply using the PDE. Precisely as when we introduced weak solutions above, we have to go back to the physics behind the model to find a criterion. A common way of singling out the physically relevant solution is to introduce some form of entropy condition. The name comes from the equations of gas dynamics, where these conditions have a physical interpretation. The basic idea is to introduce a direction of time. Note that the original conservation law (4.11) is reversible, in the sense that if $u(x,t)$ is a solution, then so is $u(-t,-x)$. Thus the qualitative behavior forward and backward in time should be identical. This is in contrast to the heat equation $u_t = u_{xx}$. As we saw in Chapter 3, the heat equation is well-posed forward in time, but not backward in time. Thus, there is a ‘preferred’ sense of direction. This is also illustrated by the fact that for the heat equation the $L^2$-norm is strictly decreasing for solutions in $\mathcal{S}(\mathbb{R})$ (see Section 3.1.4), whereas it is conserved for Schwartz class solutions of the conservation law (4.11). Arguing that there always is some diffusion or dissipation in a real physical system, one imposes the condition that the ‘correct’ solution should be a limit as $\varepsilon \to 0^+$ of a solution $u^\varepsilon$ of the equation

$$u_t + f(u)_x = \varepsilon u_{xx}, \quad \varepsilon > 0.$$  \hspace{1cm} (4.18)

Since this is difficult to verify directly, we now derive a necessary condition on $u$ for this to hold, which is easier to check. Suppose that $\eta$ is a $C^1$ function and define $q$ by $q'(u) =$
Then it is easily confirmed that if $u$ is a classical solution of the conservation law, then $η(u)_t + q(u)_x = 0$. Suppose now that $η$ is $C^2$ and convex so that $η''(u) ≥ 0$. We will show that $η(u)_t + q(u)_x ≤ 0$ in a suitable weak sense if $u$ is a weak solution obtained as a limit of $C^2$ solutions of $(4.18)$. We assume that $u^ε → u$ in $L^1_{loc}(R^2_+)$ and that $u^ε$ is locally bounded uniformly in $ε$. Note that we don’t prove that such a convergence result holds; the main point is to motivate condition $(4.19)$ below. Multiplying $(4.18)$ by $η'(u^ε)φ$, where $φ ∈ C^∞_0(R^2_+)$ with $φ ≥ 0$, we find that

$$0 = \iiint (u^ε_t + f(u^ε)_x - εu^ε_{xx})η'(u^ε)φ dx dt$$

$$= \iiint (η(u^ε)_t + q(u^ε)_x - εη'(u^ε)u^ε_{xx})φ dx dt$$

$$= \iiint (-η(u^ε)φ_t - q(u^ε)φ_x) dx dt + ε \iiint (η''(u^ε)(u^ε_{xx})^2 - η(u^ε)φ_{xx}) dx dt,$$

so that

$$\iiint (-η(u^ε)φ_t - q(u^ε)φ_x) dx dt ≤ ε \iiint η(u^ε)φ_{xx} dx dt.$$ 

In the limit as $ε → 0^+$ we therefore find that

$$\iiint (-η(u)φ_t - q(u)φ_x) dx dt ≤ 0 \quad (4.19)$$

for every $φ ∈ C^∞_0(R^2_+)$ with $φ ≥ 0$. This is sometimes expressed as saying that $η(u)_t + q(u)_x ≤ 0$ in the sense of distributions. The pair $(η, q)$ is called an entropy-flux pair.

**Definition 4.13.** A weak solution $u$ is said to be entropy admissible if $(4.19)$ holds for every entropy-flux pair $(η, q)$.

For a fixed number $u_*$, the function $η_δ(u) = (|u - u_*|^2 + δ^2)^{1/2}$ is convex and $C^∞$. It converges to $|u - u_*|$ as $δ → 0$. Choosing the corresponding flux function $q_δ$ appropriately, we find that

$$q_δ(u) = \int_{u_*}^u \frac{f'(s)(s - u_*)}{((s - u_*)^2 + δ^2)^{1/2}} ds$$

$$\rightarrow \int_{u_*}^u f'(s) \operatorname{sgn}(s - u_*) ds$$

$$= (f(u) - f(u_*)) \operatorname{sgn}(u - u_*)$$

pointwise as $δ → 0$ by dominated convergence. Note that $η(u) = |u - u_*|$ is convex and smooth everywhere except at $u_*$, and that $q(u) = (f(u) - f(u_*)) \operatorname{sgn}(u - u_*)$ satisfies $q'(u) = f'(u)η'(u)$ for $u ≠ u_*$. Thus $(η, q)$ can be thought of as a Lipschitz continuous entropy-flux
pair. Approximating \((\eta, q)\) with \((\eta_\delta, q_\delta)\) in (4.20) and passing to the limit using the dominated convergence theorem, one obtains the inequality
\[
\int_{\mathbb{R}^2} (|u - u_*| \phi_t + \text{sgn}(u - u_*) (f(u) - f(u_*)) \phi_x) \, dx \, dt \geq 0
\]
for every \(u_* \in \mathbb{R}\). This is Kružkov’s admissibility condition. One can in fact show that Kružkov’s condition is equivalent to the entropy condition in Definition 4.13.

The entropy condition turns out to be exactly what is needed to prove existence and uniqueness. The proof of the following theorem is beyond the scope of these notes. See e.g. Bressan [4] and Hörmander [11].

**Theorem 4.14.** Assume that \(u_0 \in L^\infty(\mathbb{R})\) and that \(f\) is locally Lipschitz continuous. There exists a unique entropy admissible weak solution \(u \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty(\mathbb{R} \times [0, \infty))\) of the Cauchy problem (4.11)–(4.12). If in addition \(u_0 \in L^1(\mathbb{R})\), then \(u \in C([0, \infty); L^1(\mathbb{R}))\) and the map \(S(t) : u_0 \mapsto u(\cdot, t)\) from initial data to solution extends to a continuous semigroup on \(L^1(\mathbb{R})\) with \(\|S(t)u_0 - S(t)v_0\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}\).

We consider now the Riemann problem and show how the admissibility condition can be used to single out a unique solution. The argument used to prove the Rankine-Hugoniot condition shows that (4.19) leads to the jump condition
\[
[\eta(u)] = c[\eta(u)]
\]
for every \(C^1\) entropy-flux pair \((\eta, q)\). Once again, the condition is automatically satisfied by a piecewise \(C^1\) solution. This means e.g. that a rarefaction wave is always admissible. A limiting procedure shows again that the inequality continues to hold if we take \(\eta(u) = |u - u_*|\) and \(q(u) = \text{sgn}(u - u_*) (f(u) - f(u_*))\), whence we obtain
\[
\text{sgn}(u_r - u_*)(f(u_r) - f(u_*)) - \text{sgn}(u_l - u_*)(f(u_l) - f(u_*)) \leq c(|u_r - u_*| - |u_l - u_*|).
\]
Suppose that \(u_l < u_r\) and take \(u_* = \alpha u_l + (1 - \alpha) u_r\) for \(\alpha \in [0, 1]\), so that \(u_* \in [u_l, u_r]\). We then obtain the chain of inequalities
\[
\begin{align*}
f(u_r) + f(u_l) - 2f(u_*) & \leq c(u_r + u_l - 2u_*) \\
\Rightarrow f'(u_r) + f(u_l) - 2f(u_*) & \leq c(1 - 2\alpha)(u_r - u_l) \\
\Rightarrow f(u_r) + f(u_l) - 2f(u_*) & \leq (1 - 2\alpha)(f(u_r) - f(u_l)) \\
\Rightarrow \alpha f(u_r) + (1 - \alpha) f(u_l) & \leq f(\alpha u_l + (1 - \alpha) u_r),
\end{align*}
\]
where we used the Rankine-Hugoniot condition in the third line. We leave it as an exercise to show that the inequality is reversed if \(u_r < u_l\). The conditions
\[
\alpha f(u_r) + (1 - \alpha) f(u_l) \leq f(\alpha u_l + (1 - \alpha) u_r), \quad \alpha \in [0, 1],
\]
if \(u_l < u_r\), and
\[
\alpha f(u_r) + (1 - \alpha) f(u_l) \geq f(\alpha u_l + (1 - \alpha) u_r), \quad \alpha \in [0, 1],
\]
if \( u_r < u_l \), are called the Lax shock conditions. They have a very simple geometric interpretation, saying that the graph of \( f \) must lie above the line segment between \((u_l, f(u_l))\) and \((u_r, f(u_r))\) in the first case and below the line segment in the second case. In particular, if \( f \) is convex, we obtain the condition

\[
u_r < u_l
\]

for a shock. For the Riemann problem, this means that the shock is the correct entropy solution in the case \( u_r < u_l \), whereas the rarefaction wave is the correct entropy solution in the case \( u_l < u_r \). The condition is reversed if \( f \) is concave.

**Example 4.15.** Consider the traffic model from Example 4.9, written in the normalized form

\[
u_t + f(u)_x = 0,
\]

where \( f(u) = u(1 - u) \) is concave. This means that a shock wave will form if \( u_l < u_r \) (meaning that the density is greater in the direction of the traffic). Consider e.g. the case when \( u_r = 1 \) meaning that the cars are standing still ahead. This might e.g. model a queue forming at a red light. The velocity of the shock will then be \( c = (f(u_r) - f(u_l))/(u_r - u_l) < 0 \) (assuming that \( u_l \in (0, 1) \)), so the shock wave is moving backwards, that is cars driving towards the traffic light are coming to a sudden stop. If on the other hand \( u_l > u_r \) we get a rarefaction wave. If \( u_l = 1 \), this might model what happens when the light turns green.
Chapter 5

Distributions and Sobolev spaces

In the previous chapter we encountered weak solutions of PDE. The key was to multiply the equation by a ‘test function’ \( \varphi \in C_0^\infty(\mathbb{R}^d) \) and integrate by parts to put all the derivatives on \( \varphi \). The name test function refers to the fact that the equation is ‘tested’ against \( \varphi \). This gives a suitable definition of weak solutions in which the solution \( u \) doesn’t need to be differentiable. If \( u \) is a sufficiently regular weak solution, one shows that it is a classical solution, meaning that the PDE holds pointwise. What is involved here is the concept of ‘weak’ or ‘distributional’ derivatives. The weak derivative of a non-differentiable function is in general not a function, but a distribution. The purpose of this chapter is partly to clarify what this means. Distributions or ‘generalized functions’ have a long prehistory. The theory was made rigorous in the middle of the 20th century, in particular by Laurent Schwartz (1915–2002). Distributions are indispensable in the modern theory of linear partial differential equations.

Distributions are not quite as useful for nonlinear equations, since one can in general not define the product of two distributions. Moreover, the space of distributions is not a normed vector space. For this purpose we introduce a generalization of Lebesgue spaces called Sobolev spaces after Sergei L’vovich Sobolev (1908–1989). Sobolev was also involved in the early theory of distributions. The Sobolev space \( H^k \) simply consists of \( L^2 \) functions whose weak derivatives up to order \( k \) lie in \( L^2 \). As we will see, \( H^k \) is a Banach space and moreover an algebra if \( k \) is sufficiently large. Another important reason for using Sobolev spaces is that they behave well with respect to the Fourier transform. This is not the case for spaces based on the supremum norm.

The present chapter can only serve as brief introduction. For more details on distributions we refer to Friedlander [9], Duistermaat and Kolk [6], Hörmander [12], Rauch [20] and Strichartz [25]. For more on Sobolev spaces we refer to the standard reference Adams and Fournier [1]. In particular, this reference discusses Sobolev spaces on subsets of \( \mathbb{R}^d \), where the regularity of the boundary plays an important role. In these notes we shall only discuss Sobolev spaces on \( \mathbb{R}^d \).
5.1 Distributions

Note that a locally integrable function \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \) gives rise to a linear functional\(^1\) \( \ell_u : C_0^\infty(\mathbb{R}^d) \to \mathbb{C} \) defined by \( \ell_u(\phi) = \langle u, \phi \rangle \), where

\[
\langle u, \phi \rangle := \int_{\mathbb{R}^d} u(x) \phi(x) \, dx. \tag{5.1}
\]

Note that this differs from the inner product in \( L^2(\mathbb{R}^d) \) in that there is no complex conjugate on \( \phi \) and that we don’t require \( u, \phi \in L^2(\mathbb{R}^d) \). It is clear at least if \( u \) is continuous that \( \ell_u \) determines \( u \) uniquely, since if \( \ell_u(\phi) = 0 \) for all \( \phi \in C_0^\infty(\mathbb{R}^d) \) then \( u \equiv 0 \). Below we shall prove that this is also the case if \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \).

In keeping with the notation of Laurent Schwartz, we denote the space of test functions \( C_0^\infty(\mathbb{R}^d) \) by \( \mathcal{D}(\mathbb{R}^d) \).

**Definition 5.1.** A distribution \( \ell \) is a linear functional on \( \mathcal{D}(\mathbb{R}^d) \) which is continuous in the following sense. If \( \{ \phi_n \} \subset \mathcal{D}(\mathbb{R}^d) \) has the following properties:

1. there exists a compact set \( K \subset \mathbb{R}^d \) such that \( \text{supp } \phi_n \subset K \) for all \( n \), and
2. there exists a function \( \phi \in \mathcal{D}(\mathbb{R}^d) \) such that for all \( \alpha \in \mathbb{N}^d \), \( \partial^\alpha \phi_n \to \partial^\alpha \phi \) uniformly,

then \( \ell(\phi_n) \to \ell(\phi) \) as \( n \to \infty \). The space of distributions is denoted \( \mathcal{D}'(\mathbb{R}^d) \).

**Remark 5.2.** \( \mathcal{D}'(\mathbb{R}^d) \) is simply the space of continuous linear functionals on \( \mathcal{D}(\mathbb{R}^d) \) when \( \mathcal{D}(\mathbb{R}^d) \) is given the topology defined by (1) and (2) in the above definition. When (1) and (2) hold, we say that \( \phi_n \to \phi \) in \( \mathcal{D}(\mathbb{R}^d) \). One can define \( \mathcal{D}(\Omega) \) if \( \Omega \) is an open subset of \( \mathbb{R}^d \) in a similar fashion.

The most common ways that distributions appear is through the expression (5.1). The fact that this expression defines a distribution follows from the estimate

\[
|\langle u, \phi \rangle| \leq \left( \int_K |u(x)| \, dx \right) \|\phi\|_{L^\infty(\mathbb{R}^d)},
\]

if \( \phi \in \mathcal{D}(\mathbb{R}^d) \) with \( \text{supp } \phi \subset K \). Let us prove that (5.1) determines the distribution \( \ell_u \) uniquely. Note that by linearity, it suffices to show that if \( u = 0 \) as an element of \( L^1_{\text{loc}}(\mathbb{R}^d) \) if \( \ell_u = 0 \).

**Theorem 5.3.** Let \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \). If \( \langle u, \phi \rangle = 0 \) for all \( \phi \in \mathcal{D}(\mathbb{R}^d) \), then \( u(x) = 0 \) a.e.

**Proof.** It suffices to prove that \( u = 0 \) a.e. in the ball \( B_R(0) \subset \mathbb{R}^d \) for any \( R > 0 \). Let \( v = \chi_{B_{2R}(0)} u \in L^1(\mathbb{R}^d) \). Then \( \int_{\mathbb{R}^d} v(x) \phi(x) \, dx = \int_{\mathbb{R}^d} u(x) \phi(x) \, dx = 0 \) for any \( \phi \in \mathcal{D}(\mathbb{R}^d) \) with \( \text{supp } \phi \subset B_{2R}(0) \). Let \( J_\varepsilon \) be a mollifier as in Theorem B.25. Then \( J_\varepsilon * v \to v \) in \( L^1(\mathbb{R}^d) \) by Theorem B.25 (4) and hence also in \( L^1(B_R(0)) \). On the other hand, \( (J_\varepsilon * v)(x) = \int_{\mathbb{R}^d} v(y) J_\varepsilon(x-y) \, dy = 0 \) for \( x \in B_R(0) \) and \( \varepsilon > 0 \) sufficiently small, since \( \text{supp } J_\varepsilon(x-\cdot) \subset B_{R+\varepsilon}(0) \). It follows that \( u = v = 0 \) a.e. in \( B_R(0) \).

---

\(^1\)A linear functional is just a linear map with values in \( \mathbb{C} \).
The above theorem implies that we can identify \( u \) with the linear functional \( \ell_u \). We will do this without further mention in what follows. We shall also use the notation \( \langle u, \varphi \rangle \) for \( u(\varphi) \) even when \( u \) is just an element of \( \mathcal{D}'(\mathbb{R}^d) \). Let us give an example of a distribution which does not correspond to a locally integrable function.

**Example 5.4.** The Dirac\(^2\) distribution \( \delta_a \) at \( a \in \mathbb{R}^d \) is defined by

\[
\langle \delta_a, \varphi \rangle := \varphi(a).
\]

It is clear that \( \langle \delta_a, \varphi \rangle = \varphi(a) \to \varphi(a) = \langle \delta_a, \varphi \rangle \) if \( \varphi \to \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \), so that \( \delta_a \) really is a distribution. It is also clear that \( \delta_a \) doesn’t correspond to a locally integrable function, since \( \langle \delta_a, \varphi \rangle = 0 \) if \( \varphi(a) = 0 \). If \( \langle \delta_a, \varphi \rangle = \langle u, \varphi \rangle \) for some \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \), this would imply that \( u(x) = 0 \) a.e. and hence that \( \langle \delta_a, \varphi \rangle = \langle u, \varphi \rangle = 0 \) for all \( \varphi \in \mathcal{D}(\mathbb{R}^d) \).

We also need to know what is meant by convergence of a sequence of distributions.

**Definition 5.5.** We say that \( u_n \to u \) in \( \mathcal{D}'(\mathbb{R}^d) \) if \( \langle u_n, \varphi \rangle \to \langle u, \varphi \rangle \) for all \( \varphi \in \mathcal{D}(\mathbb{R}^d) \).

**Example 5.6.** Consider a function \( K \in L^1(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} K(x) \, dx = 1 \) and set \( K_\varepsilon(x) = \varepsilon^{-d} K(\varepsilon^{-1} x) \) for \( \varepsilon > 0 \). Then \( K_\varepsilon \to \delta_0 \) as \( \varepsilon \to 0 \) since \( \langle K_\varepsilon, \varphi \rangle = \int_{\mathbb{R}^d} K_\varepsilon(y) \varphi(y) \, dy = (K_\varepsilon * \mathcal{D}(\varphi))(0) \to \varphi(0) \) as \( \varepsilon \to 0^+ \) by Theorem B.27, where \( \mathcal{D}(\varphi)(x) = \varphi(-x) \). In particular, we have that \( K_t \to \delta_0 \) as \( t \to 0^+ \), where \( K_t \) is the heat kernel.

### 5.2 Operations on distributions

We now discuss how to extend various operations on functions to distributions. To explain the method, we consider translations in \( x \). Given a function \( \varphi \in \mathcal{D}(\mathbb{R}^d) \) and a vector \( h \in \mathbb{R}^d \), we define \( (\tau_h \varphi)(x) = \varphi(x-h) \). If \( \varphi, \psi \in \mathcal{D}(\mathbb{R}^d) \) we have that

\[
\langle \tau_h \varphi, \psi \rangle = \int_{\mathbb{R}^d} \varphi(x-h) \psi(x) \, dx = \int_{\mathbb{R}^d} \varphi(y) \psi(y+h) \, dy = \langle \varphi, \tau_{-h} \psi \rangle.
\]

It therefore makes sense to define

\[
\langle \tau_h u, \varphi \rangle := \langle u, \tau_{-h} \varphi \rangle \tag{5.2}
\]

if \( u \) is just a distribution. Note that \( \tau_{-h} \varphi_n \to \tau_{-h} \varphi \) if \( \varphi_n \to \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \), so that (5.2) indeed defines a distribution \( \tau_h u \). It is clear that the new definition agrees with the old one if \( u \in \mathcal{D}(\mathbb{R}^d) \) in the sense that the distribution \( \tau_h u \) defined by (5.2) can be represented by the function \( u(\cdot - h) \).

**Example 5.7.** We have \( \tau_h \delta_0 = \delta_h \), since

\[
\langle \tau_h \delta_0, \varphi \rangle = \langle \delta_h, \tau_{-h} \varphi \rangle = \varphi(0+h) = \varphi(h).
\]

\(^2\)After the physicist Paul Dirac (1902–1984).
The above extension of the translation operator is an example of a general principle.

**Proposition 5.8.** Suppose that \( L \) is a linear operator on \( \mathcal{D}(\mathbb{R}^d) \) which is continuous in the sense that \( \varphi_n \to \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \) implies \( L(\varphi_n) \to L(\varphi) \). Suppose, moreover, that there exists a continuous linear operator \( L^T \) on \( \mathcal{D}(\mathbb{R}^d) \), with the property that \( \langle L(\varphi), \psi \rangle = \langle \varphi, L^T(\psi) \rangle \) for all \( \varphi, \psi \in \mathcal{D}(\mathbb{R}^d) \). Then \( L \) extends to a continuous linear operator on \( \mathcal{D}'(\mathbb{R}^d) \) given by

\[
\langle L(u), \varphi \rangle = \langle u, L^T(\varphi) \rangle \quad \text{for all} \quad u \in \mathcal{D}'(\mathbb{R}^d) \quad \text{and} \quad \varphi \in \mathcal{D}(\mathbb{R}^d). \tag{5.3}
\]

**Proof.** The continuity of \( L^T \) shows that \( L(u) \) defined in (5.3) is a distribution. It is also clear from the definition of the transpose that (5.3) defines an extension of the original operator \( L \). Finally, if \( u_n \to u \) in \( \mathcal{D}'(\mathbb{R}^d) \), then

\[
\langle L(u_n), \varphi \rangle = \langle u_n, L^T(\varphi) \rangle \to \langle u, L^T(\varphi) \rangle = \langle L(u), \varphi \rangle,
\]

so the extension is continuous on \( \mathcal{D}'(\mathbb{R}^d) \). \( \square \)

We call the operator \( L^T \) the transpose of \( L \). Note that this might differ from the \( L^2 \) adjoint since there is no complex conjugate on \( \psi \) in \( \langle \varphi, \psi \rangle \). We now give a list of common operators and their transposes. Here \( (\sigma_\lambda \varphi)(x) = \varphi(\lambda x) \) is dilation by \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( (M_\psi \varphi)(x) = \psi(x) \varphi(x) \) multiplication by \( \psi \in C^\infty(\mathbb{R}^d) \).

<table>
<thead>
<tr>
<th>( L )</th>
<th>( L^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_h )</td>
<td>( \tau_{-h} )</td>
</tr>
<tr>
<td>( \sigma_\lambda )</td>
<td>(</td>
</tr>
<tr>
<td>( M_\psi )</td>
<td>( M_\psi )</td>
</tr>
<tr>
<td>( \partial^\alpha )</td>
<td>( (\lambda)^{\alpha} \partial^\alpha )</td>
</tr>
</tbody>
</table>

As a corollary to Proposition 5.8 we therefore have the following result.

**Proposition 5.9.** Each of the operators \( \tau_h, \sigma_\lambda, M_\psi \) and \( \partial^\alpha \) on \( \mathcal{D}(\mathbb{R}^d) \) have unique extensions to continuous maps on \( \mathcal{D}'(\mathbb{R}^d) \). The extensions are given by \( \langle Lu, \varphi \rangle = \langle u, L^T \varphi \rangle \) where \( L \) and \( L^T \) are as in the above table.

Note the restriction to multiplication with smooth functions. In general we can’t define the product of a distribution and a non-smooth function \( \psi \) since then \( \varphi \psi \notin \mathcal{D}(\mathbb{R}^d) \). In particular, one can in general not define the product of two distributions. We will get back to this question later on.

**Example 5.10.** The Heaviside\(^3\) function

\[
H(x) = \begin{cases} 
1 & x > 0, \\
0 & x \leq 0,
\end{cases}
\]

\(^3\)After the physicist Oliver Heaviside (1850–1925).
is locally integrable and hence $H \in \mathcal{D}'(\mathbb{R})$. The distributional derivative of $H$ is calculated as follows:

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = - \int_{0}^{\infty} \varphi'(x) \, dx = \varphi(0) = \langle \delta_{0}, \varphi \rangle.$$ 

It follows that $H' = \delta_{0} \in \mathcal{D}'(\mathbb{R})$. Note that the pointwise derivative of $H$ is zero everywhere except at 0, where it is undefined. We can continue the process and calculate higher derivatives of $H$. E.g.

$$\langle H'', \varphi \rangle = \langle \delta_{0}', \varphi \rangle = - \langle \delta_{0}, \varphi' \rangle = - \varphi'(0).$$

### 5.3 Tempered distributions

One operation which cannot be defined on $\mathcal{D}'(\mathbb{R}^{d})$ is the Fourier transform. As the following lemma shows, $\mathcal{F}$ is its own transpose.

**Lemma 5.11.** For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^{d})$, we have that

$$\langle \mathcal{F}(\varphi), \psi \rangle = \langle \varphi, \mathcal{F}(\psi) \rangle.$$ 

**Proof.** The result can be proved using Lemma 2.15, but it is simpler to give a direct proof. By Fubini’s theorem we have that

$$\langle \mathcal{F}(\varphi), \psi \rangle = (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} \varphi(x) e^{-ix \cdot \xi} \, dx \right) \psi(\xi) \, d\xi$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} \varphi(x) \left( \int_{\mathbb{R}^{d}} \psi(\xi) e^{-ix \cdot \xi} \, d\xi \right) \, dx$$

$$= \langle \varphi, \mathcal{F}(\psi) \rangle.$$ 

The problem with defining $\langle \mathcal{F}(u), \varphi \rangle = \langle u, \mathcal{F}(\varphi) \rangle$ is that $\mathcal{F}$ doesn’t map $\mathcal{D}(\mathbb{R}^{d})$ to itself (compact support is lost). The key is to instead use $\mathcal{S}(\mathbb{R}^{d})$ as the space of test functions.

**Definition 5.12.** A continuous linear functional on $\mathcal{S}(\mathbb{R}^{d})$ is called a *tempered distribution*.

The space of tempered distributions is denoted $\mathcal{S}'(\mathbb{R}^{d})$.

Convergence in $\mathcal{S}'(\mathbb{R}^{d})$ is defined analogously to convergence in $\mathcal{D}'(\mathbb{R}^{d})$.

**Definition 5.13.** We say that $u_{n} \to u$ in $\mathcal{S}'(\mathbb{R}^{d})$ if $u_{n}(\varphi) \to u(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^{d})$.

Continuity of an element $u \in \mathcal{S}'(\mathbb{R}^{d})$ means that

$$\langle u, \varphi_{n} \rangle \to \langle u, \varphi \rangle$$

if $\varphi_{n} \to \varphi$ in $\mathcal{S}(\mathbb{R}^{d})$. If $u \in \mathcal{S}'(\mathbb{R}^{d})$, then $\langle u, \varphi \rangle$ is well-defined if $\varphi \in \mathcal{S}(\mathbb{R}^{d})$. Moreover, if $\varphi_{n} \to \varphi \in \mathcal{S}(\mathbb{R}^{d})$, then clearly $\varphi_{n} \to \varphi$ in the topology of $\mathcal{S}(\mathbb{R}^{d})$. Consequently, the restriction of a tempered distribution to $\mathcal{S}'(\mathbb{R}^{d})$ is an element of $\mathcal{D}'(\mathbb{R}^{d})$. Moreover, $u$ is uniquely

4Sometimes called a *temperate distribution*. 
The result in Proposition 5.8 remains true if \( D \) and \( \psi \) are replaced by \( D' \) and \( \psi' \) respectively, as long as \( \psi \) and its derivatives grow at most polynomially.

**Example 5.14.**

1. The Dirac distribution \( \delta_u \) belongs to \( \mathcal{D}'(\mathbb{R}^d) \).

2. Any locally integrable function \( u \) with \( (1 + |x|^2)^{-N} u \in L^1(\mathbb{R}^d) \) for some \( N \) belongs to \( \mathcal{D}'(\mathbb{R}^d) \) since
   \[
   \int_{\mathbb{R}^d} |u(x)| \varphi(x) \, dx \leq \int_{\mathbb{R}^d} (1 + |x|^2)^{-N} |u(x)| (1 + |x|^2)^N |\varphi(x)| \, dx \\
   \leq \|(1 + |\cdot|^2)^{-N} u\|_{L^1(\mathbb{R}^d)} \|(1 + |\cdot|^2)^N \varphi\|_{L^\infty(\mathbb{R}^d)}.
   \]

3. In particular, any \( u \in L^p(\mathbb{R}^d) \), \( p \in [1, \infty] \), belongs to \( \mathcal{D}'(\mathbb{R}^d) \) since \( (1 + |\cdot|^2)^{-N} u \in L^1(\mathbb{R}^d) \) if \( N \) is sufficiently large by Hölder’s inequality.

4. \( u(x) = e^{\lambda x} \) is in \( \mathcal{D}'(\mathbb{R}^d) \) but not in \( \mathcal{D}'(\mathbb{R}^d) \). Strictly speaking this means that the functional \( \varphi \mapsto \langle u, \varphi \rangle \), \( \varphi \in \mathcal{D}(\mathbb{R}^d) \), can not be extended to an element of \( \mathcal{D}'(\mathbb{R}^d) \). This can e.g. be seen by approximating a function \( \varphi \in \mathcal{D}(\mathbb{R}^d) \) satisfying \( \varphi(x) = e^{-|x|^2} / 2 \) for large \( |x| \) with a sequence in \( \mathcal{D}(\mathbb{R}^d) \) in the topology of \( \mathcal{D}(\mathbb{R}^d) \).

The proof of the following result is exactly the same as the proof of Proposition 5.8.

**Proposition 5.15.** The result in Proposition 5.8 remains true if \( \mathcal{D}(\mathbb{R}^d) \) is replaced by \( \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{D}'(\mathbb{R}^d) \) by \( \mathcal{S}'(\mathbb{R}^d) \).

As a consequence, Proposition 5.9 holds also for \( \mathcal{S}'(\mathbb{R}^d) \) if we in the definition of the multiplication operator assume that \( \psi \in \mathcal{S}(\mathbb{R}^d) \), so that \( \psi \varphi \in \mathcal{S}(\mathbb{R}^d) \) if \( \varphi \in \mathcal{S}(\mathbb{R}^d) \). More generally, one can assume that \( \psi \) and its derivatives grow at most polynomially.

**Proposition 5.16.** Let \( \psi \in C^\infty(\mathbb{R}^d) \) and assume that \( \psi \) and its derivative grow at most polynomially. Each of the operators \( \tau_h, \sigma_\lambda, M_\psi \) and \( \partial^\alpha \) on \( \mathcal{D}'(\mathbb{R}^d) \) extend to continuous operators on \( \mathcal{S}'(\mathbb{R}^d) \).

Alternatively, one can view these operators as restrictions of the corresponding operators on \( \mathcal{D}'(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \).

The main purpose of introducing tempered distributions is to extend the Fourier transform. The fact that \( \mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \) is continuous shows that following definition makes sense.
Definition 5.17. The Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ is the continuous operator $\mathcal{F}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ defined by

$$\langle \mathcal{F}(u), \phi \rangle := \langle u, \mathcal{F}(\phi) \rangle,$$

for $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. The inverse Fourier transform is the continuous operator $\mathcal{F}^{-1}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ defined by

$$\langle \mathcal{F}^{-1}(u), \phi \rangle := \langle u, \mathcal{F}^{-1}(\phi) \rangle.$$

It follows easily from the definition of $\mathcal{F}^{-1}$ on $\mathcal{S}(\mathbb{R}^d)$ that $\mathcal{F}^{-1} = \mathcal{R}\mathcal{F} = \mathcal{F}\mathcal{R}$. As usual, we write $\hat{u}$ for $\mathcal{F}(u)$ also if $u \in \mathcal{S}'(\mathbb{R}^d)$. The following properties are analogous to the ones in Proposition 2.6.

Proposition 5.18. For $u \in \mathcal{S}'(\mathbb{R}^d)$, we have that

$$\begin{align*}
\tau_{-y}u & \overset{\mathcal{F}}{\mapsto} e^{iy\cdot\xi} \hat{u}, \\
e^{iy\cdot\xi}u & \overset{\mathcal{F}}{\mapsto} \tau_y \hat{u}, \\
\sigma_\lambda u & \overset{\mathcal{F}}{\mapsto} |\lambda|^{-d} \sigma_{1/\lambda} \hat{u}, \\
\partial_x^\alpha u & \overset{\mathcal{F}}{\mapsto} (i\xi)^\alpha \hat{u}, \\
x^\alpha u & \overset{\mathcal{F}}{\mapsto} (i\partial_x)^\alpha \hat{u},
\end{align*}$$

where $y \in \mathbb{R}^d$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. The identities follow by combining the definitions of these operators in terms of their transposes with the corresponding properties for Schwartz class functions. We illustrate the method by proving the first identity and leave the others to the reader. Considering $\phi$ as a function of $\xi$ we have that

$$\langle \mathcal{F}(\tau_{-y}u), \phi \rangle = \langle \tau_{-y}u, \phi \rangle = \langle u, \tau_y \phi \rangle = \langle u, \mathcal{F}(e^{iy\cdot\xi} \phi) \rangle = \langle \hat{u}, e^{iy\cdot\xi} \phi \rangle = \langle e^{iy\cdot\xi} \hat{u}, \phi \rangle. \quad \square$$

Theorem 5.19. $\mathcal{F}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is an isomorphism with inverse $\mathcal{F}^{-1}$.

Proof. We simply note that

$$\langle \mathcal{F}^{-1}(\mathcal{F}(u)), \phi \rangle = \langle \mathcal{F}(u), \mathcal{F}^{-1}(\phi) \rangle = \langle u, \mathcal{F}\mathcal{F}^{-1}(\phi) \rangle = \langle u, \phi \rangle.$$

Similarly, $\langle \mathcal{F}^{-1}(u), \phi \rangle = \langle u, \mathcal{F}^{-1}(\phi) \rangle = \langle u, \phi \rangle. \quad \square$

Example 5.20. We have $\langle \hat{\delta}_0, \phi \rangle = \langle \delta_0, \phi \rangle = \phi(0) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \phi(x) \, dx$, so $\hat{\delta}_0 = (2\pi)^{-d/2} \delta_0$. By Fourier’s inversion formula it follows that $\mathcal{F}(1) = \mathcal{R}\mathcal{F}^{-1}(1) = (2\pi)^{d/2} \mathcal{F}(\delta_0) = (2\pi)^{d/2} \delta_0$. 
In order to extend convolutions to $\mathcal{S}'(\mathbb{R}^d)$ we first prove that $\varphi *$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^d)$ for a fixed $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

**Lemma 5.21.** Assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and that $\psi_n \to \psi$ in $\mathcal{S}(\mathbb{R}^d)$. Then $\varphi * \psi_n \to \varphi * \psi$ in $\mathcal{S}(\mathbb{R}^d)$.

**Proof.** From the convolution theorem it follows that $\mathcal{F}(\varphi * \psi_n) = (2\pi)^{d/2} \hat{\varphi} \hat{\psi}_n$. The result follows since $M_\varphi, \mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous operators on $\mathcal{S}(\mathbb{R}^d)$. \(\square\)

The transpose of $\varphi *$ is computed by changing the order of integration in $\langle \varphi * u, \psi \rangle$:

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi(x-y)u(y)\,dy \right) \psi(x)\,dx = \int_{\mathbb{R}^d} u(y) \left( \int_{\mathbb{R}^d} \varphi(x-y)\psi(x)\,dx \right) \,dy
\]

if $u \in \mathcal{S}(\mathbb{R}^d)$, so that $(\varphi *)^T = \mathcal{R}(\varphi) *$, where as usual $\mathcal{R}(\varphi)(x) = \varphi(-x)$.

**Proposition 5.22.** For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the operator $\varphi *$ extends to a continuous operator on $\mathcal{S}'(\mathbb{R}^d)$ given by $\langle \varphi * u, \psi \rangle := \langle u, \mathcal{R}(\varphi) * \psi \rangle$. The extension satisfies

\[
\mathcal{F}(\varphi * u) = (2\pi)^{d/2} \hat{\varphi} \hat{u},
\]

\[
\partial^\alpha (\varphi * u) = (\partial^\alpha \varphi) * u = \varphi * (\partial^\alpha u).
\]

**Proof.** The existence of the extension is a consequence of Proposition 5.15 and Lemma 5.21. The identity $\mathcal{F}(\varphi * u) = (2\pi)^{d/2} \hat{\varphi} \hat{u}$ follows from the calculation

\[
\langle \mathcal{F}(\varphi * u), \psi \rangle = \langle \varphi * u, \psi \rangle
\]

\[
= \langle u, \mathcal{R}(\varphi) * \psi \rangle
\]

\[
= \langle u, \mathcal{F}^{-1}(\mathcal{R}(\varphi) * \psi) \rangle
\]

\[
= \langle \hat{u}, \mathcal{F}^{-1}(\mathcal{R}(\varphi) * \psi) \rangle
\]

\[
= \langle \hat{u}, (2\pi)^{d/2} \hat{\varphi} \psi \rangle
\]

\[
= \langle (2\pi)^{d/2} \hat{\varphi} \hat{u}, \psi \rangle,
\]

in which the convolution theorem for $\mathcal{S}(\mathbb{R}^d)$ and the relation $\mathcal{F}^{-1} = \mathcal{F} \mathcal{R} = \mathcal{R} \mathcal{F}$ have been used. The second identity follows from similar manipulations and its proof is left to the reader. \(\square\)

Since convolution is commutative on $\mathcal{S}(\mathbb{R}^d)$ we shall also write $u * \varphi$ instead of $\varphi * u$. Keeping $u$ fixed and varying $\varphi$ we obtain a continuous linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Note that $u * v$ is in general not well-defined if $u, v \in \mathcal{S}'(\mathbb{R}^d)$.

**Example 5.23.** We have $\langle \delta_0 * \varphi, \psi \rangle = \langle \delta_0, \mathcal{R}(\varphi) * \psi \rangle = \int_{\mathbb{R}^d} \varphi(x) \psi(x)\,dx = \langle \varphi, \psi \rangle$. Thus, $\delta_0 *$ is the identity on $\mathcal{S}(\mathbb{R}^d)$. Similarly, $(\partial^\alpha \delta_0) * = \partial^\alpha$ on $\mathcal{S}(\mathbb{R}^d)$.

**Example 5.24.** If $K_\varepsilon$ is as in Example 5.6, we find that $K_\varepsilon * \varphi \rightharpoonup \delta_0 * \varphi = \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$. This agrees with what we already know about $K_\varepsilon$. 
5.4 Approximation with smooth functions

There is an important question that we have avoided. In Propositions 5.9 and 5.15 we didn’t say anything about the uniqueness of the extensions. The extensions are in fact unique, but to prove this one has to show that \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( \mathcal{D}’(\mathbb{R}^d) \) in the first case and that \( \mathcal{D}(\mathbb{R}^d) \) is dense \( \mathcal{A}(\mathbb{R}^d) \) in the second case. This is done by the usual procedure of mollification and multiplication with smooth ‘cut-off’ functions, but there are some technical details. We will present the proofs for the interested reader. The section may be skipped on a first reading since the results are not used elsewhere in the notes.

We begin by noting that the definition \( (u * \phi, \psi) := \langle u, \mathcal{R}(\phi) * \psi \rangle \) also makes sense when \( u \in \mathcal{D}’(\mathbb{R}^d) \) and \( \phi, \psi \in \mathcal{D}(\mathbb{R}^d) \), since then \( \mathcal{R}(\phi) * \psi \in \mathcal{D}(\mathbb{R}^d) \) (see Section B.5). There is however another natural way of defining convolutions. We show that these two definitions agree.

**Lemma 5.25.** For \( u \in \mathcal{D}’(\mathbb{R}^d) \) and \( \phi \in \mathcal{D}(\mathbb{R}^d) \) we have that \( u * \phi \in C^\infty(\mathbb{R}^d) \) with

\[
(u * \phi)(x) = \langle u, \tau, \mathcal{R}(\phi) \rangle = \langle u, \phi(x - \cdot) \rangle.
\]

**Proof.** Since \( x \mapsto \phi(x - \cdot) \) is continuous from \( \mathbb{R}^d \) to \( \mathcal{D}(\mathbb{R}^d) \) it follows that \( x \mapsto \langle u, \phi(x - \cdot) \rangle \in C(\mathbb{R}^d) \). Moreover, \( \lim_{h \to 0} h^{-1} \langle \phi(x + he_j - \cdot) - \phi(x - \cdot) \rangle = \partial_j \phi(x - \cdot) \) in \( \mathcal{D}(\mathbb{R}^d) \) for each \( x \) and \( j \), and \( x \mapsto \partial_j \phi(x - \cdot) \) is continuous from \( \mathbb{R}^d \) to \( \mathcal{D}(\mathbb{R}^d) \). From this it follows that \( x \mapsto \langle u, \phi(x - \cdot) \rangle \in C^1(\mathbb{R}^d) \). Proceeding by induction, one shows that \( x \mapsto \langle u, \phi(x - \cdot) \rangle \in C^\infty(\mathbb{R}^d) \). The result therefore follows if we prove the equality \( (u * \phi)(x) = \langle u, \phi(x - \cdot) \rangle \). Let \( v(x) = \langle u, \phi(x - \cdot) \rangle \). Using Riemann sums we find that

\[
\langle v, \psi \rangle = \int_{\mathbb{R}^d} v(x) \psi(x) \, dx = \lim_{n \to \infty} \sum_{\alpha \in \mathbb{Z}^d} \psi(\alpha/n) \langle u, \phi(\alpha/n - \cdot) \rangle n^{-d}
\]

\[
= \left\langle u, \lim_{n \to \infty} \sum_{\alpha \in \mathbb{Z}^d} \psi(\alpha/n) \phi(\alpha/n - \cdot) n^{-d} \rightangle
\]

\[
= \langle u, \mathcal{R}(\phi) * \psi \rangle
\]

for \( \psi \in \mathcal{D}(\mathbb{R}^d) \), where we have used the fact that \( \sum_{\alpha \in \mathbb{Z}^d} \psi(\alpha/n) \phi(\alpha/n - \cdot) n^{-d} \to \mathcal{R}(\phi) * \psi \) in \( \mathcal{D}(\mathbb{R}^d) \) (left as an exercise for the reader). The expression on the last line is simply \( \langle u * \phi, \psi \rangle \), so this proves the equality \( u * \phi = v \). \( \square \)

**Theorem 5.26.** \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( \mathcal{D}’(\mathbb{R}^d) \) and \( \mathcal{D}’(\mathbb{R}^d) \).

**Proof.** We first approximate \( u \in \mathcal{D}’(\mathbb{R}^d) \) with the sequence \( \psi(\cdot/n)u \in \mathcal{D}’(\mathbb{R}^d) \), where \( \psi \in \mathcal{D}(\mathbb{R}^d) \) with \( \psi(x) = 1 \) for \( |x| \leq 1 \). It is clear that \( \psi(\cdot/n) \phi \to \phi \) in \( \mathcal{D}(\mathbb{R}^d) \) if \( \phi \in \mathcal{D}(\mathbb{R}^d) \) (they are equal for large \( n \)) and hence \( \psi(\cdot/n)u \to u \) in \( \mathcal{D}’(\mathbb{R}^d) \). Similarly, if \( u \in \mathcal{D}’(\mathbb{R}^d) \), the sequence converges to \( u \) in \( \mathcal{D}’(\mathbb{R}^d) \). We can therefore replace \( u \) by \( \psi u \), for some function \( \psi \in \mathcal{D}(\mathbb{R}^d) \), in what follows. We define \( u_n = (\psi u) * J_{1/n} \), where \( J \) is an even mollifier (see
Section B.5). By the previous lemma it follows that \( u_n \in C^\infty(\mathbb{R}^d) \) as an element of \( \mathcal{D}'(\mathbb{R}^d) \). Moreover, \( u_n \) has compact support since \( u_n(x) = \langle \psi u, J_{1/n}(x-\cdot) \rangle = \langle u, \psi J_{1/n}(x-\cdot) \rangle = 0 \) if \( \text{dist}(x, \text{supp } \psi) \geq 1/n \). Finally,

\[
\langle u_n, \varphi \rangle = \langle \psi u, J_{1/n} * \varphi \rangle = \langle \psi u, \tilde{\psi}(J_{1/n} * \varphi) \rangle,
\]

where \( \tilde{\psi} \in \mathcal{D}(\mathbb{R}^d) \) with \( \tilde{\psi} = 1 \) on \( \text{supp } \psi \). We have that \( \partial^\alpha (J_{1/n} * \varphi) = J_{1/n} * \partial^\alpha \varphi \rightarrow \partial^\alpha \varphi \) uniformly. Hence \( \tilde{\psi}(J_{1/n} * \varphi) \rightarrow \tilde{\psi} \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \) and \( u_n \rightarrow \psi u \) in \( \mathcal{D}(\mathbb{R}^d) \). To prove the same convergence in the topology of \( \mathcal{D}'(\mathbb{R}^d) \), we first note that \( \langle (\psi u) * J_{1/n}, \varphi \rangle \) is given by \( \langle w, \varphi \rangle \), where \( w(x) = \langle u, J_{1/n}(x-\cdot) \rangle \) is in \( \mathcal{D}(\mathbb{R}^d) \), whenever \( \varphi \in \mathcal{D}(\mathbb{R}^d) \). However, since \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( \mathcal{D}(\mathbb{R}^d) \) and \( w \) has compact support, the same is true by an approximation argument whenever \( \varphi \in \mathcal{D}(\mathbb{R}^d) \). We still have that \( \partial^\alpha (J_{1/n} * \varphi) = J_{1/n} * \partial^\alpha \varphi \rightarrow \partial^\alpha \varphi \) uniformly and hence \( \tilde{\psi}(J_{1/n} * \varphi) \rightarrow \tilde{\psi} \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \). It follows that \( u_n \rightarrow \psi u \) in \( \mathcal{D}'(\mathbb{R}^d) \).

**Corollary 5.27.** The extensions in Propositions 5.9 and 5.15 are unique.

**Example 5.28.** Consider the three-dimensional wave equation. For initial data \( u_0 = 0 \) and \( u_1 \in \mathcal{S}(\mathbb{R}^d) \) the solution is expressed on the Fourier side as \( \frac{\sin(t|\xi|)}{|\xi|} \hat{u}_1(\xi) \). Note that \( \frac{\sin(t|\cdot|)}{|\cdot|} \in \mathcal{S}'(\mathbb{R}^d) \). By Proposition 5.22 we therefore have that

\[
u(t,x) = (2\pi)^{-3/2} \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) * u_1
\]

in the sense of distributions. At least if \( u_1 \in \mathcal{D}(\mathbb{R}^d) \), we can make sense of \( (2\pi)^{-3/2} \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) * u_1 \) as

\[
\langle \nu_t, u_1(x-\cdot) \rangle,
\]

where \( \nu_t = (2\pi)^{-3/2} \mathcal{F}^{-1} \left( \frac{\sin(t|\xi|)}{|\xi|} \right) \in \mathcal{S}'(\mathbb{R}^d) \). The distribution \( \nu_t \) is nothing but \( tM_t \) from Section 3.2, that is,

\[
\langle \nu_t, \varphi \rangle = tM_t(\varphi) = \frac{1}{4\pi t} \int_{|x|=|t|} \varphi(x) dS(x).
\]

### 5.5 Sobolev spaces

**Definition 5.29.** The Sobolev space \( H^k(\mathbb{R}^d) \) is the space of \( u \in L^2(\mathbb{R}^d) \) such that \( \partial^\alpha u \in L^2(\mathbb{R}^d) \) for any \( |\alpha| \leq k \), where the derivatives are interpreted in the sense of distributions.

The notation \( W^{k,2}(\mathbb{R}^d) \) is also frequently used instead of \( H^k(\mathbb{R}^d) \). One then regards \( H^k(\mathbb{R}^d) \) as a member of a more general family \( W^{k,p}(\mathbb{R}^d) \) of Sobolev spaces, in which \( L^2(\mathbb{R}^d) \) is replaced by \( L^p(\mathbb{R}^d) \).

Note that

\[
(u,v)_{H^k(\mathbb{R}^d)} := \sum_{|\alpha| \leq k} \int_{\mathbb{R}^d} \partial^\alpha u(x) \partial^\alpha v(x) dx
\]
defines an inner product on $H^k(\mathbb{R}^d)$. Using the completeness of $L^2(\mathbb{R}^d)$ and the definition of distributional derivatives, it is easy to prove that $H^k(\mathbb{R}^d)$ is in fact a Hilbert space with this inner product. Alternatively, one can prove this by first characterizing $H^k(\mathbb{R}^d)$ using the Fourier transform. The following result is a direct consequence of Parseval’s formula.

**Lemma 5.30.** Let $P_k(\xi) = \sum_{|\alpha| \leq k} \xi^{2\alpha}$. Then $u \in L^2(\mathbb{R}^d)$ is in $H^k(\mathbb{R}^d)$ if and only if

$$\int_{\mathbb{R}^d} P_k(\xi) |\hat{u}(\xi)|^2 d\xi < \infty. \quad (5.4)$$

Moreover,

$$\|u\|_{H^k(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} P_k(\xi) |\hat{u}(\xi)|^2 d\xi. \quad (5.5)$$

**Proof.** If $u \in L^2(\mathbb{R}^d)$, then its distributional derivatives satisfy $\mathcal{F}(\partial^\alpha u) = (i\xi)^\alpha \hat{u}$ by Proposition 5.18. If $u \in H^k(\mathbb{R}^d)$, we therefore obtain the equality (5.5) which implies (5.4). On the other hand, if (5.4) holds, we obtain that all the derivatives of order less than or equal to $k$ are in $L^2(\mathbb{R}^d)$. We therefore obtain that $u \in H^k(\mathbb{R}^d)$ and that (5.5) holds.

This result suggests a generalization of the Sobolev spaces. We introduce the notation $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Note that $P_k(\xi) \sim \langle \xi \rangle^{2k}$ in the sense that there exist constants $C_1, C_2 > 0$ such that $C_1 \langle \xi \rangle^{2k} \leq P_k(\xi) \leq C_2 \langle \xi \rangle^{2k}$ for all $\xi \in \mathbb{R}^d$. We can therefore replace $P_k(\xi)$ by $\langle \xi \rangle^{2k}$ in the definition of the norm of $H^k(\mathbb{R}^d)$. This allows us to extend the definition of Sobolev spaces to non-integer indices and to negative indices.

**Definition 5.31.** Let $s \in \mathbb{R}$. $H^s(\mathbb{R}^d)$ is the space of tempered distributions $u$ such that $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d)$ with

$$\|u\|_{H^s(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty. \quad (5.6)$$

These spaces are usually called *fractional Sobolev spaces* (even though $s$ doesn’t have to be rational). From now on we will use (5.6) as the norm on $H^s(\mathbb{R}^d)$, even when $s$ is a non-negative integer. Note that $\mathcal{S}(\mathbb{R}^d) \subset H^s(\mathbb{R}^d) \subset H^t(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ when $s < t$, each inclusion being continuous.

**Theorem 5.32.** $H^s(\mathbb{R}^d)$ is a Hilbert space with inner product

$$(u, v)_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.$$ 

**Proof.** It is clear that $(\cdot, \cdot)_{H^s(\mathbb{R}^d)}$ defines an inner product on $H^s(\mathbb{R}^d)$ with $(u, u)_{H^s(\mathbb{R}^d)} = \|u\|_{H^s(\mathbb{R}^d)}^2$, so we only need to prove the completeness. This follows since the map $T : u \mapsto \langle \xi \rangle^s \hat{u}$ is an isometric isomorphism from $H^s(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

Note that the elements of $H^s(\mathbb{R}^d)$ need not be functions when $s < 0$. 


Example 5.33. The distribution $\delta_0 \in \mathcal{D}'(\mathbb{R}^d)$ satisfies $\delta_0 = (2\pi)^{-d/2}$. Using polar coordinates, we find that
\[ \int_{\mathbb{R}^d} (1 + |\xi|^2)^s d\xi < \infty \]
if and only if $s < -d/2$. Hence $\delta_0 \in H^s(\mathbb{R}^d)$ if and only if $s < -d/2$.

Riesz’ representation theorem (Theorem A.12) implies that the dual space of $H^s(\mathbb{R}^d)$ can be identified with $H^{-s}(\mathbb{R}^d)$ itself. The element $v \in H^s(\mathbb{R}^d)$ is here identified with the bounded linear functional
\[ H^s(\mathbb{R}^d) \ni u \mapsto (u,v)_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{v}(\xi) d\xi. \tag{5.7} \]
The dual can also be identified with $H^{-s}(\mathbb{R}^d)$ using an extension of the usual pairing between $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. Rewriting
\[ \langle w, \varphi \rangle = \int_{\mathbb{R}^d} \hat{w}(\xi) \phi(-\xi) d\xi = \int_{\mathbb{R}^d} \langle \xi \rangle^{-s} \hat{w}(\xi) \langle \xi \rangle^s \phi(-\xi) d\xi, \]
we see that $\langle \cdot, \cdot \rangle$ extends to a bilinear map $H^{-s}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \to \mathbb{C}$. It is also clear that the functional defined by $v \in H^s(\mathbb{R}^d)$ according to (5.7) is identical to the functional $u \mapsto \langle w, u \rangle$ where $w := \Lambda^{2s} \mathcal{G} v$, in which $\mathcal{G} v = \overline{v}$ for $v \in \mathcal{S}(\mathbb{R}^d)$ and is extended by duality to $\mathcal{D}'(\mathbb{R}^d)$, and $\Lambda^{2s} := \mathcal{F}^{-1} M_{\xi}^{2s} \mathcal{F} : H^s(\mathbb{R}^d) \to H^{-s}(\mathbb{R}^d)$. Whenever one identifies the dual with either $H^s(\mathbb{R}^d)$ or $H^{-s}(\mathbb{R}^d)$ one has to keep in mind how the identification is done.

Proposition 5.34. $C_0^\infty(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.

Proof. We can approximate $\langle \xi \rangle^s \hat{u}$ arbitrarily well by $\varphi \in C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$. Hence, $u$ is approximated well by $\mathcal{F}^{-1} (\langle \xi \rangle^{-s} \varphi) \in \mathcal{S}(\mathbb{R}^d)$ in $H^s(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ it follows that $C_0^\infty(\mathbb{R}^d)$ is also dense in $H^s(\mathbb{R}^d)$.

In particular this implies that $H^s(\mathbb{R}^d)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $\| \cdot \|_{H^s(\mathbb{R}^d)}$. This gives an alternative way of introducing Sobolev spaces.

Sobolev spaces are typically used in connection with PDE. One usually proves the existence of a solution in some Sobolev space. It is of interest to know when the solution is sufficiently differentiable in the usual sense to define a classical solution of the PDE.

Definition 5.35. $\mathcal{C}^k(\mathbb{R}^d)$ is the space of $k$ times continuously differentiable functions $u : \mathbb{R}^d \to \mathbb{C}$, such that $\lim_{|x| \to \infty} |\partial^\alpha u(x)| = 0$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, endowed with the norm $\|u\|_{\mathcal{C}^k(\mathbb{R}^d)} := \max_{|\alpha| \leq k} \sup_{\mathbb{R}^d} |\partial^\alpha u(x)|$.

Lemma 5.36. $\mathcal{C}^k(\mathbb{R}^d)$ is a closed subspace of the Banach space $C_b^k(\mathbb{R}^d)$.

Proof. Let $\{u_n\} \subset \mathcal{C}^k(\mathbb{R}^d)$ with $u_n \to u$ in $C_b^k(\mathbb{R}^d)$. For $\varepsilon > 0$ we can find a number $N$ such that $\|u_n - u\|_{\mathcal{C}^k(\mathbb{R}^d)} < \varepsilon$ if $n \geq N$. Choosing $R$ such that $|\partial^\alpha u_N(x)| < \varepsilon$ for $|x| \geq R$ and $|\alpha| \leq k$, we find that $|\partial^\alpha u(x)| \leq |\partial^\alpha u_N(x)| + \|u - u_N\|_{\mathcal{C}^k(\mathbb{R}^d)} < 2\varepsilon$ if $|x| \geq R$. Hence, $u \in \mathcal{C}^k(\mathbb{R}^d)$.\qed
Note that this means that $\mathcal{C}^k(\mathbb{R}^d)$ is itself a Banach space.

**Theorem 5.37 (Sobolev embedding theorem).** Let $k \geq 0$. $H^s(\mathbb{R}^d)$ is continuously embedded in $\mathcal{C}^k(\mathbb{R}^d)$ when $s > k + \frac{d}{2}$, meaning that $H^s(\mathbb{R}^d) \subset \mathcal{C}^k(\mathbb{R}^d)$ and that the inclusion map from $H^s(\mathbb{R}^d)$ to $\mathcal{C}^k(\mathbb{R}^d)$ is continuous.

**Proof.** We begin by proving the inequality

$$\|\varphi\|_{\mathcal{C}^k_b(\mathbb{R}^d)} \leq C \|\varphi\|_{H^s(\mathbb{R}^d)}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ where $C = C(k,s,d)$. For $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ it follows from Hölder’s inequality that

$$|\partial^\alpha \varphi(x)| \leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} |\xi|^k |\hat{\varphi}(\xi)| d\xi$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{|\xi|^k}{(1 + |\xi|^2)^{s/2}} \left((1 + |\xi|^2)^{s/2} |\hat{\varphi}(\xi)|\right) d\xi$$

$$\leq (2\pi)^{-d/2} \left( \int_{\mathbb{R}^d} \frac{|\xi|^{2k}}{(1 + |\xi|^2)^{2s}} d\xi \right)^{1/2} \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi \right)^{1/2}$$

$$= C \|\varphi\|_{H^s(\mathbb{R}^d)},$$

where $C = (2\pi)^{-d/2} \|\xi|^k (\xi)^{-s} \|_{L^2(\mathbb{R}^d)} < \infty$ since $|\xi|^{2k} (\xi)^{-2s} \leq 2|\xi|^{2(k-s)}$ for large $|\xi|$ and $2(k-s) < -d$. Taking the maximum over all $\alpha$ we find that

$$\|\varphi\|_{\mathcal{C}^k_b(\mathbb{R}^d)} \leq C \|\varphi\|_{H^s(\mathbb{R}^d)}.$$  \(5.8\)

Pick $\{\varphi_n\} \subset \mathcal{S}(\mathbb{R}^d)$ with $\varphi_n \to f$ in $H^s(\mathbb{R}^d)$ as $n \to \infty$. The inequality $\|\varphi_n - \varphi_m\|_{\mathcal{C}^k_b(\mathbb{R}^d)} \leq C \|\varphi_n - \varphi_m\|_{H^s(\mathbb{R}^d)}$ follows from (5.8) and implies that $\{\varphi_n\}$ is a Cauchy sequence in $\mathcal{C}^k_b(\mathbb{R}^d)$.

Since $\mathcal{C}^k(\mathbb{R}^d)$ is a Banach space it follows that $\{\varphi_n\}$ converges to some element $g$ in $\mathcal{C}^k(\mathbb{R}^d)$. But then $\{\varphi_n\}$ converges both to $f$ and $g$ in $\mathcal{S}'(\mathbb{R}^d)$ so $f = g$ as distributions. It follows from Theorem 5.3 that $f = g$ a.e. Taking limits, one obtains (5.8) with $f$ substituted for $\varphi$. \(\square\)

**Remark 5.38.** The elements of $H^s(\mathbb{R}^d)$ are actually equivalence classes, not functions. The Sobolev embedding theorem implies that, for $s > k + d/2$, there is a continuous member of each such class. As usual we identify the equivalence class with the continuous representative. Finally, we prove that $H^s(\mathbb{R}^d)$ is an algebra if $s > d/2$.

**Theorem 5.39.** $H^s(\mathbb{R}^d)$ is an algebra if $s > d/2$, meaning that $u, v \in H^s(\mathbb{R}^d) \Rightarrow uv \in H^s(\mathbb{R}^d)$ and that there exists $C = C(s,d) > 0$ such that

$$\|uv\|_{H^s(\mathbb{R}^d)} \leq C \|u\|_{H^s(\mathbb{R}^d)} \|v\|_{H^s(\mathbb{R}^d)}.$$ \(5.9\)

We need the following lemma.
Lemma 5.40. The inequality

\[ \langle \xi + \eta \rangle^s \leq 2^{s-1}(\langle \xi \rangle^s + \langle \eta \rangle^s) \]

holds for \( s \geq 1 \) and \( \xi, \eta \in \mathbb{R}^d \).

Proof. First note that

\[ \langle \xi + \eta \rangle = (1 + |\xi + \eta|^2)^{1/2} \leq (1 + |\xi|^2)^{1/2} + |\eta| \leq \langle \xi \rangle + \langle \eta \rangle \quad (5.10) \]

by the triangle inequality in \( \mathbb{R}^{d+1} \) applied to the vectors \((1, \xi)\) and \((0, \eta)\). The function \( t \mapsto t^s \), \( t \geq 0 \), is convex for \( s \geq 1 \) (the derivative is strictly increasing), and hence

\[ \left( \frac{t_1}{2} + \frac{t_2}{2} \right)^s \leq \frac{t_1^s}{2} + \frac{t_2^s}{2}, \]

so that

\[ (t_1 + t_2)^s \leq 2^{s-1}(t_1^s + t_2^s), \quad (5.11) \]

when \( t_1, t_2 \geq 0 \). The result follows by combining (5.10) and (5.11). \( \square \)

Proof of Theorem 5.39. Note that \( \mathcal{F}(uv) = \mathcal{R}F^{-1}(uv) = (2\pi)^{-d/2} \mathcal{R}(F^{-1}(u) * F^{-1}(v)) = (2\pi)^{-d/2}(\hat{u} * \hat{v}) \) for \( u, v \in \mathcal{S}(\mathbb{R}^d) \) by the convolution theorem. By continuity, we also have that \( \mathcal{F}(uv) = (2\pi)^{-d/2}(\hat{u} * \hat{v}) \) when \( u, v \in L^2(\mathbb{R}^d) \), and hence when \( u, v \in H^s(\mathbb{R}^d), s > d/2 \). We therefore have that

\[ (2\pi)^{d/2}\|uv\|_{H^s(\mathbb{R}^d)} = \|\langle \xi \rangle^s(\hat{u} * \hat{v})\|_{L^2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \|\hat{u} * \hat{v}\|^2 \, d\xi \right)^{1/2}. \]

Note that

\[ \langle \xi \rangle^s(\hat{u} * \hat{v})(\xi) = \int_{\mathbb{R}^d} \langle \xi \rangle^s|\hat{u}(\xi - \eta)||\hat{v}(\eta)| \, d\eta. \]

By the previous lemma,

\[ \langle \xi \rangle^s = \langle \xi - \eta + \eta \rangle^s \leq 2^{s-1}(\langle \xi - \eta \rangle^s + \langle \eta \rangle^s). \]

Hence,

\[ \langle \xi \rangle^s(\hat{u} * \hat{v})(\xi) \leq 2^{s-1} \left( \int_{\mathbb{R}^d} \langle \xi - \eta \rangle^s|\hat{u}(\xi - \eta)||\hat{v}(\eta)| \, d\eta \right)^{1/2} \]

By the special case (B.3) of Young’s inequality, we therefore obtain that

\[ \|\langle \xi \rangle^s(\hat{u} * \hat{v})\|_{L^2(\mathbb{R}^d)} \leq C(\|\langle \xi \rangle^s\hat{u}\|_{L^2(\mathbb{R}^d)} \|\hat{v}\|_{L^1(\mathbb{R}^d)} + \|\hat{u}\|_{L^1(\mathbb{R}^d)} \|\langle \xi \rangle^s\hat{v}\|_{L^2(\mathbb{R}^d)}). \]

Since \( \|\langle \xi \rangle^s\hat{u}\|_{L^2(\mathbb{R}^d)} = \|u\|_{H^s(\mathbb{R}^d)} \) and

\[ \|\hat{u}\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle \xi \rangle^{-s}\langle \xi \rangle^s|\hat{u}(\xi)| \, d\xi \]

\[ \leq \left( \int_{\mathbb{R}^d} \langle \xi \rangle^{-2s} \, d\xi \right)^{1/2} \left( \int_{\mathbb{R}^d} \langle \xi \rangle^{2s}|\hat{u}(\xi)|^2 \, d\xi \right)^{1/2} \]

\[ \leq C\|u\|_{H^s(\mathbb{R}^d)}, \]

where we have used the assumption that \( 2s > d \), the result follows. \( \square \)
Chapter 6

Dispersive equations

In this chapter we will focus on nonlinear dispersive wave equations of the form
\[ u_t - a(D_x)u = F(u, D_x u). \]
Here \( F \) is some nonlinear function of \( u \) and \( D_x u \), while \( a(D_x) \) is a differential operator and \( D_x = \nabla / i \). We assume that \( F \) vanishes to second order at the origin. By the equation being dispersive, we mean that the linear part \( u_t = a(D_x)u \) is dispersive. Recall that in the one-dimensional case this means that the equation has sinusoidal waves
\[ e^{i(kx - \omega(k)t)} = e^{ik(x - c(k)t)}, \]
where \( c(k) = \omega(k)/k \in \mathbb{R} \) is a non-constant function of \( k \). Since the dispersion relation is given by \( \omega(k) = i a(k) \), one should assume that \( a(k) \) is a polynomial of order 2 or higher, which takes purely imaginary values for \( k \in \mathbb{R} \).

1 In the higher dimensional case, the sinusoidal waves take the form
\[ e^{i(k \cdot x - \omega(k)t)} = e^{ik \cdot (x - c(k)t)}, \]
with \( c(k) = \omega(k)k/|k|^2 = i a(k)k/|k|^2 \). In this case it is more difficult to define what is meant by a dispersive equation. Even for the wave equation one obtains that there is some weak dispersion in the sense that \( c(k) = \pm k/|k| \) depends on the direction of \( k \).

At the very least, one should assume that \( a \) takes imaginary values when \( k \in \mathbb{R}^d \).

To illustrate the ideas we will concentrate on two particular examples:

1. the nonlinear Schrödinger (NLS) equation: \( iu_t + \Delta u = \sigma |u|^{p-1} u \), where \( \sigma = \pm 1 \) and \( p \geq 3 \) is an odd integer;

2. the Korteweg-de Vries (KdV) equation: \( u_t + 6uu_x + u_{xxx} = 0 \).

Taking NLS as an example, we first consider the Cauchy problem
\[ \begin{align*}
    iu_t + \Delta u &= f(x,t), \\
    u(x,0) &= u_0(x).
\end{align*} \tag{6.1} 
\]
with \( u(x,0) = u_0(x) \). Using Duhamel’s formula, we express the solution \( u = L(u_0, f) \) in terms of \( u_0 \) and \( f \), where \( L \) is a linear operator. A solution of the NLS equation is then obtained by solving the fixed-point equation \( u = L(u_0, \sigma |u|^{p-1} u) \) using Banach’s fixed point theorem. In order to carry out this procedure, we need to identify a suitable function space in which to study (6.1). We begin with a short introduction to calculus in Banach spaces.

---

1 If \( a(k) = i a_0 + i a_1 k \), with \( a_0, a_1 \in \mathbb{R} \setminus \{0\} \), we obtain the transport equation \( v_t = a_1 \partial_x v \) after replacing \( u \) by \( v := e^{-i a_0 t} u \). This equation is therefore not truly dispersive.

2 The wave equation is second order in time, and doesn’t quite fit into the above framework.
6.1 Calculus in Banach spaces

We restrict ourselves to functions of one real variable with values in a Banach space \((X, \| \cdot \|)\).

**Definition 6.1.** Let \(I\) be an open interval in \(\mathbb{R}\).

1. A function \(u: I \to X\) is said to be continuous at \(t_0 \in I\) if \(\lim_{t \to t_0} u(t) = u(t_0)\). It is said to be continuous if it is continuous at all points of \(I\). \(C(I; X)\) is the vector space of continuous functions \(I \to X\).

2. A function \(u: I \to X\) is said to be differentiable at \(t_0\) with derivative \(u'(t_0) \in X\) if
   \[
   \lim_{h \to 0} \frac{u(t_0 + h) - u(t_0)}{h} = u'(t_0)
   \]
   in \(X\). \(u\) is said to be differentiable if it is differentiable at all points in \(I\). \(C^k(I; X)\) is the vector space of \(k\) times continuously differentiable functions \(u: I \to X\).

If \(I\) is any interval (not necessarily open), we say that \(u \in C^k(I; X)\) if it is \(k\) times continuously differentiable in the interior of \(I\) and all the derivatives extend continuously in \(X\) to the endpoints. From Proposition 6.2 below it then follows that \(u\) has one-sided derivatives at the endpoints.\(^3\)

Assume that \(I\) is a compact interval. One easily verifies that \(t \mapsto \|u(t)\|\) is continuous if \(u \in C(I; X)\). From this it follows that \(\|u(\cdot)\|\) is bounded. \(C(I; X)\) is therefore a normed vector space with
\[
\|u\|_{C(I; X)} = \sup_{t \in I} \|u(t)\|_X.
\]
The proof that \(C(I)\) is a Banach space can be repeated word for word to prove that \(C(I; X)\) is a Banach space. The assumption that \(X\) is a Banach space is essential here.

The mean-value theorem is replaced by the following result.

**Proposition 6.2.** Assume that \(u \in C([a, b]; X)\) and that \(u\) is differentiable on \((a, b)\). Then
\[
\|u(b) - u(a)\| \leq (b - a) \sup_{t \in (a, b)} \|u'(t)\|.
\]

**Proof.** If \(\sup_{t \in (a, b)} \|u'(t)\| = \infty\) there is nothing to prove, so we can assume that it is finite. Let \(C > \sup_{t \in (a, b)} \|u'(t)\|\). Take \(\alpha, \beta \in (a, b)\) with \(\alpha < \beta\) and let \(I = \{t \in [\alpha, \beta]: \|u(t) - u(\alpha)\| \leq C(t - \alpha)\}\). \(I\) is non-empty since it contains \(\alpha\). It is also closed since \(u\) is continuous. Hence it has a largest element \(\beta_*\). If \(\beta_* < \beta\), we obtain that
\[
\|u(\beta_* + h) - u(\alpha)\| = \|u(\beta_* + h) - u(\beta_*) + u(\beta_*) - u(\alpha)\| \\
\leq \|u(\beta_* + h) - u(\beta_*)\| + C(\beta_* - \alpha) \\
\leq Ch + C(\beta_* - \alpha) \\
= C(\beta_* + h - \alpha),
\]
\(^3\)For instance, if \(u'(a) = \lim_{t \to a^+} u'(t)\), it follows from Proposition 6.2 applied to \(v(t) = u(t) - u(a) - u'(a)(t - a)\) that \(\|u(a + h) - u(a) - u'(a)h\| = o(h)\) as \(h \to 0^+\).
if \( h > 0 \) is sufficiently small, since \( \|u(\beta_\ast + h) - u(\beta)\| \leq h\|u'(t)\| + o(h) \) as \( h \to 0 \) according to the definition of differentiability and the triangle inequality. It follows that \( \beta_\ast + h \in I \), and therefore \( \beta_\ast = \beta \). Hence, \( \|u(\beta) - u(\alpha)\| \leq C(\beta - \alpha) \) and since \( u \) is continuous on \([a,b]\) it follows that \( \|u(b) - u(a)\| \leq C(b - a) \). As \( C > \sup_{t \in [a,b]} \|u'(t)\| \) is arbitrary we obtain the desired result. \( \Box \)

We immediately obtain the following corollary.

**Corollary 6.3.** Let \( u \in C^1(I;X) \) with \( u'(t) = 0 \) on \( I \). Then \( u(t) \) is constant on \( I \).

We also need to integrate Banach-space-valued functions. For our purposes it suffices to consider continuous functions defined on a compact interval \( I \), for which the theory is very simple. There is a also a generalization of the Lebesgue integral for Banach-space-valued functions called the *Bochner integral*, but we will not need this here (see e.g. Yosida [29]). A step function \( s: I \to X \) is a piecewise constant function from \( I \) to \( X \), that is

\[
s = \sum_{k=1}^N \chi_{I_k}(t) s_k, \tag{6.2}
\]

where \( \{I_k\}_{k=1}^N \) is a partition of \( I \) into pairwise disjoint intervals \( I = \bigcup_{k=1}^N I_k \) and \( I_j \cap I_k = \emptyset \) if \( j \neq k \). We define

\[
\int_I s(t) \, dt = \sum_{k=1}^N s_k |I_k|,
\]

where \( |I_k| \) is the length of the interval \( I_k \). The representation (6.2) is not unique, but the integral is independent of the particular representation. One easily verifies that the integral is linear and that the triangle inequality \( \|\int_I s(t) \, dt\| \leq \int_I \|s(t)\| \, dt \) holds. The integral of a continuous function \( u \) is defined by approximating \( u \) with step functions. Precisely as for the Riemann integral, one uses the fact that a continuous function on a compact set is uniformly continuous.\(^4\)

**Proposition 6.4.** Assume that \( f \in C(I;X) \). Then there exists a sequence of step functions \( \{s_n\} \) such that \( s_n(t) \to f(t) \) uniformly on \( I \) (\( \sup_{t \in I} \|s_n(t) - f(t)\|_X \to 0 \) as \( n \to \infty \)). The sequence \( \{\int_I s_n(t) \, dt\} \) converges in \( X \) and the limit, denoted \( \int_I f(t) \, dt \), is the same for any sequence of step functions which converges uniformly to \( f \) on \( I \).

**Proof.** The existence of such a sequence is proved by taking \( s_n(t) = \sum_{1 \leq k \leq N_n} f(t_k) \chi_{I_{n,k}} \) with \( t_k \in I_{n,k} \) for any sequence of partitions \( \{I_{n,k}\}_{k=1}^{N_n} \) of \( I \) with \( \max_k |I_{n,k}| \to 0 \) as \( n \to \infty \). Since a continuous function on a compact interval is uniformly continuous, one easily shows that \( s_n(t) \to f(t) \) uniformly on \( I \). But this means that \( \int_I \|s_n(t) - f(t)\| \, dt \to 0 \) and hence \( \int_I \|s_n(t) - s_m(t)\| \, dt \to 0 \) as \( n, m \to \infty \) by the triangle inequality in \( X \). Using the triangle inequality for the integral, we find that \( \{\int_I s_n(t) \, dt\} \) is a Cauchy sequence, and hence that it has a limit in \( X \). The fact that this limit is independent of the particular sequence of step functions follows again by an application of the triangle inequality. \( \Box \)

\(^4\)This can be proved using the Bolzano-Weierstraß theorem, exactly as for real-valued functions.
We define \( \int_a^b u(t) \, dt := -\int_a^b u(t) \, dt \) when \( a < b \). The proof of the following proposition is left to the reader.

**Proposition 6.5.** Let \( X, Y \) be Banach spaces, \( I = [a, b] \) with \( a < b < c \), \( f, g \in C(I; X) \), \( \alpha, \beta \in \mathbb{C} \), and \( T : X \to Y \) a bounded linear operator. Then

1. \( \int_I (\alpha f(t) + \beta g(t)) \, dt = \alpha \int_I f(t) \, dt + \beta \int_I g(t) \, dt; \)
2. \( \int_I f(t) \, dt = \int_I^b f(t) \, dt + \int_I^c f(t) \, dt; \)
3. \( \| \int_I f(t) \, dt \|_X \leq \| \int_I^b f(t) \|_X \, dt; \)
4. \( T(\int_I f(t) \, dt) = \int_I T f(t) \, dt. \)

The usual relation between derivatives and integrals holds.

**Proposition 6.6.** Let \( u \in C([a, b]; X) \). Then \( \int_a^t u(\tau) \, d\tau \) is differentiable on \((a, b)\) with

\[
\frac{d}{dt} \int_a^t u(\tau) \, d\tau = u(t).
\]

**Proof.** We simply note that

\[
\frac{1}{h} \left( \int_a^{t+h} u(\tau) \, d\tau - \int_a^t u(\tau) \, d\tau \right) - u(t) = \frac{1}{h} \int_t^{t+h} (u(\tau) - u(t)) \, d\tau.
\]

If \( \varepsilon > 0 \) is given, we can choose \( \delta > 0 \) such that \( \|u(\tau) - u(t)\| \leq \varepsilon \) if \( |\tau - t| \leq \delta \). We then obtain that

\[
\left\| \frac{1}{h} \int_t^{t+h} (u(\tau) - u(t)) \, d\tau \right\| \leq \frac{1}{h} \int_t^{t+h} \|u(\tau) - u(t)\| \, d\tau \leq \varepsilon
\]

if \( |h| \leq \delta \). The result follows.

Combining this with Corollary 6.3 we obtain the following result.

**Corollary 6.7.** Let \( f \in C(I; X) \) and \( t_0 \in I \). Then \( u \in C^1(I; X) \) with \( u'(t) = f(t) \) on \( I \) if and only if \( u(t) = \int_{t_0}^t f(\tau) \, d\tau + u_0 \) for some \( u_0 \in X \).

As an application of the above, we record the following existence and uniqueness result for ordinary differential equations in Banach spaces.

**Theorem 6.8.** Let \( I = [0, b] \) for some \( b > 0 \). Assume that \( f \in C(X \times I; X) \) and that for every \( R > 0 \) there exists a constant \( L \geq 0 \) such that \( \|f(x, t) - f(y, t)\| \leq L \|x - y\| \) for \( t \in I \) and \( x, y \in B_R(0) \subset X \). There exists a \( T > 0 \) such that the Cauchy problem

\[
u'(t) = f(u(t), t), \quad u(0) = u_0,
\]

has a unique solution \( u \in C^1([0, T]; X) \).

The proof of the theorem is precisely the same as for ODE in \( \mathbb{R}^n \). The reader is invited to fill in the details. The theorem has very limited applicability to PDE, however. Consider e.g., the nonlinear Schrödinger equation \( u_t = i\Delta u - i\sigma |u|^{p-1}u \). Taking \( X = H^s(\mathbb{R}^d) \) as a Sobolev space, one finds that \( f(u, t) = i\Delta u - i\sigma |u|^{p-1}u \) does not define a continuous function from \( X \times I \) to \( X \). As we will see, a modification of the method is on the other hand useful for solving the NLS equation.
6.2 NLS

The nonlinear Schrödinger equation shows up in many different models for physical phenomena. The main reason is that it appears as a universal model when one looks for solutions of wave packet type. To explain this we concentrate on one example, namely the KdV equation. Several other interesting examples are provided by Sulem & Sulem [26].

Example 6.9. We look for an approximate solution of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ of the form

$$u(x,t) = \varepsilon A_1(X,T)e^{i(kx - \omega t)} + \varepsilon A_1(X,T)e^{-i(kx - \omega t)} + \varepsilon^2 A_2(X,T)e^{2i(kx - \omega t)} + \varepsilon^2 A_0(X,T),$$

where $T = \varepsilon^2 t$, $X = \varepsilon(x + 3k^2 t)$ and $\omega = -k^3$. The functions $A_j$ are complex-valued, and the bars indicate complex conjugation. Note that $u$ is therefore real-valued. The parameter $\varepsilon$ is assumed to be small, so that $u$ is given to leading order by the wave packet $\varepsilon A_1(X,T)e^{i(kx - \omega t)} + \varepsilon A_1(X,T)e^{-i(kx - \omega t)}$. Note that this can be rewritten in the form $2\varepsilon R(X,T)\cos(kx - \omega t + \theta(X,T))$, where $R = |A_1|$ and $\theta = \text{arg} A_1$. Since the variable $T$ changes at a slower rate than $X$, which in turn changes at a slower rate than $kx - \omega t$, we roughly have an envelope $2\varepsilon R(X,T)$ travelling with group velocity $c_g = -3k^2$ and a carrier wave $\cos(kx - \omega t + \theta(X,T))$ with wave number $k$ and phase velocity $c = -k^2$. To see how the NLS equation arises, we substitute the above Ansatz into the equation. Equating terms of the same order in $\varepsilon$, we arrive at the following equations:

$$k^2 A_0 = -2|A_1|^2,$$

$$\partial_T A_1 = -3ik \partial_X^2 A_1 - 6ik(A_2 A_1 + A_1 A_0),$$

$$k^2 A_2 = A_1^2,$$

where two integration constants have been set to 0. When these equations are satisfied, we formally find that $u_t + 6uu_x + u_{xxx} = O(\varepsilon^4)$. Combining the three equations, we obtain the cubic one-dimensional NLS equation

$$i\partial_T A_1 - 3k \partial_X^2 A_1 + \frac{6}{k} A_1 |A_1|^2 = 0$$

(the minus sign in front of the second order derivative can be removed by changing $T$ to $-T$). For a rigorous justification of this approximation we refer to Schneider [23].

6.2.1 Linear theory

We first consider the Cauchy problem for the inhomogeneous Schrödinger equation

$$\begin{cases}
    iu_t + \Delta u = f, & t > 0, \\
    u = u_0, & t = 0,
\end{cases}$$

(6.3)
where $f$ and $u_0$ are given functions. If $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $f \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d))$ there is a unique solution $u \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d))$ given by

$$u(t) = S(t)u_0 - i \int_0^t S(t-\tau)f(\tau)\,d\tau$$

(6.4)

according to Theorem 3.36. Here $S(t)\phi = \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{\phi})$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$. Note that we can write the solution of (6.3) with $f \equiv 0$ as

$$u(x,t) = \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{u}_0) = K_t * u_0,$$

where $K_t = (2\pi)^{-d/2}\mathcal{F}^{-1}(e^{-it|\xi|^2}) \in \mathcal{S}'(\mathbb{R}^d)$.

**Lemma 6.10.** Let $z \in \mathbb{C}$ with $\text{Re } z > 0$. Then

$$\mathcal{F}(e^{-\frac{|z|^2}{2}})(\xi) = \frac{1}{z^{d/2}}e^{-\frac{|\xi|^2}{z}}$$

(6.5)

where $z^{d/2}$ is defined using the principal branch of the square root if $d$ is odd.

**Proof.** The assumption $\text{Re } z > 0$ implies that $e^{-\frac{|z|^2}{2}} \in \mathcal{S}(\mathbb{R}^d)$ so that $\mathcal{F}(e^{-\frac{|z|^2}{2}}) \in \mathcal{S}(\mathbb{R}^d)$. Moreover, the rapid decay implies that

$$\mathcal{F}(e^{-\frac{|z|^2}{2}})(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2} - iz \cdot \xi} \,dx$$

is complex differentiable as function of $z$ for each fixed $\xi \in \mathbb{R}^d$ (the order of complex differentiation and integration can be interchanged). We therefore have that $z \mapsto \mathcal{F}(e^{-\frac{|z|^2}{2}})$ is analytic in the right half-plane $\text{Re } z > 0$. On the other hand, the function $z^{-d/2}e^{-\frac{|\xi|^2}{2z}}$ is also analytic in the right half-plane and the two functions agree when $\text{Im } z = 0$ (Example 2.7). The unique continuation principle for analytic functions therefore proves that they are equal in the whole right half-plane. \(\square\)

**Proposition 6.11.** The formula (6.5) extends to $z \in \mathbb{C}$ with $\text{Re } z = 0$ and $z \neq 0$.

**Proof.** Let $z$ be as in the statement of the proposition. By the previous lemma we have that

$$\langle e^{-\frac{|z+\varepsilon|^2}{2}}, \phi \rangle = \langle \mathcal{F}(e^{-\frac{|z+\varepsilon|^2}{2}}), \varphi \rangle = \langle (z+\varepsilon)^{-d/2}e^{-\frac{|\xi|^2}{2(z+\varepsilon)}}, \varphi \rangle$$

for $\varepsilon > 0$. Letting $\varepsilon \to 0^+$ we find that

$$\langle e^{-\frac{|z|^2}{2}}, \phi \rangle = \langle z^{-d/2}e^{-\frac{|\xi|^2}{2z}}, \varphi \rangle.$$ \(\square\)

**Theorem 6.12.** For $u_0 \in \mathcal{S}(\mathbb{R}^d)$ the solution of (6.3) can be written

$$(S(t)u_0)(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{|y-x|^2}{4t}} u_0(y) \,dy$$

(6.6)

for $t \neq 0$. 

Proof. From the previous proposition we find that

\[(2\pi)^{d/2} K_{i}(x) = \mathcal{F}^{-1}(e^{-i|\xi|^2})(x) = \mathcal{F}(e^{-\frac{2|m|^2}{t}})(-x) = \frac{1}{(2\pi t)^{d/2}} e^{\frac{i|x|^2}{4t}}.\]

We therefore find that

\[S(t)u_0 = \left(\frac{1}{2\pi it}\right)^{d/2} e^{-\frac{i|x|^2}{4t}} * u_0\]

in the sense of distributions, meaning that

\[\langle S(t)u_0, \varphi \rangle = \left\langle \left(\frac{1}{2\pi it}\right)^{d/2} e^{-\frac{i|x|^2}{4t}}, \mathcal{F}(u_0) * \varphi \right\rangle,
\]

for all \(\varphi \in \mathcal{S}(\mathbb{R}^d)\). On the other hand, the convolution integral in (6.6) converges and defines a continuous, bounded function of \(x\) for each \(t \neq 0\). Since \(u_0(x-y)\varphi(x)\) is in \(L^1(\mathbb{R}^d \times \mathbb{R}^d)\), we can change the order of integration to show that

\[
\left\langle \frac{1}{(2\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{i|x|^2}{4t}} u_0(y) \, dy, \varphi \right\rangle = \left\langle \frac{1}{(2\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{i|y|^2}{4t}} u_0(\cdot - y) \, dy, \varphi \right\rangle \\
= \int_{\mathbb{R}^d} \left( \frac{1}{(2\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{i|y|^2}{4t}} u_0(x-y) \, dy \right) \varphi(x) \, dx \\
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} u_0(x-y) \varphi(x) \, dx \right) \frac{1}{(2\pi it)^{d/2}} e^{-\frac{i|x|^2}{4t}} \, dy \\
= \left\langle \left(\frac{1}{(2\pi it)^{d/2}} e^{-\frac{i|x|^2}{4t}}, \mathcal{F}(u_0) * \varphi \right) \right\rangle,
\]

so that the identity (6.7) actually holds pointwise. This proves (6.6). \(\square\)

**Proposition 6.13.** Let \(u_0 \in \mathcal{S}(\mathbb{R}^d)\). Then

\[\|S(t)u_0\|_{L^p(\mathbb{R}^d)} \leq \left(\frac{2\pi |t|}{4\pi^p}ight)^{-\frac{d}{2}} \|u_0\|_{L^q(\mathbb{R}^d)}\]

for \(p \in [2, \infty]\), where \(1/p + 1/q = 1\).

**Proof.** For \(p = 2\) we have that \(q = 2\) and the result follows from Parseval’s formula since \(e^{-i|\xi|^2}\) has modulus 1. For \(p = \infty\), the result follows from the representation formula (6.6). For \(p \in (2, \infty)\) the result follows from the Riesz-Thorin interpolation theorem (Theorem B.28). \(\square\)

Recall that solutions of the heat equation also have decay properties (see Section 3.1.4). The decay for the Schrödinger equation is of a different nature. This can be seen by looking at an individual Fourier mode \(\hat{u}(t)e^{i\xi \cdot x}\). For the heat equation the solution has the form \(\hat{u}(0)e^{ix \cdot \xi - |\xi|^2 t}\), while the corresponding solution of the Schrödinger equation is \(\hat{u}(0)e^{ix \cdot \xi - |\xi|^2 t}\). Thus, the individual Fourier modes decay for the heat equation, but not for Schrödinger’s equation. In the latter case, the decay is instead related to the fact that different modes travel with
different speeds, that is, dispersion. An initially localized solution consists of a continuous linear combination of different Fourier modes. The dispersion of these modes causes the decay.

The solution formula (6.4) can be extended to \( u_0 \in S' (\mathbb{R}^d) \) and \( f \in C([0, T); S' (\mathbb{R}^d)) \) if one interprets the integral in a suitable weak sense. Here we shall content ourselves with data in Sobolev spaces. We assume that \( u_0 \in H^s (\mathbb{R}^d) \) and that \( f \in C([0, T); H^s (\mathbb{R}^d)) \) for some \( T > 0 \). We begin by recording some properties of the propagator \( S(t) \).

**Proposition 6.14.** The family of operators \( \{ S(t) \}_{t \in \mathbb{R}} \) defined for \( u \in H^s (\mathbb{R}^d) \) by \( S(t) u = \mathcal{F}^{-1} (e^{-i|\xi|^2 t} \hat{u}) \) satisfies the following properties:

1. \( S(t) \) is a unitary operator on \( H^s (\mathbb{R}^d) \);
2. \( S(t) \) commutes with partial derivatives, meaning that \( \partial_j (S(t) u) = S(t) (\partial_j u) \);
3. \( S(0) = \text{Id} \);
4. \( S(t_1 + t_2) = S(t_1) S(t_2) \);
5. the map \( t \mapsto S(t) \) is strongly continuous, meaning that \( t \mapsto S(t) u \) is in \( C(\mathbb{R}; H^s (\mathbb{R}^d)) \) for every \( u \in H^s (\mathbb{R}^d) \);
6. for \( u \in H^s (\mathbb{R}^d) \) we have that \( t \mapsto S(t) u \) is in \( C^1 (\mathbb{R}; H^{s-2} (\mathbb{R}^d)) \) with
   \[
   \frac{d}{dt} (S(t) u) = i\Delta S(t) u.
   \]

**Proof.** We work on the Fourier transform side, where \( S(t) \) corresponds to multiplication with \( e^{-i|\xi|^2 t} \). Properties (1)–(4) now follow from the facts that \( e^{-i|\xi|^2 t} \) has modulus 1 and satisfies \( e^{-i|\xi|^2 (t + t_2)} = e^{-i|\xi|^2 t_1} e^{-i|\xi|^2 t_2} \), while \( \partial_j \) corresponds on the Fourier side to multiplication with \( i\xi_j \). Property (5) is proved by noting that \( t \mapsto e^{-i|\xi|^2 t} \) is continuous for fixed \( \xi \) and applying the dominated convergence theorem. Property (6) is also proved using the dominated convergence theorem and the facts that \( \partial_j e^{-i|\xi|^2 t} = -i|\xi|^2 e^{-i|\xi|^2 t} \) and \( \mathcal{F} (\Delta u) = -|\xi|^2 \hat{u} \).

Properties (3)–(5) are the defining properties of a strongly continuous group of operators on a normed vector space (it is also assumed that the operators are bounded).

**Proposition 6.15.** Assume that \( f \in C([0, T); H^s (\mathbb{R}^d)) \) and that \( u_0 \in H^s (\mathbb{R}^d) \). There exists a unique solution \( u \in C([0, T); H^s (\mathbb{R}^d)) \cap C^1 ([0, T); H^{s-2} (\mathbb{R}^d)) \) of (6.3) given by (6.4).

**Proof.** Writing (6.4) in the form \( u(t) = S(t) u_0 - i S(t) \int_0^t S(-\tau) f(\tau) d\tau \) and using Proposition 6.14 it is easy to verify that \( u \) defines a solution in \( C([0, T); H^s (\mathbb{R}^d)) \cap C^1 ([0, T); H^{s-2} (\mathbb{R}^d)) \) of (6.3). On the other hand, if \( v \) is another solution, then \( w(t) := S(-t) (u(t) - v(t)) \) satisfies the differential equation \( w'(t) = 0 \) with \( w(0) = 0 \), so \( w(t) = 0 \) for all \( t \) by Corollary 6.3. Hence \( u(t) \equiv v(t) \).
6.2.2 Nonlinear theory

We now return to the nonlinear Schrödinger equation

\[iu_t + \Delta u = \sigma |u|^{p-1}u\]  \hspace{1cm} (6.8)

where \(\sigma = \pm 1\) and \(p \geq 3\) is an odd integer. The two different cases, \(\sigma = -1\) and \(\sigma = 1\), are referred to as focusing and defocusing, respectively. The distinction between these two cases will become clear later on.

Local well-posedness of the Cauchy problem

The first goal is to prove the local well-posedness of the Cauchy problem

\[
\begin{cases}
  iu_t + \Delta u = \sigma |u|^{p-1}u, & t > 0, \\
  u = u_0, & t = 0,
\end{cases}
\]

(6.9)

We assume that \(u_0 \in H^s(\mathbb{R}^d)\) for some \(s > d/2\). We are looking for a solution \(u \in C([0,T);H^s(\mathbb{R}^d)) \cap C^1([0,T);H^{s-2}(\mathbb{R}^d))\). Repeated use of Theorem 5.39 shows that \(|u|^{p-1}u = u^{(p+1)/2} \bar{u}^{(p-1)/2} \in H^s(\mathbb{R}^d)\) if \(u \in H^s(\mathbb{R}^d)\), where we also used the fact that \(\bar{u} \in H^s(\mathbb{R}^d)\).

Lemma 6.16. Let \(s > d/2\) and \(f(z) = \sigma |z|^{p-1}z\). The map \(u \mapsto f(u)\) is a locally Lipschitz continuous on \(H^s(\mathbb{R}^d)\), meaning that for each \(R > 0\) there exists a constant \(C\) such that \(\|f(u) - f(v)\|_{H^s(\mathbb{R}^d)} \leq C\|u - v\|_{H^s(\mathbb{R}^d)}\), whenever \(u, v \in B_R(0) \subset H^s(\mathbb{R}^d)\).

Proof. Since \(f(z)\) is a polynomial in \(z\) and \(\bar{z}\), we can find two polynomials \(P_1(z_1, z_2, z_3, z_4)\) and \(P_2(z_1, z_2, z_3, z_4)\), such that

\[|u|^{p-1}u - |v|^{p-1}v = P_1(u, v, \bar{u}, \bar{v})(u - v) + P_2(u, v, \bar{u}, \bar{v})(\bar{u} - \bar{v}).\]

This proves that

\[\|f(u) - f(v)\|_{H^s(\mathbb{R}^d)} \leq C(\|P_1(u, v, \bar{u}, \bar{v})\|_{H^s(\mathbb{R}^d)} + \|P_2(u, v, \bar{u}, \bar{v})\|_{H^s(\mathbb{R}^d)})\|u - v\|_{H^s(\mathbb{R}^d)}.\]

Using Theorem 5.39 we find that \(\|P_j(u, v, \bar{u}, \bar{v})\|_{H^s(\mathbb{R}^d)}\), \(j = 1, 2\), is bounded by a constant (depending on \(R\)) when \(\|u\|_{H^s(\mathbb{R}^d)}, \|v\|_{H^s(\mathbb{R}^d)} \leq R\).

It follows that \(t \mapsto f(u(t))\) is in \(C([0,T);H^s(\mathbb{R}^d))\) if \(u \in C([0,T);H^s(\mathbb{R}^d))\). We cannot directly view (6.8) as an ODE in a \(H^s(\mathbb{R}^d)\) since \(\Delta\) is not a bounded operator on \(H^s(\mathbb{R}^d)\). Proposition 6.15 shows that the problem is equivalent to

\[u(t) = S(t)u_0 - i \int_0^t S(t - \tau)f(u(\tau))\,d\tau,\]

(6.10)

where \(S(t)\) is the propagator for the linear Schrödinger equation.

Definition 6.17. A function \(u \in C([0,T);H^s(\mathbb{R}^d))\) is called a strong solution of (6.9) if it satisfies (6.10).
It follows from Proposition 6.15 that a strong solution $u$ is also in $C^1([0, T); H^{s-2}(\mathbb{R}^d))$ and solves (6.8) for each $t \in (0, T)$ with $u(x, 0) = u_0(x)$. Just as for finite-dimensional ODE it is simpler to work with the integral equation (6.10) than the original problem. Note however that (6.10) differs from the usual integral equation

$$u(t) = u_0 + i \int_0^t (\Delta u(\tau) - f(u(\tau))) \, d\tau,$$

which we would obtain if we were to treat the problem as an infinite-dimensional ODE. The advantage with (6.10) is that the right hand side defines a continuous map on $C([0, T); H^s(\mathbb{R}^d))$ for $s > d/2$; this is not the case for (6.11).

**Theorem 6.18.** Let $s > d/2$. For any $u_0 \in H^s(\mathbb{R}^d)$, there exists a time $T_1 = T_1(\|u_0\|_{H^s(\mathbb{R}^d)}) > 0$ such that the Cauchy problem (6.9) has a unique strong solution $u \in C([0, T_1]; H^s(\mathbb{R}^d))$.

**Proof.** The strategy is to prove that the nonlinear operator defined by the right hand side of (6.10) is a contraction if the time interval is sufficiently small. We choose $X$ as the Banach space

$$X = C([0, T_1]; H^s(\mathbb{R}^d))$$

where $T_1$ is to be determined and let

$$K = \{u \in X : \|u\|_X \leq 2R\},$$

where $\|u_0\|_{H^s(\mathbb{R}^d)} < R$. $K$ is a closed subset of the Banach space $X$ and hence a complete metric space. We need to prove that $F(K) \subseteq K$ and that $\|F(u) - F(v)\|_X \leq \theta \|u - v\|_X$ for some $\theta \in (0, 1)$, where $F(u)$ denote the right hand side of (6.10). Note that $\|S(\cdot)u_0\|_X = \|u_0\|_{H^s(\mathbb{R}^d)} < R$. Using Lemma 6.16 we find that there exists a constant $C = C(R) > 0$ such that

$$\|f(u) - f(v)\|_{H^s(\mathbb{R}^d)} \leq C\|u - v\|_{H^s(\mathbb{R}^d)}$$

for $u, v \in H^s(\mathbb{R}^d)$ with $\|u\|_{H^s(\mathbb{R}^d)}, \|v\|_{H^s(\mathbb{R}^d)} \leq 2R$. Therefore

$$\left\| \int_0^t S(t - \tau) (f(u(\tau)) - f(v(\tau))) \, d\tau \right\|_{H^s(\mathbb{R}^d)} \leq CT\|u - v\|_{H^s(\mathbb{R}^d)}.$$

Taking $T = 1/(2C)$, the contraction property follows with $\theta = 1/2$. We thus have $\|F(u) - F(v)\|_X \leq \frac{1}{2}\|u - v\|_X$. Taking $v = 0$, it follows that $\|F(u)\|_X \leq \|F(0)\|_X + \frac{1}{2}\|u\|_X$, $\|u_0\|_{H^s(\mathbb{R}^d)} + \frac{1}{2}\|u\|_X \leq 2R$, so that $F(K) \subseteq K$. From Banach’s fixed point theorem (Theorem A.15) we therefore find a unique strong solution $u \in K$. It remains to prove that the solution is unique in $X$. This follows if we can prove that any solution in $X$ is in fact in $K$. To see this, we estimate $\|F(u)\|_{H^s(\mathbb{R}^d)} = \|f(u) - f(0)\|_{H^s(\mathbb{R}^d)} \leq C\|u - 0\|_{H^s(\mathbb{R}^d)} = C\|u\|_{H^s(\mathbb{R}^d)}$ and

$$\|F(u)(t)\|_{H^s(\mathbb{R}^d)} \leq \|u_0\|_{H^s(\mathbb{R}^d)} + \int_0^t \|f(u(\tau))\|_{H^s(\mathbb{R}^d)} \, d\tau$$

$$\leq R + 2RCt = (1 + t/T)R,$$
as long as $u(t)$ remains in the ball of radius $2R$ around the origin in $H^s(\mathbb{R}^d)$. But this implies that $\|u(t)\|_{H^s(\mathbb{R}^d)} = \|F(u(t))\|_{H^s(\mathbb{R}^d)} \leq 2R$ when $0 \leq t \leq T$, since $\|u(t)\|_{H^s(\mathbb{R}^d)}$ is continuous with $\|u(0)\|_{H^s(\mathbb{R}^d)} < R$. In other words $u \in K$. The proof is complete.

Note that $T_1$ only depends on the Lipschitz constant $C$ in Lemma 6.16. Since the optimal constant is an increasing function of $R$, we can assume that $T_1$ is a decreasing function of $\|u_0\|_{H^s(\mathbb{R}^d)}$.

There is nothing special about the point $t = 0$. If the initial condition is prescribed at some point $t_0$ we instead obtain the integral equation

$$u(t) = S(t - t_0)u_0 - i \int_0^t S(t - \tau)f(u(\tau)) \, d\tau.$$ 

Suppose that $u$ is a strong solution on $[0, T_1]$ and that $v$ is a strong solution on $[T_1, T_2]$ with initial data $v_0 = u(T_1)$ at $t = T_1$. Set $w(t) = u(t)$ on $[0, T_1]$ and $w(t) = v(t)$ on $[T_1, T_2]$, so that $w \in C([0, T_2]; H^s(\mathbb{R}^d))$. Clearly $w$ is a strong solution on $[0, T_1]$. It is also a strong solution on $[0, T_2]$ since

$$S(t - T_1)u(T_1) - i \int_0^{T_1} S(t - \tau)f(v(\tau)) \, d\tau$$

$$= S(t - T_1)S(T_1)u_0 - i \int_0^{T_1} S(t - T_1)S(T_1 - \tau)f(u(\tau)) \, d\tau - i \int_0^{T_1} S(t - \tau)f(v(\tau)) \, d\tau$$

$$= S(t)u_0 - i \int_0^t S(t - \tau)f(w(\tau)) \, d\tau,$$

for $t \in [T_1, T_2]$ by Proposition 6.14.

Suppose that $u, v \in C([0, T]; H^s(\mathbb{R}^d))$ are two strong solutions for some $T > T_1$. By the above theorem, the solutions coincide on $[0, T_1]$. Let $T_* \leq T$ be the maximal time such that $u(t) = v(t)$ on $[0, T_*)$. By continuity it follows that they also coincide at $T_*$. But then $T_* = T$, for otherwise we could solve the equation with initial data $u(T_*) = v(T_*)$ at $t = T_*$. The uniqueness assertion of Theorem 6.18 proves that $u(t) = v(t)$ in a neighbourhood of $T_*$, giving a contradiction. This proves that we have uniqueness on an arbitrary time interval. It also implies that we can extend the local solution to some maximal interval in a unique way. Suppose that $T$ is the maximal existence time, meaning that there is a solution on $[0, T)$, which can not be extended beyond $T$. If $T < \infty$ we have that $\lim_{t \to T} \|u(t)\|_{H^s(\mathbb{R}^d)} = \infty$. Indeed, otherwise there exists an $R > 0$ and a sequence $t_j \to T$ such that $\|u(t_j)\|_{H^s(\mathbb{R}^d)} \leq R$. If we solve the equation with initial data $u(t_j)$ at time $t_j \in [0, T)$, the solution will exist at least on a time interval $[t_j, t_j + T_1]$ where $T_1 = T_1(\|u(t_j)\|_{H^s(\mathbb{R}^d)}) \geq T_1(R) > 0$ is the existence time provided by Theorem 6.18. Choosing $j$ sufficiently large we obtain a contradiction.

We can also prove that the solution depends continuously on the initial data. Indeed, if $u_0, v_0 \in H^s(\mathbb{R}^d)$, we obtain that

$$\|u(t) - v(t)\|_{H^s(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{H^s(\mathbb{R}^d)} + \int_0^t C\|u(\tau) - v(\tau)\|_{H^s(\mathbb{R}^d)} \, d\tau,$$
if $u, v \in C([0,T];H^s(\mathbb{R}^d))$ are the corresponding solutions ($T$ is chosen strictly less than the maximal existence time for each solution), $\|u(t)\|_{H^s(\mathbb{R}^d)}, \|v(t)\|_{H^s(\mathbb{R}^d)} \leq R$ on $[0,T]$ and $C$ is a Lipschitz constant for $f(u)$ on $\{u \in H^s(\mathbb{R}^d): \|u\|_{H^s(\mathbb{R}^d)} \leq R\}$. It follows from Grönwall’s inequality that

$$\|u(t) - v(t)\|_{H^s(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{H^s(\mathbb{R}^d)} e^{Ct}, \quad 0 \leq t \leq T.$$ 

In conclusion, we have the following result.

**Theorem 6.19.** There exists a maximal time $T > 0$ such that the Cauchy problem (6.9) has a strong solution in $C([0,T];H^s(\mathbb{R}^d))$. The solution is unique in this class and if $T < \infty$ then $\lim_{t \to T} \|u(t)\|_{H^s(\mathbb{R}^d)} = \infty$. The solution depends continuously on the initial data in the sense described above.

**Global existence theory and conservation laws**

We now turn to the global existence theory. For simplicity we will assume that $d = 1$ from now on. One of the main ways of proving global existence is to use conserved quantities. Define the functionals

$$M(u) = \frac{1}{2} \int_{\mathbb{R}} |u|^2 \, dx$$

and

$$E(u) = \int_{\mathbb{R}} \left( \frac{1}{2} |u_t|^2 + \frac{\sigma}{p+1} |u|^{p+1} \right) \, dx.$$ 

One can formally show that $M(u(t))$ and $E(u(t))$ are independent of $t$ by differentiating with respect to $t$, using the equation and integrating by parts (see Theorem 6.23 below). The parameters $p, \sigma$ don’t play a big role in the local well-posedness theory. They are extremely important for the global theory, however. This can be realized by inspecting the conserved functional $E$ above. If $\sigma = 1$, we can estimate $\|u\|_{H^1(\mathbb{R})} \leq 2(M(u) + E(u))$. Thus we can prevent the blow-up of the $H^1$ norm and obtain a global existence result for $H^1$ solutions. In order to do this, we must first rigorously prove that $E(u)$ and $M(u)$ are conserved for such solutions.

If $u_0 \in H^s(\mathbb{R})$ with $s > 1/2$, Theorem 6.19 guarantees the existence of a maximal solution $u \in C([0,T_\ast];H^s(\mathbb{R}^d))$. Let $1/2 < r < s$. Since $H^s(\mathbb{R}) \subset H^r(\mathbb{R})$, we also obtain the existence of a solution in $C([0,T_r];H^r(\mathbb{R}^d))$, where a priori we could have $T_\ast \neq T_r$. By the uniqueness assertion of the proposition, the two solutions must agree on their common domain of definition and clearly $T_\ast \leq T_r$. One could however imagine that the solution remains in $H^s(\mathbb{R})$ only up to the time $T_\ast < T_r$ and that it thereafter looses regularity (consider e.g. the weak solutions discussed in Chapter 4). We shall now prove that this cannot happen.

**Lemma 6.20.** Let $1/2 < r < s$. Then

$$\|uv\|_{H^r(\mathbb{R})} \leq C(\|u\|_{H^s(\mathbb{R})}\|v\|_{H^r(\mathbb{R})} + \|u\|_{H^s(\mathbb{R})}\|v\|_{H^r(\mathbb{R})}).$$

---

$^5$ They do play a role if one considers initial data with lower regularity.
Proof. From the proof of Theorem 5.39, we find that
\[
\|uv\|_{H^r(\mathbb{R})} \leq C(\|u\|_{H^r(\mathbb{R})}\|v\|_{L^1(\mathbb{R})} + \|\dot{u}\|_{L^1(\mathbb{R})}\|v\|_{H^r(\mathbb{R})}).
\]
On the other hand \(\|\dot{u}\|_{L^1(\mathbb{R})} \leq C\|u\|_{H^r(\mathbb{R})}\) by the same proof. \(\square\)

Remark 6.21. One actually has the stronger result
\[
\|uv\|_{H^r(\mathbb{R}^d)} \leq C(\|u\|_{H^r(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{L^\infty(\mathbb{R}^d)}\|v\|_{H^r(\mathbb{R}^d)})
\]
for \(s \geq 0\), but the proof requires more sophisticated methods (see Tao [27]).

Theorem 6.22 (Persistence of regularity for NLS). Let \(u_0 \in H^s(\mathbb{R})\), where \(1/2 < r < s\). Then \(T_s(u_0) = T_r(u_0)\).

Proof. Let \(T < T_r(u_0)\), so that \(C := \sup_{t \in [0,T]} \|u(t)\|_{H^r(\mathbb{R})} < \infty\). Repeated application of Lemma 6.20 shows that
\[
\|u(t)\|_{H^r(\mathbb{R}^d)} \leq \|u_0\|_{H^r(\mathbb{R})} + \int_0^t \|f(u(\tau))\|_{H^r(\mathbb{R})} \, d\tau
\leq \|u_0\|_{H^r(\mathbb{R})} + K \int_0^t \|u(\tau)\|_{H^{p-1}(\mathbb{R})}^{p-1} \|u(\tau)\|_{H^r(\mathbb{R})} \, d\tau
\leq \|u_0\|_{H^r(\mathbb{R})} + KC^{p-1} \int_0^t \|u(\tau)\|_{H^r(\mathbb{R})} \, d\tau.
\]
It follows from Grönwall’s inequality that
\[
\|u(t)\|_{H^r(\mathbb{R})} \leq \|u_0\|_{H^r(\mathbb{R})} e^{KC^{p-1}t}, \quad 0 \leq t < \min\{T,T_s\},
\]
so that \(T \leq T_s(u_0) \leq T_r(u_0)\). Since \(T < T_r(u_0)\) is arbitrary we obtain that \(T_s(u_0) = T_r(u_0)\). \(\square\)

Theorem 6.23. Let \(u \in C([0,T);H^1(\mathbb{R}))\) be a strong solution of the NLS equation. Then the quantities \(M(u(t))\) and \(E(u(t))\) are independent of \(t\).

Proof. First assume that \(u_0 \in H^3(\mathbb{R})\) and let \(u \in C([0,T);H^3(\mathbb{R})) \cap C^1([0,T);H^1(\mathbb{R}))\) be the corresponding solution. We have that
\[
\frac{d}{dt} M(u(t)) = \Re \int_\mathbb{R} u_t \bar{u} \, dx
= \Re \int_\mathbb{R} i(\partial_x u - \sigma |u|^{p-1} u) \bar{u} \, dx
= - \Re \int_\mathbb{R} i(|\partial_x|^2 + \sigma|u|^{p+1}) \, dx
= 0
\]
Dispersive equations

(note that $u \in \dot{C}^2(\mathbb{R})$, so that the integration by parts is justified). Similarly,

\[
\frac{d}{dt} E(u(t)) = \text{Re} \int_{\mathbb{R}} (u_t \overline{u_x} + \sigma |u|^{p-1} u_t \overline{n}) \, dx \\
= - \text{Re} \int_{\mathbb{R}} u_t (u_{xx} - \sigma |u|^{p-1} u) \, dx \\
= - \text{Re} \int_{\mathbb{R}} i u_{xx} - \sigma |u|^{p-1} |u|^2 \, dx \\
= 0.
\]

If $u_0 \in H^1(\mathbb{R})$ we approximate it in $H^1(\mathbb{R})$ by a sequence $\{u_{0,n}\} \subset H^3(\mathbb{R})$. By Theorems 6.18 and 6.22, there exist corresponding solutions $\{u_n\} \subset C([0,T];H^3(\mathbb{R}))$, where $T > 0$ can be taken independent of $n$. We also have that

\[
\lim_{n \to \infty} \|u_n - u\|_{C([0,T];H^1(\mathbb{R}))} = 0,
\]

by Theorem 6.18. By the above reasoning we have that $E(u_n(t)) = E(u_{0,n})$ and $M(u_n(t)) = M(u_{0,n})$ for all $t \in [0,T]$. Since the maps $u \mapsto E(u)$ and $u \mapsto M(u)$ are continuous from $H^1(\mathbb{R})$ to $\mathbb{R}$, it follows that

\[
E(u(t)) = E(u_0), \quad M(u(t)) = M(u_0).
\]

\textbf{Theorem 6.24.} If $\sigma = 1$ and $u_0 \in H^s(\mathbb{R})$, $s \geq 1$, the strong solution of (6.9) in $H^s(\mathbb{R})$ exists for all $t \geq 0$.

\textbf{Proof.} This follows from Theorems 6.19 and 6.23 and the inequality $\|u\|_{H^1(\mathbb{R})}^2 \leq 2(E(u) + M(u))$. \hfill $\square$

In the focusing case, we generally don’t have global existence. The problem is that one generally can’t control the $H^1$ norm using $M$ and $E$ since the second term in $E$ is negative. An exception is when the nonlinearity is cubic.

\textbf{Theorem 6.25.} If $\sigma = -1$, $p = 3$ and $u_0 \in H^s(\mathbb{R})$, $s \geq 1$, the strong solution of (6.9) in $H^s(\mathbb{R})$ exists for all $t \geq 0$.

\textbf{Lemma 6.26.} Let $u \in H^1(\mathbb{R})$. Then

\[
\|u\|_{L^\infty(\mathbb{R})}^2 \leq 2 \|u\|_{L^2(\mathbb{R})} \|u'\|_{L^2(\mathbb{R})}.
\]

\textbf{Proof.} Assume that $u \in C_0^\infty(\mathbb{R})$. Then

\[
|u(x)|^2 = 2 \text{Re} \int_{-\infty}^{x} u'(y) \overline{u(y)} \, dy,
\]

yielding

\[
\|u\|_{L^\infty(\mathbb{R})}^2 \leq 2 \|u\|_{L^2(\mathbb{R})} \|u'\|_{L^2(\mathbb{R})}.
\]

The result now follows by using the fact that $C_0^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$. \hfill $\square$
Proof of Theorem 6.25. We can estimate

\[ \int_{\mathbb{R}} |u|^{p+1} \, dx \leq \|u\|_{L^p(\mathbb{R})}^{p-1} \|u\|_{L^2(\mathbb{R})}^2 \]

\[ \leq 2^{p-1} \|u\|_{L^2(\mathbb{R})}^{p+3/2} \|u\|_{L^2(\mathbb{R})}^{p-1}. \]

If \( p = 3 \) this gives

\[ \frac{1}{4} \int_{\mathbb{R}} |u|^4 \, dx \leq \frac{1}{2} \|u\|_{L^2(\mathbb{R})}^3 \|u'\|_{L^2(\mathbb{R})} \leq \frac{1}{4} \|u'\|_{L^2(\mathbb{R})}^2 + \frac{1}{4} \|u\|_{L^2(\mathbb{R})}^6, \]

by the arithmetic-geometric inequality, so that

\[ E(u) = \frac{1}{2} \|u'\|_{L^2(\mathbb{R})}^2 - \frac{1}{4} \int_{\mathbb{R}} |u|^4 \, dx \geq \frac{1}{4} \|u'\|_{L^2(\mathbb{R})}^2 - 2M(u)^3. \]

We can therefore bound \( \|u\|_{H^1(\mathbb{R})}^2 \leq 4(E(u) + M(u)) + 8M(u)^3. \) The conservation of \( E(u) \) and \( M(u) \) now implies that \( \|u(t)\|_{H^1} \) is bounded.

We conclude by showing a blow-up result. For this we need the additional result that if the initial data satisfies \( xu_0 \in L^2(\mathbb{R}) \), then \( xu \in L^2(\mathbb{R}) \) for all \( t \) for which the solution is defined. This can e.g. be shown by proving a well-posedness result in the space

\[ H^{1,1}(\mathbb{R}) = \{ u \in H^1(\mathbb{R}) : xu \in L^2(\mathbb{R}) \}, \]

with norm \( \|u\|_{H^{1,1}(\mathbb{R})} = \left( \|u\|_{H^1(\mathbb{R})}^2 + \|xu\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \) (see Tao [27]). We also need the following result, which can formally be verified by carrying out the differentiation and integrating by parts. The rigorous justification relies on an approximation argument, as in the proof of Theorem 6.23.

Lemma 6.27. Define

\[ V(u) = \int_{\mathbb{R}} x^2 |u(x)|^2 \, dx. \]

Then

\[ \frac{d}{dt} V(u(t)) = 4 \text{Im} \int_{\mathbb{R}} x \bar{u} u_x \, dx \]

and

\[ \frac{d^2}{dt^2} V(u(t)) = 16E(u(t)) + \frac{4\sigma(p-5)}{p+1} \int_{\mathbb{R}} |u(t)|^{p+1} \, dx. \]

Theorem 6.28. Suppose that \( \sigma = -1 \) and \( p \geq 5 \), and consider \( u_0 \in H^1(\mathbb{R}) \) with \( V(u_0) \) finite. If

\[ E(u_0) < 0, \]

the solution with initial data \( u_0 \) has finite existence time.
Proof. Abbreviate $V(t) = V(u(t))$ and $E = E(u(t)) = E(u_0)$. If $p \geq 5$ and $\sigma = -1$, we obtain that $V''(t) \leq 16E$. Thus,

$$V(t) \leq 8Et^2 + V'(0)t + V(0), \quad 0 \leq t < T,$$

with $E$ negative by assumption. Hence, $T < \infty$ since we would otherwise obtain the contradiction $V(t) < 0$ for $t$ large. \hfill \square

Remark 6.29. Note that $E(\lambda u_0) < 0$ if $\lambda > 0$ is sufficiently large for a fixed function $u_0 \in H^{1,1}(\mathbb{R})$, $u_0 \neq 0$.

6.3 KdV

As mentioned in the introduction, the KdV equation was introduced in an attempt to explain the solitary water wave observed by John Scott Russell. The equation can be derived as an approximation of the equations of fluid mechanics. Since the derivation is quite long it has been placed at the end of the section.

6.3.1 Linear theory

The Cauchy problem for the linearized KdV equation is

$$\begin{cases}
  u_t + u_{xxx} = f, & t > 0, \\
  u = u_0, & t = 0,
\end{cases} \quad (6.12)$$

where $f$ and $u_0$ are given. If $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $f \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d))$ there is a unique solution $u \in C^\infty([0, \infty); \mathcal{S}(\mathbb{R}^d))$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)f(\tau) \, d\tau, \quad (6.13)$$

where $S(t)\varphi = \mathcal{F}^{-1}(e^{it\xi^3} \hat{\varphi})$ for $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Just as for the NLS equation this solution formula can be extended to $u_0 \in H^s(\mathbb{R})$ and $f \in C([0, T); H^s(\mathbb{R}))$. Using the same arguments as in the proof of Propositions 6.14 and 6.15, we obtain the following results.

Proposition 6.30. The family of operators $\{S(t)\}_{t \in \mathbb{R}}$ defined for $u \in H^s(\mathbb{R})$ by $S(t)u = \mathcal{F}^{-1}(e^{it\xi^3} \hat{u})$ is a strongly continuous group of unitary operators on $H^s(\mathbb{R})$. The operator $S(t)$ commutes with $\partial_x$ for each $t$. If $u \in H^s(\mathbb{R})$ then $t \mapsto S(t)u$ is in $C^1(\mathbb{R}; H^{s-3}(\mathbb{R}))$ with

$$\frac{d}{dt}(S(t)u) = -\partial_x^3 S(t)u.$$

Proposition 6.31. Assume that $f \in C([0, T); H^s(\mathbb{R}))$ and that $u_0 \in H^s(\mathbb{R})$. There exists a unique solution $u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-3}(\mathbb{R}))$ of (6.12) given by (6.13).
Just as in the case of the Schrödinger equation, the solution can be expressed as a convolution of \( u_0 \) with a function \( K_t(x) \) if \( f \equiv 0 \) and \( u_0 \) is sufficiently smooth (e.g. \( u_0 \in \mathcal{S}(\mathbb{R}) \)). Define the tempered distribution \( \mathcal{A}_t \in \mathcal{S}'(\mathbb{R}) \) through \( \mathcal{A}_t = (2\pi)^{-1/2} \mathcal{F}^{-1}(e^{ix^3/3}) \). We then have

\[
u = K_t * u_0
\]
in the sense of distributions, where \( K_t \) is the tempered distribution defined by

\[
k_t(x) = (3t)^{-1/3} \mathcal{A}_1((3t)^{-1/3}x)
\]

Here we have abused notation slightly by writing \( \mathcal{A}_t(\lambda x) \) instead of \( \sigma_{\lambda} \mathcal{A} \). This abuse of notation is justified since \( \mathcal{A}_t \) is in fact given by a bounded, smooth function.

**Lemma 6.32.** The tempered distribution \( \mathcal{A}_i \) is given by the smooth function defined by the path integral

\[
\frac{1}{2\pi} \int_{\text{Im} \, \zeta = \eta} e^{i\zeta^2/3 + ix\zeta} \, d\zeta,
\]

(6.14)

where \( \eta > 0 \) is arbitrary.

**Proof.** We have that \( \text{Re}(i\zeta^2/3 + ix\zeta) = -\xi^2 \eta + \eta^3/3 - x\eta \), so that the integral in (6.14) converges for each \( \eta > 0 \). Moreover, the function \( \zeta \mapsto e^{i\zeta^2/3 + ix\zeta} \) is analytic for fixed \( x \), showing that the integral is independent of \( \eta \). Since the integrand is smooth (in fact, real analytic) in \( x \) and all the derivatives decay rapidly as \( |\text{Re} \, \zeta| \to \infty \), (6.14) defines a smooth function of \( x \). It remains to show that the distribution defined by this function coincides with \( \mathcal{A}_i \). Changing the order of integration, we find that

\[
\langle \frac{1}{2\pi} \int_{\text{Im} \, \zeta = \eta} e^{i\zeta^2/3 + ix\zeta} \, d\zeta, \varphi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{i\xi^2/3 + ix\xi} \varphi(x) \, dx \right) \, d\xi
\]

\[
\to \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{i\xi^2/3 + ix\xi} \varphi(x) \, dx \right) \, d\xi
\]

\[
= \langle (2\pi)^{-1/2} e^{ix^3/3}, \mathcal{F}^{-1}(\varphi) \rangle
\]

\[
= \langle \mathcal{A}_i, \varphi \rangle
\]

as \( \eta \to 0^+ \). Since (6.14) is independent of \( \eta \), the result follows.

From (6.14) one also sees that

\[
\mathcal{A}_i''(x) - x \mathcal{A}_i(x) = 0,
\]

(6.15)

since

\[
\mathcal{A}_i''(x) - x \mathcal{A}_i(x) = -\frac{1}{2\pi} \int \text{Im} \, \zeta = \eta \left( \text{Im} \, \zeta^2 = x \right) e^{i\zeta^2/3 + ix\zeta} \, d\zeta
\]

\[
= -\frac{1}{2\pi i} \lim_{R \to \infty} \left[ e^{i\zeta^2/3 + ix\zeta} \right]_{\zeta = R + i\eta}^{\zeta = -R + i\eta}
\]

\[
= 0.
\]

Equation (6.15) is called Airy’s equation.
**Proposition 6.33.** \( \text{Ai}(x) = O(|x|^{-1/4}) \) as \( x \to -\infty \), while \( \text{Ai}(x) = O(x^{-1/4}e^{-x^{3/2}}) \) as \( x \to \infty \).

**Proof.** We begin with the case \( x \to \infty \) and thus assume that \( x > 0 \). The derivative of \( \zeta \mapsto i\zeta^3/3 + ix\zeta \) vanishes when \( \zeta^2 = -x \), that is, when \( \zeta = \pm i\sqrt{x} \). Write \( \zeta = \xi + i\eta \). Along the line \( \mathbb{R} + i\sqrt{x} \) we have

\[
 i(\xi + i\eta)^3/3 + ix(\xi + i\eta) = i\xi^3/3 - \sqrt{x}\xi^2 - 2x^{3/2}/3,
\]

and thus,

\[
 \text{Ai}(x) = \frac{e^{-2\sqrt{x}^3/3}}{2\pi} \int_{\mathbb{R}} e^{-\sqrt{x}\zeta^2 + i\xi^3/3} d\zeta. \tag{6.16}
\]

We can estimate

\[
 \left| \int_{\mathbb{R}} e^{-\sqrt{x}\zeta^2 + i\xi^3/3} d\zeta \right| \leq \int_{\mathbb{R}} e^{-\sqrt{x}\zeta^2} d\zeta = \frac{\sqrt{\pi}}{x^{1/4}},
\]

yielding

\[
 |\text{Ai}(x)| \leq \frac{e^{-2\sqrt{x}^3/3}}{2\sqrt{\pi}x^{1/4}}, \quad x > 0. \tag{6.17}
\]

One has to work a bit harder to find the asymptotic behaviour when \( x \to -\infty \). Note first that (6.16) continues to hold when \( |\arg x| < \pi \). So does the estimate (6.17), if one replaces \( x^{3/2} \) with \( \text{Re}x^{3/2} \) and \( x^{1/4} \) with \( (\text{Re}\sqrt{x})^{1/2} \) in the right hand side. Next note that \( \text{Ai}(\omega x) \) is also a solution of (6.15) if \( \omega \) is a cubic root of unity, that is, if \( \omega = \omega_k \), where \( \omega_k = e^{2\pi i/3} \), \( k = 0, 1, 2 \). We claim that any two of these solutions are linearly independent. This follows since

\[
 \text{Ai}(0) = \frac{1}{2\pi} \int_{\mathbb{R} + i} e^{i\xi^3/3} d\xi = \frac{1}{\pi} \text{Re} \int_0^{\infty} e^{i\pi/6} e^{-t^{3/3}} dt = \frac{3^{-1/6} \Gamma(1/3)}{2\pi},
\]

\[
 \text{Ai}'(0) = \frac{1}{2\pi} \int_{\mathbb{R} + i} i\xi e^{i\xi^3/3} d\xi = -\frac{1}{\pi} \text{Im} \int_0^{\infty} e^{i\pi/3} e^{-t^{3/3}} dt = -\frac{3^{-1/6} \Gamma(2/3)}{2\pi},
\]

where we have deformed the path \( \mathbb{R} + i \) into \( \{e^{-i\pi/6} t : -\infty \leq t < 0\} \cup \{e^{i\pi/6} t : 0 \leq t < \infty\} \). Hence, \( \text{Ai}(\omega x) \) has the value \( \text{Ai}(0) \) at 0, while the derivative is \( \omega \text{Ai}'(0) \), confirming the linear independence. We also have

\[
 \sum_{k=0}^3 \omega_k \text{Ai}(\omega_k x) = 0,
\]

since the value of this solution of the Airy equation at 0 is \( \text{Ai}(0) \sum_{k=0}^3 \omega_k = 0 \) and its derivative is \( \text{Ai}'(0) \sum_{k=0}^3 \omega_k^2 = 0 \). Expressing \( \text{Ai}(-r) = -\omega_1 \text{Ai}(-\omega_1 r) - \omega_2 \text{Ai}(-\omega_2 r), r > 0, \) with \( \arg(-\omega_1 r) = -\pi/3 \) and \( \arg(-\omega_2 r) = \pi/3 \), we therefore find that

\[
 |\text{Ai}(-r)| \leq |\text{Ai}(-\omega_1 r)| + |\text{Ai}(-\omega_2 r)| \leq C r^{-1/4}, \quad r > 0,
\]

for some constant \( C > 0 \). \( \square \)
Remark 6.34. With a little more work one can also prove that \( \text{Ai}(-r) = \pi^{-1/2} r^{-1/4} (\sin(2r^{3/2}/3 + \pi/4) + O(r^{-3/2})) \) as \( r \to \infty \), which shows that the Airy function oscillates rapidly for large negative values of \( x \). The treatment of the Airy function given here follows the approach in Hörmander [12].

Corollary 6.35. There exists a constant \( C > 0 \) such that

\[
\|S(t)u_0\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{1}{2} \left( \frac{1 - 1}{p} \right)} \|u_0\|_{L^q(\mathbb{R})}
\]

for \( u_0 \in \mathscr{S}(\mathbb{R}) \) and \( p \in [2, \infty] \), where \( 1/p + 1/q = 1 \).

6.3.2 Nonlinear theory

We next consider the Cauchy problem

\[
\begin{align*}
  u_t + 6uu_x + u_{xxx} &= 0, \quad t > 0, \\
  u &= u_0, \quad t = 0,
\end{align*}
\]

(6.18)

where \( u \) is assumed to be real-valued. One could consider more general nonlinearities of the form \( u^k u_x \), where \( k \geq 1 \) is an integer, but we shall concentrate on the case \( k = 1 \) for simplicity. The local results hold for any power of the nonlinearity, while the global results depend heavily on \( k \).

Local well-posedness

The Cauchy problem for KdV is harder than for NLS, the reason being that the nonlinearity contains derivatives of \( u \). We begin by stating the main result. The proof is conceptually easy but contains some technical details.

Theorem 6.36. Let \( u_0 \in H^k(\mathbb{R}) \), \( k \geq 2 \). There exists a maximal time \( T > 0 \) such that the Cauchy problem (6.18) has a strong solution in \( C([0,T);H^k(\mathbb{R})) \). The solution is unique in this class and if \( T < \infty \), then \( \lim_{t \to T} \|u(t)\|_{H^k(\mathbb{R})} = \infty \). The solution depends continuously on the initial data.

Remark 6.37. The reason for using Sobolev spaces of integer order is to simplify the proof. Using the same methods one can prove local well-posedness in \( H^s(\mathbb{R}) \) with \( s > 3/2 \).

We first prove uniqueness.

Lemma 6.38. Let \( u, v \in C([0,T];H^2(\mathbb{R})) \cap C^1([0,T];H^{-1}(\mathbb{R})) \) be solutions of (6.18) with initial data \( u_0 \in H^2(\mathbb{R}) \). Then \( u = v \).
Proof. Since $w := u - v$ satisfies the equation $w_t + 3((u + v)w)_x + w_{xxx} = 0$, it follows that

\[
\frac{d}{dt} \|w(t)\|_{L^2(\mathbb{R})}^2 = 2\langle w_{xxx}(t) - 3((u(t) + v(t))w(t))_x, w \rangle \\
= 6 \int_{\mathbb{R}} w_x(x,t)w(x,t)(u(x,t) + v(x,t))
\]

\[
= -3 \int_{\mathbb{R}} w^2(x,t)(u_x(x,t) + v_x(x,t))
\]

\[
\leq C \|w(t)\|_{L^2(\mathbb{R})}^2,
\]

where $C = 3 \sup_{0 \leq t \leq T} (\|u_x(\cdot,t)\|_{L^\infty(\mathbb{R})} + \|v_x(\cdot,t)\|_{L^\infty(\mathbb{R})}) < \infty$ due to the Sobolev embedding $H^2(\mathbb{R}) \subset C^1_b(\mathbb{R})$. Note that we have to interpret $f(t) := w_{xxx}(t) - 3((u(t) + v(t))w(t))_x$ as an element of $H^{-1}(\mathbb{R})$ in the first line. The expression $\langle f(t), w(t) \rangle$ is well-defined since $w(t) \in H^2(\mathbb{R}) \subset H^1(\mathbb{R})$. Using Grönwall’s inequality and the fact that $w(0) = 0$, it follows that $w(t) \equiv 0$ in $[0, T]$.

\[\square\]

Remark 6.39. If we instead assume that $u, v \in C([0, T];H^4(\mathbb{R})) \cap C^1([0, T];H^1(\mathbb{R}))$, then all the derivatives can be interpreted in the classical sense. The reason for working with the lower regularity is that we need it to prove global existence.

The strategy for proving the existence of a local solution is to replace the Cauchy problem (6.18) by the approximate problem

\[
\begin{align*}
\begin{cases}
u_t + 6uu_x + u_{xxx} + \varepsilon u_{xxxx} = 0, & t > 0, \\
u = u_0, & t = 0,
\end{cases}
\end{align*}
\]

(6.19)

with $\varepsilon > 0$. This is basically the same idea which was involved when we discussed uniqueness for weak solutions of conservation laws in Section 4.3.3. In fact, one could instead have used $u_t + 6uu_x + u_{xxx} = \varepsilon u_{xx}$, although this gets more technical.

The strategy is the following.

1. Prove that (6.19) has a solution $u_\varepsilon$ defined on some time interval $[0, T(\varepsilon))$.

2. Prove that $\{u_{\varepsilon_n}\}$ converges to a solution of (6.18) as $n \to \infty$, where $\{\varepsilon_n\}$ is a sequence of positive numbers which converges to 0.

There are two technical complications here. First of all, one must show that the lifespan $T(\varepsilon_n)$ is bounded away from 0 as $n \to \infty$. Secondly, we will only prove convergence in $C([0, T];H^{k-2}(\mathbb{R}))$ if $u_0 \in H^k(\mathbb{R})$ in the second step. In order to prove the existence of a solution in $C([0, T];H^k(\mathbb{R}))$ we approximate the initial data by smooth functions and use the KdV equation to construct a sequence of approximate solutions which converges in the stronger norm.
The Cauchy problem (6.19) can be reformulated as the fixed point equation\textsuperscript{6}

\[ u(t) = S_\varepsilon(t) u_0 - 3 \int_0^t S_\varepsilon(t - \tau) (u_\tau^2), \quad dx, \]  

(6.20)

where \( S_\varepsilon(t) = e^{-i(\varepsilon \partial_x^4 + \partial_x^2)}. \) We write \( \hat{S}_\varepsilon(t) = e^{-i(\varepsilon \xi^4 - i \xi^2)}. \)

**Lemma 6.40.**

\[ \| \partial_t S_\varepsilon(t) f \|_{H^k(\mathbb{R})} \leq C((\varepsilon t)^{-1/2} + (\varepsilon t)^{-1/4}) \| f \|_{H^{k-1}(\mathbb{R})}. \]

**Proof.** The result follows by estimating \( \| \xi \hat{S}_\varepsilon(t) \|_{L^\infty(\mathbb{R})} \) and \( \| \xi^2 \hat{S}_\varepsilon(t) \|_{L^\infty(\mathbb{R})} \) since

\[
\mathcal{F}(\partial_t S_\varepsilon(t) f)(\xi) = i \xi \hat{S}_\varepsilon(t) \hat{f}(\xi) \quad \text{and} \quad \mathcal{F}(\partial_x^2 S_\varepsilon(t) f)(\xi) = -\xi^2 \hat{S}_\varepsilon(t) \hat{f}(\xi).
\]

We have

\[
\| \xi \hat{S}_\varepsilon(t) \|_{L^\infty} = \| \xi e^{-t \xi^4} \|_{L^\infty} = (\varepsilon)^{-1/4} \| (\varepsilon)^{1/4} \xi e^{-t \xi^4} \|_{L^\infty} \leq C(\varepsilon t)^{-1/4}.
\]

In a similar way one proves that \( \| \xi^2 \hat{S}_\varepsilon(t) \|_{L^\infty(\mathbb{R})} \leq C(\varepsilon t)^{-1/2}. \)

**Lemma 6.41.** Let \( T > 0, \varepsilon > 0 \) and \( f \in C([0, T]; H^k(\mathbb{R})). \) There exists \( C = C(T, \varepsilon) \) such that

\[ \| \int_0^t \partial_t S_\varepsilon(t - \tau) f(\tau) \, d\tau \|_{H^k(\mathbb{R})} \leq C \| f \|_{C([0, T]; H^{k-1}(\mathbb{R}))}, \quad t \in [0, T]. \]

**Proof.** This follows immediately by moving the norm inside the integral and applying the previous lemma. Here it is essential that \( \tau \mapsto (t - \tau)^{-1/2} \) is integrable on \([0, t]\)

**Theorem 6.42.** Let \( k \geq 2 \) be an integer and \( u_0 \in H^k(\mathbb{R}). \) For any \( \varepsilon > 0 \) there exists a unique maximal strong solution \( u \in C([0, T); H^k(\mathbb{R})). \) of (6.19) with \( u(0) = u_0. \) If \( T < \infty, \) we have that \( \lim_{t \to T} \| u(t) \|_{H^k(\mathbb{R})} = \infty. \) The solution has the additional property that \( u \in C(\infty, (0, T); H^m(\mathbb{R})). \)

**Proof.** The proof of the existence and uniqueness of a solution \( u \in C([0, T); H^k(\mathbb{R})). \) follows as for the NLS equation in view of Lemma 6.41. To prove the additional regularity property of the solution, we note that \( S_\varepsilon(t) u_0 \in C^\infty((0, \infty); H^m(\mathbb{R})). \) for any \( m \) due to the rapid decay of \( \hat{S}_\varepsilon(t) \) in \( \xi \) for \( t > 0. \) On the other hand, \( \int_0^t \partial_t S(t - \tau) u_\tau^2(\tau) \, d\tau \in C([0, T); H^{k+1}(\mathbb{R})). \) by Lemma 6.41. It follows that \( u \in C((0, T); H^{k+1}(\mathbb{R})). \) Solving the equation with initial data \( u(t_0) \in H^{k+1}(\mathbb{R}), \) for any \( t_0 \in (0, T), \) we see by the same argument that \( u \in C((t_0, T); H^{k+2}(\mathbb{R})). \) Since \( t_0 \) is

---

\textsuperscript{6}The proof of the equivalence requires some modifications of the method in the proof of Proposition 6.15 since \( S_\varepsilon(-t) \) is not bounded for \( t > 0. \) To prove the uniqueness of the solution of the inhomogeneous linear problem one can instead use energy estimates.
arbitrary it follows that \( u \in C((0, T); H^{k+2}(\mathbb{R})) \) and by induction that \( u \in C((0, T); H^m(\mathbb{R})) \) for any \( m \geq 2 \). By differentiating under the integral sign, one now finds that

\[
u(t) = S_\epsilon(t-t_0)u(t_0) - 3 \int_{t_0}^t \partial_x S_\epsilon(t-\tau)u^2(\tau) \, d\tau \in C^\alpha((t_1, T); H^m(\mathbb{R}))
\]

for any \( m \) and any \( t_0 \in (0, T) \). Since \( t_0 \) is arbitrary it follows that \( u \in C^\alpha((0, T); H^m(\mathbb{R})) \) for any \( m \geq 2 \).

As mentioned earlier, the maximal existence time \( T = T(\epsilon) \) is a function \( \epsilon \). Our first task is to prove that it doesn’t shrink to 0 as \( \epsilon \to 0^+ \). To do so, we must simply bound the \( H^k \) norm of the solution. We begin with a lemma related to Grönwall’s inequality.

**Lemma 6.43.** Assume that \( f \in C[0, T] \) with \( f \geq 0 \), and that there exist \( a, b, \alpha \in \mathbb{R} \) with \( a > 0 \) and \( \alpha > 1 \) such that

\[
f(t) \leq a + b \int_0^t f^\alpha(\tau) \, d\tau
\]

when \( t \in [0, T] \). Then

\[
f(t) \leq (a^{1-\alpha} - (\alpha - 1)bt)^{\frac{1}{1-\alpha}}.
\]

**Proof.** Let \( F(t) = a + b \int_0^t f^\alpha(\tau) \, d\tau \), so that \( F \in C^1([0, T]) \) with \( F'(t) = a + bf^\alpha(t) \) and \( F(t) > 0 \), \( f(t) \leq F(t) \) for all \( t \in [0, T] \). Then

\[
F'(t) \leq bF^\alpha(t), \quad t \in [0, T],
\]

which implies that \( F'(t)F^{-\alpha}(t) \leq b \). Integrating this inequality, we find that

\[
\frac{1}{1-\alpha} \left( F^{1-\alpha}(t) - F^{1-\alpha}(0) \right) \leq bt,
\]

yielding

\[
F^{1-\alpha}(t) \geq F(0)^{1-\alpha} - (\alpha - 1)bt = a^{1-\alpha} - (\alpha - 1)bt
\]

and hence

\[
f(t) \leq F(t) \leq (a^{1-\alpha} - (\alpha - 1)bt)^{\frac{1}{1-\alpha}}.
\]

**Lemma 6.44.** \( T(\epsilon) \) is bounded away from 0 as \( \epsilon \to 0^+ \).

**Proof.** Let \( 0 \leq j \leq k \). For \( t > 0 \) we have that

\[
\frac{d}{dt}\|\partial_x^ju(t)\|_{L^2(\mathbb{R})}^2 = -2(\partial_x^j u, \partial_x^j (\epsilon u_{xxxx} + u_{xxx} + 3(u^2)_x))_{L^2(\mathbb{R})}
\]

\[
= -2\epsilon\|\partial_x^{j+2}u\|_{L^2(\mathbb{R})}^2 + 2(\partial_x^{j+1} u, \partial_x^{j+2}u)_{L^2(\mathbb{R})} - 6(\partial_x^j u, \partial_x^{j+1}(u^2))_{L^2(\mathbb{R})}
\]

\[
\leq -6(\partial_x^j u, \partial_x^{j+1}(u^2))_{L^2(\mathbb{R})}
\]

\[
= -6 \sum_{m=0}^{j+1} \binom{j+1}{m} (\partial_x^j u, \partial_x^m u \partial_x^{j+1-m}u)_{L^2(\mathbb{R})}.
\]
If $1 \leq m \leq j - 1$, we can estimate the terms in the sum by $\| \partial_x^j u \|_{L^2(\mathbb{R})} \| \partial_x^m u \|_{L^\infty(\mathbb{R})} \| \partial_x^{j-1-m} u \|_{L^2(\mathbb{R})} \leq C \| u \|_{H^3(\mathbb{R})}$, where the Sobolev embedding theorem is used. The case $m = j$ can be estimated by $\| \partial_x^j u \|_{L^2(\mathbb{R})}^2 \| \partial_x u \|_{L^\infty(\mathbb{R})} \leq C \| u \|_{H^3(\mathbb{R})}^3$. This leaves the term $-12 \langle \partial_x^j u, u \partial_x^{j+1} u \rangle$. Integration by parts yields that

$$-2 \langle \partial_x^j u, u \partial_x^{j+1} u \rangle = \int_{\mathbb{R}} (\partial_x^j u)^2 \partial_x u \, dx \leq \| \partial_x u \|_{L^\infty(\mathbb{R})} \| \partial_x^j u \|_{L^2(\mathbb{R})}^2 \leq C \| u \|_{H^3(\mathbb{R})}^3.$$ 

Altogether we find that

$$\frac{d}{dt} \| u(t) \|_{H^k(\mathbb{R})}^2 \leq C \| u(t) \|_{H^3(\mathbb{R})}^3$$

for $t \in (0, T)$. Integrating this inequality from $t_0$ to $t$ and letting $t_0 \to 0^+$, we find that

$$\| u(t) \|_{H^k(\mathbb{R})}^2 \leq \| u_0 \|_{H^k(\mathbb{R})}^2 + C \int_0^t \| u(\tau) \|_{H^3(\mathbb{R})}^3 \, d\tau.$$ 

Applying the previous lemma with $f(t) = \| u(t) \|_{H^k(\mathbb{R})}^2$, $a = \| u_0 \|_{H^k(\mathbb{R})}^2$, $b = C$ and $\alpha = 3/2$, we find that

$$\| u(t) \|_{H^k(\mathbb{R})}^2 \leq \left( \| u_0 \|_{H^k(\mathbb{R})}^{-1} - \frac{1}{2} Ct \right)^{-2}.$$ 

This proves that the maximal existence time is bounded from below by $T_0 = 2 \| u_0 \|_{H^k(\mathbb{R})}^{-1}/C$, for otherwise the left hand side would blow up at some time $t < T_0$. This would result in a contradiction, since the right hand side doesn’t blow up until $T_0$. $\square$

We now consider a sequence $\{ \varepsilon_n \}$ with $\varepsilon_n \to 0^+$ and a sequence $\{ u_n \}$ of solutions in $C([0, T]; H^k(\mathbb{R}))$ of (6.19) with $\varepsilon = \varepsilon_n$ and $u_n(0) = u_0 \in H^k(\mathbb{R})$, where $k \geq 4$. By the previous lemma we can assume that the sequence $\{ \| u_n \|_{C([0, T]; H^k(\mathbb{R}))} \}$ is bounded.

**Lemma 6.45.** The sequence $\{ u_n \}$ converges in $C([0, T]; H^{k-2}(\mathbb{R}))$ to a strong solution $u \in C([0, T]; H^{k-2}(\mathbb{R}))$ of (6.18).

**Proof.** We show that $\{ u_n \}$ is a Cauchy sequence in $C([0, T]; H^{k-2}(\mathbb{R}))$. Let $w := u_n - u_m$. Then

$$w_t + 3((u_n + u_m)w)_x + w_{xxx} + (\varepsilon_n - \varepsilon_m)u_{nxxx} + \varepsilon_m w_{xxxx} = 0.$$ 

One finds that

$$\frac{1}{2} \frac{d}{dt} \| \partial_x^j w(t) \|_{L^2(\mathbb{R})}^2 = (\varepsilon_m - \varepsilon_n)(\partial_x^{j+2} w, \partial_x^{j+2} u_n)_{L^2(\mathbb{R})} - \varepsilon_m \langle \partial_x^{m+2} w, \partial_x^{m+2} u \rangle_{L^2(\mathbb{R})} - 3(\partial_x^j w, \partial_x^{j+1} ((u_n + u_m)w))_{L^2(\mathbb{R})} \leq (\varepsilon_m - \varepsilon_n)(\partial_x^{j+2} w, \partial_x^{j+2} u_n)_{L^2(\mathbb{R})} - 3(\partial_x^j w, \partial_x^{j+1} ((u_n + u_m)w))_{L^2(\mathbb{R})}.$$ 

The first term can be estimated by $\| \varepsilon_m - \varepsilon_n \| w \|_{C([0, T]; H^k(\mathbb{R}))} \| u_n \|_{C([0, T]; H^k(\mathbb{R}))}$, while the second and the third term can be estimated by $\| w(t) \|_{H^k(\mathbb{R})}^2 \| \partial_x u_n \|_{C([0, T]; H^k(\mathbb{R}))}$, since

$$2(\partial_x^j w, (u_n + u_m)\partial_x^{j+1} w)_{L^2(\mathbb{R})} = -\int_{\mathbb{R}} (\partial_x^j w)^2 (u_n + u_m) \, dx.$$
Summing over all \( j \leq k - 2 \), we therefore find that

\[
\frac{d}{dt} \| w(t) \|_{H^{k-2}(\mathbb{R})}^2 \leq C(\| \varepsilon_m - \varepsilon_n \| + \| w(t) \|_{H^{k-2}(\mathbb{R})}^2).
\]

It follows from Grönwall’s inequality that

\[
\| u_n(t) - u_m(t) \|_{H^{k-2}(\mathbb{R})}^2 \leq C(\| \varepsilon_m - \varepsilon_n \| (e^{Ct} - 1) + \| u_n(0) - u_m(0) \|_{H^{k-2}(\mathbb{R})}^2 e^{Ct}).
\]

Taking the supremum over \([0, T]\) and using the fact that \( \varepsilon_n, \varepsilon_m \to 0 \) as \( n, m \to \infty \), we find that \( \{ u_n \} \) is Cauchy in \( C([0, T]; H^{k-2}(\mathbb{R})) \). Letting \( \varepsilon \to 0 \) in (6.20), we find that the limit \( u \in C([0, T]; H^{k-2}(\mathbb{R})) \) is a strong solution of (6.18).

**Proof of Theorem 6.36.** The uniqueness of the solution follows from Lemma 6.38. We approximate \( u_0 \) by a sequence \( \{ u_{0n} \} \subset C_0^\infty(\mathbb{R}) \) in the \( H^k(\mathbb{R}) \) norm. Since \( C_0^\infty(\mathbb{R}) \subset H^{k+2}(\mathbb{R}) \), we can find a solution \( u_n \in C([0, T_n]; H^k(\mathbb{R})) \) with initial data \( u_{0n} \) for each \( n \). It remains to prove that we can take \( T_n = T_0 \), for some \( T_0 > 0 \) which is independent of \( n \), and that \( \{ u_n \} \) converges in \( C([0, T_0]; H^{k-2}(\mathbb{R})) \). To do this, we use the same arguments as in Lemmas 6.44 and 6.45. The only purpose of working in \( H^{k-2}(\mathbb{R}) \) instead of \( H^k(\mathbb{R}) \) in Lemma 6.45 was to handle the term \( \varepsilon u_{xxxx} \). With this term gone, we find that there exists a \( T_0 > 0 \) such that

\[
\frac{d}{dt} \| u_n(t) - u_m(t) \|_{H^k(\mathbb{R})}^2 \leq C \| u_n(t) - u_m(t) \|_{H^k(\mathbb{R})}^2, \quad t \in [0, T_0],
\]

with \( C = C(\| u_0 \|_{H^k(\mathbb{R})}) \), and hence that

\[
\| u_n - u_m \|_{C([0, T_0]; H^k(\mathbb{R}))}^2 \leq \| u_{0n} - u_{0m} \|_{H^k(\mathbb{R})}^2 e^{CT_0},
\]

showing that \( \{ u_n \} \) is Cauchy in \( C([0, T_0]; H^k(\mathbb{R})) \). This proves the existence of a local solution.

If \( u_0, v_0 \in B_\epsilon(0) \subset H^k(\mathbb{R}) \) the same argument shows that

\[
\| u_n - v_n \|_{C([0, T_0]; H^k(\mathbb{R}))}^2 \leq \| u_{0n} - v_{0n} \|_{H^k(\mathbb{R})}^2 e^{CT_0},
\]

where \( u_{0n}, v_{0n} \) are smooth initial data approximating \( u_0 \) and \( v_0 \) and \( u_n, v_n \) the corresponding solutions on the time interval \([0, T_0]\) with \( T_0 = T_0(\epsilon) \). Passing to the limit, we find that

\[
\| u - v \|_{C([0, T_0]; H^k(\mathbb{R}))}^2 \leq \| u_0 - v_0 \|_{H^k(\mathbb{R})}^2 e^{CT_0},
\]

proving that the solutions depend continuously on the initial data over the interval \([0, T_0]\). The argument used for the NLS equation shows that the solution can be extended to a maximal solution \( u \in C([0, T]; H^k(\mathbb{R})) \). Since the time interval in the above continuity argument just depends on the norm of the initial data, one can extend it to any interval \([0, T_0]\) on which the two solutions exist.

\( \square \)
Global existence and conservation laws

We now turn to the question of global existence. As for the NLS equation, this will be given a positive answer due to the existence of suitable conservation laws. Define

\[ I(u) = \int_{\mathbb{R}} u \, dx, \]

\[ M(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 \, dx, \]

\[ E(u) = \int_{\mathbb{R}} \left( \frac{1}{2} u_x^2 - u^3 \right) \, dx, \]

\[ F(u) = \int_{\mathbb{R}} \left( \frac{1}{2} u_{xx}^2 - 5uu_x^2 + \frac{5}{2} u^4 \right) \, dx. \]

These are formally conserved for the KdV equation. The first conservation law follows by writing the equation as

\[ u_t + (3u^2 + u_{xx})_x = 0. \]

Assuming that \(3u^2 + u_{xx} \to 0\) as \(x \to \pm \infty\), we obtain that \(I(u)\) is conserved. Multiplying the KdV equation by \(u\) we obtain

\[ uu_t + 6u^2 u_x + uu_{xxx} = 0, \]

which can be rewritten

\[ \partial_t \left( \frac{1}{2} u^2 \right) + \partial_x \left( uu_x + 2u^3 - \frac{1}{2} u_x^2 \right) = 0. \]

This gives the conservation of \(M(u)\). The conservation of \(E(u)\) follows by subtracting \(3u^2 \times (\text{KdV})\) from \(u_x \times \partial_x (\text{KdV})\). After some work one obtains that

\[ \partial_t \left( \frac{1}{2} u_x^2 - u^3 \right) + \partial_x \left( 6uu_x^2 + uu_{xxxx} - 3u^2 u_{xx} - \frac{1}{2} u_{xx}^2 - \frac{9}{2} u^4 \right) = 0. \]

This formally results in the conservation of \(E(u)\). The conservation of \(F(u)\) can be proved in a similar manner, although the calculations are rather tedious. By adding \(u_{xxx} \times \partial_x^2 (\text{KdV})\), \(-5u_x^2 \times (\text{KdV})\), \(-10uu_x \times \partial_x (\text{KdV})\) and \(5u^3 \times (\text{KdV})\) one finds after a long calculation that

\[ \partial_t \left( \frac{1}{2} u_{xx}^2 - 5uu_x^2 + \frac{5}{2} u^4 \right) + \partial_x R(u) = 0, \]

where

\[ R(u) = u_{xx}u_{xxxx} - \frac{1}{2} u_{xx}^2 + 5u_x^2 u_{xx} + 8uu_{xx}^2 - 10uu_xu_{xxxx} + 10u^3u_{xx} - 45u^2 u_x^2 + 12u^5. \]

The conserved quantities \(I(u), M(u)\) and \(E(u)\) are related to mass, momentum and energy for water waves. The quantity \(F(u)\) has no direct physical interpretation. The existence of this conserved quantity is related to the complete integrability of the KdV equation, a property which will be discussed further in the next chapter.
If $u \in H^2(\mathbb{R})$, the quantities $M(u)$, $E(u)$ and $F(u)$ are well-defined by Sobolev’s embedding theorem. By repeating the procedure in the proof of Theorem 6.23 one obtains the following result.

**Proposition 6.46.** Let $u \in C([0,T);H^2(\mathbb{R}))$ be a strong solution of the KdV equation. The quantities $M(u(t))$, $E(u(t))$ and $F(u(t))$ are independent of $t$.

For the proof one needs a persistence of regularity result. We denote by $T_k(u_0)$ the maximal existence time of the $H^k$ solution of KdV with initial data $u_0 \in H^k(\mathbb{R})$.

**Lemma 6.47 (Persistence of regularity for KdV).** Let $u_0 \in H^k(\mathbb{R})$, where $k \geq 2$. Then $T_k(u_0) = T_2(u_0)$.

To illustrate the idea we consider the proof that $T_3(u_0) = T_2(u_0)$. One formally has that
\[
\frac{d}{dt} \|u_{x}\|_{L^2(\mathbb{R})}^2 = (u_{xxx}, -6(uu_x)_{xxx} - u_{xxxxx})L^2(\mathbb{R}) = (u_{xxx}, -6(uu_x)_{xxx})L^2(\mathbb{R}).
\]

The last expression can be written using Leibniz’ formula as a linear combination of terms of the form $(u_{xxx}, uu_{xxx})L^2(\mathbb{R})$, $(u_{xxx}, u_{xx}u_{xxx})L^2(\mathbb{R})$, $(u_{xxx}, u^2_{xxx})L^2(\mathbb{R})$. Integration by parts reveals that $2(u_{xxx}, uu_{xxx})L^2(\mathbb{R}) = -2(u_{xxx}, uu_{xxx})L^2(\mathbb{R})$, while $(u_{xxx}, u^2_{xxx})L^2(\mathbb{R}) = (u_{xxx}uu_{xx})L^2(\mathbb{R}) = 0$. We can estimate $(u_{xxx}, uu_{xxx})L^2(\mathbb{R}) \leq \|u_x\|_{L^\infty(\mathbb{R})}\|u_{xxx}\|_{L^2(\mathbb{R})} \leq C\|u\|_{H^2(\mathbb{R})}\|u_{xxx}\|_{L^2(\mathbb{R})}$. Since $\|u\|_{H^2(\mathbb{R})}$ is bounded on $[0,T]$ for any $T < T_2(u_0)$ we obtain by Grönwall’s inequality that $\|u(t)\|_{H^3(\mathbb{R})}$ is bounded on $[0,T]$ and hence that $T_3(u_0) = T_2(u_0)$. The calculation doesn’t quite make sense, since $u_{xxx}(t) \notin C^1([0,T];L^2(\mathbb{R}))$ in general. This can be taken care of by first proving the result for (6.19) and then letting $\varepsilon \to 0^+$. One shows similarly that $T_{k+1}(u_0) = T_k(u_0) = T_2(u_0)$ for all $k \geq 2$.

**Theorem 6.48.** For $u_0 \in H^k(\mathbb{R})$, $k \geq 2$, the solution of (6.18) exists for all $t \geq 0$.

**Proof.** We have that
\[
\|u\|_{H^1(\mathbb{R})}^2 \leq 2(E(u) + M(u)) + 4\|u\|_{L^\infty(\mathbb{R})}M(u)
\]
\[
\leq 2(E(u) + M(u)) + C\|u\|_{H^1(\mathbb{R})}^2M(u)
\]
\[
\leq 2(E(u) + M(u)) + \frac{1}{2}\|u\|_{H^1(\mathbb{R})}^2 + \frac{C^2}{2}M(u)^2,
\]

where we have used the arithmetic-geometric inequality. This shows that
\[
\frac{1}{2}\|u(t)\|_{H^1(\mathbb{R})}^2 \leq 2(E(u(t)) + M(u(t))) + \frac{C^2}{2}M(u(t)) = 2(E(u(0)) + M(u(0))) + \frac{C^2}{2}M(u(0))
\]
is bounded. Similarly, we find that
\[
\|u_{xx}\|_{L^2(\mathbb{R})}^2 \leq 2F(u) + 10\|u\|_{L^\infty(\mathbb{R})}\|u\|_{H^1(\mathbb{R})}^2 + 10\|u\|_{L^\infty(\mathbb{R})}^2M(u)
\]
\[
\leq 2F(u) + C(\|u\|_{H^1(\mathbb{R})}^3 + \|u\|_{H^1(\mathbb{R})}^4).
\]
Which together with the bound on $\|u(t)\|_{H^1(\mathbb{R})}$ and the conservation of $F(u(t))$ shows that $\|u(t)\|_{H^2(\mathbb{R})}$ is bounded for all $t$. We conclude using Theorem 6.36, that if $u_0 \in H^2(\mathbb{R})$, then the corresponding solution is defined for all $t$, that is, $u \in C([0, \infty); H^2(\mathbb{R}))$. If $u_0 \in H^k(\mathbb{R})$, with $k \geq 2$, we conclude by Lemma 6.47 that $u \in C([0, T); H^k(\mathbb{R}))$.

6.3.3 Derivation of the KdV equation

The usual mathematical model for water waves is the following. We assume that the water is bounded from below by a rigid flat bottom $z = 0$ and above by a moving boundary $z = \eta(x, y, t)$ (we assume that it the graph of a function). The fluid domain is $\Omega_t := \{(x, y, z) \in \mathbb{R}^3 : 0 < z < \eta(x, y, t)\}$. We denote the surface by $S_t := \{(x, y, z) \in \mathbb{R}^3 : z = \eta(x, y, t)\}$ and the bottom by $B := \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. The flow of the water is described by the velocity field $u = (u_1, u_2, u_3)$. The velocity field satisfies Euler’s equation

$$u_t + (u \cdot \nabla)u = F - \frac{1}{\rho} \nabla p,$$

where $\rho$ is the density, $p$ the pressure and $F$ the body forces acting on the fluid. In addition, the density satisfies the continuity equation

$$\rho_t + \nabla \cdot (\rho u) = 0.$$

For water it is reasonable to assume that the density is constant, in which case the last equation reduces to

$$\nabla \cdot u = 0. \quad (6.22)$$

Moreover, we assume that the only force is gravity, so that $F = (0, 0, -g)$, where $g$ is the gravitational constant of acceleration. We assume in addition that all functions are independent of $y$, so that the waves move only in the horizontal direction $x$ and are uniform in the $y$-direction. The equations of motion then reduce to

$$\begin{cases}
v_x + w_z = 0, \\
v_t + vv_x + wv_z = -\frac{1}{\rho} p_x, \\
w_t + vw_x + wv_z = -\frac{1}{\rho} p_z - g, \quad (6.23)
\end{cases}$$

if we set $u_1 = v$, $u_2 = 0$ and $u_3 = w$. In addition to (6.23) one has to impose boundary conditions at the surface and the bottom. These are usually assumed to be of the form

$$\begin{cases}
w = 0 & \text{on } B, \\
w = \eta_t + v\eta_x & \text{on } S_t, \\
p = p_{\text{atm}} & \text{on } S_t, \quad (6.24)
\end{cases}$$

where $p_{\text{atm}}$ is the constant atmospheric pressure at the water surface. The first condition simply means that there is no flow of water across the bottom. The second condition means that surface moves with the fluid, so that it always contains the same fluid particles. If $(x(t), z(t))$
describes the position of a water particle, then the condition for a water particle to remain on the surface is that \( z(t) = \eta(x(t), t) \). Differentiating this with respect to \( t \), one obtains \( \dot{z}(t) = \eta_t(x(t), t) + \eta_x(x(t), t) \dot{x}(t) \). Since \( \dot{x} = v \) and \( \dot{z} = w \), the second boundary condition follows. There are two important difficulties to note here. First of all, the domain \( \Omega_t \) depends on \( t \) since the surface is time-dependent. Even more importantly, the surface is not prescribed, it is one of the unknowns of the problem. We therefore have to solve PDE in a domain which is unknown! This is an example of a free boundary problem.

To derive the KdV equation, we assume that the waves we are describing have small amplitude. We therefore introduce a small parameter \( \varepsilon \) (the amplitude parameter) and write the equation for the surface in the form \( z = h_0 + h_0 \varepsilon \eta(x, t) \), where \( h_0 \) is the depth of the undisturbed water. Rescaling the other variables in an appropriate way, one obtains the new set of equations

\[
\begin{align*}
  v_x + w_z &= 0, \\
  v_t + \varepsilon (vv_x + vw_z) &= -p_x, \\
  \varepsilon (w_t + \varepsilon (vw_x + vw_z)) &= -pz, \\
  w &= 0 \quad \text{on } z = 0 \\
  p &= \eta \quad \text{on } z = 1 + \varepsilon \eta, \\
  w &= \eta_x + \varepsilon \eta_x \quad \text{on } z = 1 + \varepsilon \eta.
\end{align*}
\]

The appropriate change of variables is

\[
\begin{align*}
  x &= \frac{h_0}{\sqrt{\varepsilon}} \tilde{x}, \\
  z &= h_0 \tilde{z}, \\
  t &= \frac{h_0}{g} \frac{1}{\sqrt{\varepsilon}} \tilde{t}, \\
  v &= \sqrt{gh_0 \varepsilon} \tilde{v}, \\
  w &= \sqrt{gh_0 \varepsilon^{3/2}} \tilde{w}, \\
  p &= p_{\text{atm}} + \rho gh_0 (1 - \tilde{z}) + \rho gh_0 \varepsilon \tilde{p} \\
  \eta &= h_0 + h_0 \varepsilon \tilde{\eta}.
\end{align*}
\]

Equations (6.25) are obtained from (6.23)–(6.24) by using the tilde-variables. We have dropped the tildes in (6.25) for notational convenience. For an explanation of this change of variables we refer to Johnson [15].

To derive the leading order approximation we set \( \varepsilon = 0 \). This yields \( v_x + w_z = 0, v_t = -p_x, p_z = 0 \) and the boundary conditions \( w = 0 \) at \( z = 0 \) and \( p = \eta, w = \eta_x \) at \( z = 1 \). A calculation which is left as an exercise to the reader then shows that

\[ \eta_{tt} = \eta_{xx} \]

(the crucial point is that \( p(x, z, t) = \eta(x, t) \)). In other words, the surface profile satisfies the wave equation to leading order. This suggests the introduction of new variables

\[ \xi = x - t, \quad \tau = \varepsilon t, \]

meaning that we are seeking a perturbation of a wave moving to the right with unit speed\(^7\). The equations (6.25) are rewritten in the form

\[
\begin{align*}
  v_\xi + w_\zeta &= 0, \\
  -v_\xi + \varepsilon (v_\tau + vv_\xi + vw_\zeta) &= -p_\xi, \\
  \varepsilon \{ -w_\xi + \varepsilon (-w_\tau + vw_\xi + vw_\zeta) \} &= -p_\zeta, \\
  w &= 0 \quad \text{on } z = 0, \\
  p &= \eta \quad \text{on } z = 1 + \varepsilon \eta, \\
  w &= -\eta_\xi + \varepsilon (\eta_\tau + v \eta_\xi) \quad \text{on } z = 1 + \varepsilon \eta.
\end{align*}
\]

\(^7\)A similar analysis can be made for waves moving to the left.
A solution is sought in the form of asymptotic expansions

\[ v = v_0 + \varepsilon v_1 + \cdots, \quad w = w_0 + \varepsilon w_1 + \cdots, \quad \eta = \eta_0 + \varepsilon \eta_1 + \cdots, \quad p = p_0 + \varepsilon p_1 + \cdots. \]

The KdV equation is obtained by substituting this in the equations and collecting terms of the same order in \( \varepsilon \). The boundary conditions on the surface \( z = 1 + \varepsilon \eta \) have to be handled by expanding \( p(\xi, 1 + \varepsilon \eta, \tau) = p(\xi, 1, \tau) + \varepsilon p_1(\xi, 1, \tau) \eta + O(\varepsilon^2) \) and similarly for the other variables. Keeping this in mind, one obtains at order 0

\[
\begin{align*}
  v_0 \xi + w_{0z} &= 0, \\
  -v_0 \xi &= -p_{0\xi}, \\
  p_{0z} &= 0,
\end{align*}
\]

which is solved by

\[ p_0 = \eta_0 = v_0, \quad w_0 = -z \eta_0 \xi \]

(one can add an arbitrary function of \((z, \tau)\) to \(v_0\)). Using this, one obtains at the next order

\[
\begin{align*}
  v_1 \xi + w_{1z} &= 0, \\
  -v_1 \xi + v_0 \tau + v_0 v_0 \xi &= -p_1 \xi, \\
  p_{1z} &= w_{0\xi},
\end{align*}
\]

with \( w_{1z} = v_1 \xi = v_0 \tau + v_0 v_0 \xi + p_1 \xi = \eta_0 \tau + \eta_0 \eta_0 \xi + \frac{1}{2} (1 - \varepsilon^2) \eta_0 \xi \xi \xi + \xi_1 \xi \), from which it follows that \( p_1 = \frac{1}{2} (1 - \varepsilon^2) \eta_0 \xi \xi \xi + 1 \). From this it follows that

\[
- w_{1z} = v_1 \xi = v_0 \tau + v_0 v_0 \xi + p_1 \xi = \eta_0 \tau + \eta_0 \eta_0 \xi + \frac{1}{2} (1 - \varepsilon^2) \eta_0 \xi \xi \xi + 1 \xi_1, 
\]

which in combination with the bottom boundary condition gives

\[ w_1 = -(\eta_0 \tau + \eta_0 \eta_0 \xi + \eta_1 \xi + \frac{1}{2} \eta_0 \xi \xi \xi + \frac{1}{6} \varepsilon^3 \eta_0 \xi \xi \xi \xi) z + \frac{1}{6} \varepsilon^3 \eta_0 \xi \xi \xi \xi \]

Substituting this in the surface boundary condition, one finally arrives at

\[-(\eta_0 \tau + \eta_0 \eta_0 \xi + \frac{1}{2} \eta_0 \xi \xi \xi + \frac{1}{6} \eta_0 \xi \xi \xi \xi - \eta_0 \eta_0 \xi \xi \xi \xi = \eta_0 \tau + \eta_0 \eta_0 \xi \]

\[ 2 \eta_0 \tau + 3 \eta_0 \eta_0 \xi + \frac{1}{3} \eta_0 \xi \xi \xi \xi = 0. \]

Up to a simple rescaling this is the KdV equation.

### 6.4 Further reading

The material presented in this chapter can only be considered as a first glimpse of the subject. In particular, we have not made any serious use of the properties of the propagator \( S(t) \) for the...
linear problems when considering the nonlinear Cauchy problem. We only used the boundedness of $S(t)$ on $H^s(\mathbb{R}^d)$. Our approach could therefore essentially be used for any equation (or even system) $u_t - a(D_x)u = f(u, D_x u)$ where the operator $a(D_x)$ satisfies Petrowsky’s condition from Chapter 3 and $f$ e.g. is a polynomial. This includes the conservation law $u_t + f(u)_x = 0$ from Chapter 4, for which $a(D_x) = 0$. With the technique used to treat the KdV equation one can prove local well-posedness. The method is of course more complicated than the method of characteristics, but has the advantage that it extends to higher dimensions and to systems.

In the last decades, an important development has been the use of finer properties of $S(t)$ in order to lower the regularity assumptions in the well-posedness results. Taking the NLS equation as an example, the basic ingredient is the estimate

$$||S(t)u_0||_{L^p(\mathbb{R}^d)} \leq (4\pi|t|)^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} ||u_0||_{L^p(\mathbb{R}^d)}$$

from Proposition 6.13. This estimate can be used to prove the continuity of the map

$$u_0 \mapsto S(t)u_0$$

from $L^2(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d; L^r(\mathbb{R}^d))$ under certain conditions on $q$ and $r$. One also has continuity of the operator

$$f \mapsto \int_0^t S(t - \tau)f(\tau) d\tau$$

from $L^q((0,T); L^p(\mathbb{R}^d))$ into $L^q(0,T; L^r(\mathbb{R}^d)) \cap C([0,T]; L^2(\mathbb{R}^d))$ under appropriate conditions on the exponents. These types of estimates go under the name of Strichartz estimates. They can e.g. be used to prove the local well-posedness of the NLS equation in $L^2(\mathbb{R}^d)$ under the condition $1 < p < 1 + 4/d$ on the exponent. The idea is to replace the space $X$ in the proof of Theorem 6.18 with a space which is compatible with the Strichartz estimates. The great advantage of having such a local well-posedness result is that it automatically implies global existence, irrespective of the sign of $\sigma$, due to the conservation of the $L^2$ norm.

For the KdV equation, the Strichartz estimates do not suffice to lower the regularity requirements. The problem is as usual the derivative in the nonlinearity. To handle this, one needs smoothing properties of the operator $f \mapsto \int_0^T S(t - \tau)f(\tau) d\tau$. A very simple example of such a smoothing result is the estimate

$$||\partial_x S(t)u_0||_{L^q(\mathbb{R}^d; L^q(\mathbb{R}^d))} \leq C||u_0||_{L^2(\mathbb{R})},$$

in which the indices in the left hand side indicate that we first integrate in $t$ and then take the essential supremum in $x$. This estimate follows by writing

$$\partial_x S(t)u_0(x) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{it\xi^3 + ix\xi} i\xi \hat{u}_0(\xi) d\xi = \frac{1}{3\sqrt{2\pi}} \int_\mathbb{R} e^{i\eta} e^{i\eta^{1/3}} i\eta^{-1/3} \hat{u}_0(\eta^{1/3}) d\eta$$

and applying Parseval’s formula with respect to the $t$ variable.

For details on well-posedness results with low regularity and applications to global existence we refer to Linares & Ponce [19], Sulem & Sulem [26] and Tao [27]. The derivation of the KdV equation from the governing equations for water waves is essentially taken from Johnson [15].
Chapter 7

Solitons and complete integrability

In the final chapter we will consider some of the remarkable properties of the KdV equation alluded to in the introduction and the previous chapter. These properties can be summarized by saying that the equation is ‘completely integrable’. It isn’t easy to give a precise meaning to this phrase, but it involves the existence of infinitely many independent conserved quantities and a way to transform the equation into a linear problem. In the context of finite-dimensional Hamiltonian systems (ODE) one can give a more precise definition (see [2]). There are many other interesting examples of completely integrable PDE, e.g. the cubic nonlinear Schrödinger equation in one dimension, but most equations do not have this property. We will focus on the KdV equation here for simplicity. The considerations in this chapter will mostly be of a formal nature. In other words, we will not specify the function spaces that the solutions have to belong to for the results to be true. It is useful in this chapter to rewrite the KdV equation in the equivalent form

\[ u_t - 6uu_x + u_{xxx} = 0, \]

which is obtained using the transformation \( u \rightarrow -u \). Transformations between different PDE play a central role in this chapter. As a first illustration, we consider the Cole-Hopf transformation.

**Example 7.1.** Consider the viscous Burgers equation

\[ u_t + uu_x = \varepsilon u_{xx}, \quad \varepsilon > 0. \]

Let \( w(x,t) = \int_0^x u(s,t) \, ds + g(t) \). If \( g \) is chosen appropriately, one finds that

\[ w_t + \frac{1}{2} w_x^2 = \varepsilon w_{xx} \]

(show this!). Setting \( v(x,t) = e^{-w(x,t)/2\varepsilon} \), one obtains

\[ v_t = -\frac{w_t}{2\varepsilon} v, \quad v_x = -\frac{w_x}{2\varepsilon} v \quad (7.1) \]

and

\[ v_{xx} = \left( -\frac{w_{xx}}{2\varepsilon} + \frac{w_x^2}{(2\varepsilon)^2} \right) v \]

\( 127 \)
so that
\[ v_t - \varepsilon v_{xx} = -\frac{1}{2\varepsilon} \left( w_t + \frac{1}{2} w_x^2 - \varepsilon w_{xx} \right) v = 0. \]

The transformation \( u \mapsto v \) is called the \textit{Cole-Hopf transformation}. Note that this transforms the \textit{nonlinear} Burgers equation to the \textit{linear} heat equation.

The existence of the Cole-Hopf transformation is very surprising and seems almost like magic. There is a way to transform the KdV equation into a linear problem, but as we will see, it is much more complicated. There is however a transformation of the KdV equation to a different nonlinear equation which will turn out to be very useful.

### 7.1 Miura’s transformation, Gardner’s extension and infinitely many conservation laws

The modified KdV equation (mKdV)
\[ u_t - 6u^2u_x + u_{xxx} = 0 \]
is obtained by changing the power of the nonlinearity in KdV. Suppose that
\[ u = v^2 + v_x. \quad (7.2) \]

Then
\[ u_t - 6uu_x + u_{xxx} = (2vv_t + v_{xt}) - 6(v^2 + v_x)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2v v_{xxx} + v_{xxxx}, \]
so that \( u \) solves the KdV equation if \( v \) is a solution of the mKdV equation (but not necessarily vice versa). The transformation \( v \mapsto u \) is known as the \textit{Miura transformation}. So far this doesn’t seem to gain very much since the mKdV equation is just as complicated as the KdV equation. We shall soon see that it in fact gives rise to an infinite number of conservation laws for the KdV equation, but first we need an extension due to Gardner. The idea is to introduce a parameter \( \varepsilon \). \textit{Gardner’s transformation} is
\[ u = v + \varepsilon v_x + \varepsilon^2 v^2. \quad (7.3) \]

Substitution of this in the KdV equation gives
\[ u_t - 6u^2u_x + u_{xxx} = (1 + \varepsilon \partial_x + 2\varepsilon^2 v)(v_t - 6(v + \varepsilon^2 v^2)v_x + v_{xxx}). \]
Hence, \( u \) is a solution of the KdV equation if \( v \) is a solution of the \textit{Gardner equation}
\[ v_t - 6(v + \varepsilon^2 v^2)v_x + v_{xxx} = 0. \]

This is in conservation form and hence
\[ \int_{\mathbb{R}} v \, dx \]
is a conserved quantity. For $\varepsilon = 0$ we have $v = u$, in which this is just the conservation of $I(u)$ (see Section 6.3). The idea is now to expand $v$ in powers of $\varepsilon$. Assume that $v$ can be expanded in a series

$$v = \sum_{n=0}^{\infty} \varepsilon^n v_k.$$  

Substituting this in (7.3), we obtain

$$u = \sum_{k=0}^{\infty} \varepsilon^k v_k + \varepsilon^0 \sum_{k=0}^{\infty} \varepsilon^k \partial_x v_k + \varepsilon^2 \left( \sum_{k=0}^{\infty} \varepsilon^k v_k \right)^2$$  

$$= \sum_{k=0}^{\infty} \varepsilon^k v_k + \varepsilon^0 \sum_{k=0}^{\infty} \varepsilon^k \partial_x v_k + \varepsilon^2 \sum_{k=0}^{\infty} \varepsilon^k \left( \sum_{j=0}^{k} v_{k-j} v_j \right)$$  

$$= v_0 + \varepsilon (v_1 + \partial_x v_0) + \sum_{k=2}^{\infty} \varepsilon^k \left( v_k + \partial_x v_{k-1} + \sum_{j=0}^{k-2} v_{k-2-j} v_j \right).$$

Thus

$$v_0 = u, \quad v_1 = -\partial_x v_0 = -u_x, \quad (7.4)$$

and

$$v_k = -\left( \partial_x v_{k-1} + \sum_{j=0}^{k-2} v_{k-2-j} v_j \right), \quad k \geq 2. \quad (7.5)$$

The first few expressions are easily computed. We have

$$v_2 = u_{xx} - u^2,$$

$$v_3 = -(u_{xx} - u^2)_x + 2uu_x,$$

$$v_4 = -(2uu_x - (u_{xx} - u^2)_x)_x - 2u(u_{xx} - u^2) - u_x^2.$$

The fact that $\int_R v \, dx = \text{const}$ moreover implies that

$$\int_R v_k(t, x) \, dx = \text{const}, \quad k \in \mathbb{N}.$$  

In general, it turns out that the relations corresponding to odd powers of $\varepsilon$ are trivial (the corresponding $v_k$ are exact differentials), whereas the even ones induce an infinite family of nontrivial conserved quantities [5]. Note that $v_2$ and $v_4$ correspond to the functionals $M(u)$ and $E(u)$ discussed in the previous chapter, respectively.

### 7.2 Soliton solutions

We now show how the Miura transformation can be used to generate new solutions of the KdV equation from known solutions. The basic idea is that the mKdV equation has a symmetry which the KdV equation does’ have: it is invariant under the reflection $v \mapsto -v$. In other
words, if \( v \) solves the mKdV equation, then so does \(-v\). In addition, the Miura transformation is not injective. For example, any solution of the Riccati equation \( v^2 + v_x = 0 \) is mapped by the Miura transformation to the 0 solution of the KdV equation. We could therefore in principle generate a new solution of the KdV equation by mapping \( u_1 = 0 \) to a solution \( v_1 \) of the Riccati equation, setting \( v_2 = -v_1 \) and \( u_2 = v_2^2 + v_{2x} = v_1^2 - v_{1x} \). One problem is that solutions of the KdV equation are not automatically mapped to solutions of the mKdV equation by the (inverse) Miura transformation. There is however a free parameter in the solution of the Riccati equation, which can be adjusted to guarantee that \( v_1 \) solves mKdV. Another problem is that the solution \( v_1 \) is only local, unless it vanishes identically. To remedy this, we introduce a parameter \( \lambda \) in Miura’s transformation, so that

\[
    u = \lambda + v^2 + v_x.
\]

The modified KdV equations is now replaced by

\[
    v_t - 6(v^2 + \lambda) v_x + v_{xxx} = 0,
\]

(meaning as usual that \( u \) solves the former equation if \( v \) satisfies the latter. The above idea for generating new solutions can now be summarized as follows.

1. Start with a known solution \( u_1 \) of the KdV equation and let \( v(x, t; f) \) be the general solution of the equation \( \lambda + v_x + v^2 = u_1 \). This will in general depend upon an arbitrary function \( f = f(t) \).

2. Adapt the function \( f \) so that \( v \) solves equation (7.6).

3. Set \( u_2 = \lambda + v^2 - v_x \). \( u_2 \) is a new solution of the KdV equation.

In the second step we note that (7.6) can be simplified using the identity \( v_x = -\lambda - v^2 + u_1 \). Indeed, we have \( v_{xx} = -2vv_x + u_{1x} \) and \( v_{xxxx} = -2v^2 - 2vv_{xx} + u_{1xx} = -2v^2 + 4v^2v_x - 2vu_{1x} + u_{1xxx} \). We can therefore rewrite (7.6) as

\[
    v_t = 6(v^2 + \lambda) v_x + 2v_x^2 - 4v^2 v_x + 2vu_{1x} - u_{1xx} \\
    = 2(v^2 + v_x) v_x + 6\lambda v_x + 2vu_{1x} - u_{1xx} \\
    = 2(u_1 - \lambda) v_x + 6\lambda v_x + 2vu_{1x} - u_{1xx} \\
    = 4\lambda v_x + 2(u_1 v)_x - u_{1xx}.
\]

**Example 7.2.** We illustrate the method for \( u_1 = 0 \). We should then choose \( v \) as a solution of the ordinary differential equation

\[
    v_x + v^2 + \lambda = 0.
\]

This can be integrated to

\[
    v(x, t) = \kappa \tanh(\kappa x + f(t)),
\]

where \( \lambda = -\kappa^2 \) (\(<0\)) and \( f \) is arbitrary. The equation for \( v_t \) becomes

\[
    v_t = -4\kappa^2 v_x,
\]
and hence \( f'(t) = -4\kappa^3 \), so that \( f(t) = -4\kappa^3 t - \kappa x_0 \), where \( x_0 \) is arbitrary. We therefore obtain the new solution

\[
u_2(x,t) = -\kappa^2 + v^2(x,t) - v_x(x,t) = -\frac{2\kappa^2}{\cosh^2(\kappa(x - x_0 - 4\kappa^2 t))}.
\]

This is the solitary wave from the introduction with wave speed \( c = 4\kappa^2 \) (recall that we changed \( u \) to \( -u \) in the KdV equation). This solution is valid only if \(|v| < \kappa\). If \(|v| > \kappa\), we obtain the singular solution

\[
v(x,t) = -\kappa \coth(\kappa(x - x_0 - 4\kappa^2 t)).
\]

The above transformation of solutions of the KdV equation to other solutions is an example of a Bäcklund transformation. Assume that \( u \) and \( v \) are functions of two variables \( x \) and \( t \) and that

\[
P(u,u_x,u_t,\ldots;x,t) = 0 \quad \text{and} \quad Q(v,v_x,v_t,\ldots;x,t) = 0
\]

are two uncoupled partial differential equations. A Bäcklund transformation for \( P \) and \( Q \) is a system of differential equations

\[
F_j(u,v,u_x,v_x,u_t,v_t,\ldots;x,t) = 0, \quad j = 1,2,
\]

which is solvable for \( u \) if and only if \( Q(v) = 0 \) and has the property that \( u \) then solves \( P(u) = 0 \), and vice versa. If \( P = Q \), then \( F_j = 0 \) is called an auto-Bäcklund transformation.

**Example 7.3.** The simplest example of a Bäcklund transformation is the Cauchy-Riemann system

\[
\begin{align*}
u_t &= v_y, \\
u_y &= -v_x,
\end{align*}
\]

which is an auto-Bäcklund transformation for Laplace’s equation.

The above transformation for the KdV equation may also be seen as an auto-Bäcklund transformation. Introducing two functions \( w_1 \) and \( w_2 \) such that \( w_{jx} = u_j \), the system defining the Bäcklund transformation can be written

\[
\begin{align*}(w_1 + w_2)_x &= 2\lambda + \frac{1}{2}(w_1 - w_2)^2, \\
(w_1 - w_2)_t &= 3(w_{1x}^2 - w_{2x}^2) - (w_1 - w_2)_{xxx}.
\end{align*}
\] (7.7)

The relation with (7.6) is that \( 2v = w_1 - w_2 \) is a solution and \( u_1 = \lambda + v^2 + v_x \), while \( u_2 = \lambda + v^2 - v_x \). The amazing thing about the auto-Bäcklund transformation is that we can continue generating new solutions from the solitary wave. In practice this turns out to be a bit difficult since we have to integrate two ordinary differential equations in each step (one in \( x \) and one in \( t \)). There is an elegant way to avoid this. The idea is to use the parameter \( \lambda \). For this it is convenient to work with (7.7). We first relabel the solutions. Let \( w_0 \) be the known solution.
We solve equation (7.7) with \( w_2 \) replaced by \( w_0 \) and with two different values of \( \lambda \), say \( \lambda_1 \) and \( \lambda_2 \). The two solutions are denoted \( w_1 \) and \( w_2 \), so that

\[
(w_1 + w_0)_x = 2\lambda_1 + \frac{1}{2}(w_1 - w_0)^2, \tag{7.8}
\]

\[
(w_2 + w_0)_x = 2\lambda_2 + \frac{1}{2}(w_2 - w_0)^2 \tag{7.9}
\]

Using these solutions, we define \( w_{12} \) and \( w_{21} \) by

\[
(w_{12} + w_1)_x = 2\lambda_2 + \frac{1}{2}(w_{12} - w_1)^2, \tag{7.10}
\]

\[
(w_{21} + w_2)_x = 2\lambda_1 + \frac{1}{2}(w_{21} - w_2)^2 \tag{7.11}
\]

**Theorem 7.4** (Permutability theorem). Assume that \( w_1 \) and \( w_2 \) are solutions of (7.8)–(7.9). There exists solutions \( w_{12} \) and \( w_{21} \) of (7.10)–(7.11) with \( w_{12} = w_{21} \).

Such a result was originally proved for a similar problem in differential geometry by Bianchi. Assume for a moment that the theorem is true, so that \( w_{12} = w_{21} = w \). Forming the combination (7.11)–(7.10)–(7.9)–(7.8), we deduce that

\[
0 = 4(\lambda_1 - \lambda_2) + \frac{1}{2}((w - w_2)^2 - (w - w_1)^2 - (w_2 - w_0)^2 + (w_1 - w_0)^2).
\]

Solving for \( w \), we obtain the formula

\[
w = w_0 - \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2}. \tag{7.12}
\]

This may be seen as a nonlinear superposition principle. From the two solutions \( u_1 \) and \( u_2 \) corresponding to \( w_1 \) and \( w_2 \), we obtain a new solution \( u = w_x \).

**Example 7.5.** We take \( w_0 = 0 \), \( w_1 = -2\tanh(x - 4t) \) and \( w_2 = -4\coth(2x - 32t) \) so that \( \lambda_1 = -1 \) and \( \lambda_2 = -4 \). We therefore obtain

\[
w(x, t) = -\frac{6}{2\tanh(2x - 32t) - \tanh(x - 4t)}.
\]

The corresponding solution of the KdV equation \( u(x, t) = w_x(x, t) \) is (after some simplifications) given by

\[
u(x, t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{(3\cosh(x - 28t) + \cosh(3x - 36t))},
\]

This is a so-called two-soliton solution. As \( t \to \pm\infty \), the solution has the asymptotic form

\[
u(x, t) \sim -8 \text{sech}^2(2(x - 16t) \mp \frac{1}{2}\log 3) - 2 \text{sech}^2(x - 4t \pm \frac{1}{2}\log 3).
\]

The solution can therefore be interpreted as describing the interaction of two solitary waves \(-8 \text{sech}^2(2(x - 16t - \frac{1}{2}\log 3))\) and \(-2 \text{sech}^2(x - 4t + \frac{1}{2})\). For large negative \( t \), the solution is basically a linear combination of these two solutions. This is not surprising since the solitary waves are very localized and are far apart. For \( t \) close to 0 the solutions interact in a nonlinear wave. The amazing thing is that the the waves come out of the interaction looking exactly the same as before. The only difference is a phase shift, so that the waves are not at the positions one would expect if the interaction were linear. It is this property that gave rise to the name solitons; the solitary waves interact almost like particles. Figure 7.2 illustrates the soliton interaction.
Figure 7.1: A two-soliton solution – before, during and after interaction. Note that $u$ has been changed to $-u$, so that this is a solution of $u_t + 6uu_x + u_{xxx} = 0$. 
Proof of Theorem 7.4. To check the validity of the permutability theorem, we simply substitute $w$ given by (7.12) back into equations (7.10) and (7.11). This results in

$$(w_1 + w_0)_x + \frac{4(\lambda_1 - \lambda_2)}{(w_1 - w_2)^2}(w_1 - w_2)_x = 2\lambda_2 + \frac{1}{2} \left( w_1 - w_0 + \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2} \right)^2$$

and

$$(w_2 + w_0)_x + \frac{4(\lambda_1 - \lambda_2)}{(w_1 - w_2)^2}(w_1 - w_2)_x = 2\lambda_1 + \frac{1}{2} \left( w_2 - w_0 + \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2} \right)^2$$

We now use (7.8) and (7.9) to rewrite this as

$$\frac{4(\lambda_1 - \lambda_2)}{(w_1 - w_2)^2}(w_1 - w_2)_x = 2(\lambda_2 - \lambda_1) + \frac{4(\lambda_1 - \lambda_2)(w_1 - w_0)}{w_1 - w_2} + \frac{1}{2} \left( \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2} \right)^2,$$

or equivalently,

$$(w_1 - w_2)_x = -\frac{1}{2}(w_1 - w_2)^2 + (w_1 - w_2)(w_1 - w_0) + 2(\lambda_1 - \lambda_2),$$

and

$$(w_1 - w_2)_x = \frac{1}{2}(w_1 - w_0)^2 - (w_2 - w_0)^2) + 2(\lambda_1 - \lambda_2).$$

This is nothing but the difference between (7.8) and (7.9). We also need to verify the second part of the Bäcklund transformation That is, we need to show that

$$w_t - 3w_x^2 + w_{xxx} = w_{2t} - 3w_{2x}^2 + w_{2xxx} = 0,$$

where we have used the fact that $w_{2x}$ solves the KdV equation. An argument similar to the one above shows that this is satisfied. The calculations are straightforward but rather lengthy. We leave them to the interested reader.

7.3 Lax pairs

We now return to the problem of transforming the KdV equation to a linear problem. As we will see, the Miura transformation is useful also in this aspect.

7.3.1 The scattering problem

In the last section, we sometimes treated (7.2) as an equation in the unknown $v$ for a given $u$. We mentioned that this is a Riccati equation, a type which can always be rewritten as a second order linear ordinary differential equation through the introduction of a potential $\varphi$ such that

$$v = \frac{\varphi'}{\varphi} = \partial_t \ln(\varphi).$$
This leads to

\[ u = \partial_x \left( \frac{\varphi_x}{\varphi} \right) + \left( \frac{\varphi_x}{\varphi} \right)^2 = \frac{\varphi_{xx} \varphi - \varphi_x^2}{\varphi^2} + \frac{\varphi_x^2}{\varphi}, \]

so that

\[ \varphi_{xx} - u \varphi = 0, \]

at least as long as \( \varphi \neq 0 \). Ingeniously, Gardner, Greene, Kruskal and Miura generalized this, and went on to investigate the \textit{scattering problem}

\[ \psi_{xx} + (\lambda - u) \psi = 0. \] (7.13)

For every fixed \( t \), and equipped with appropriate boundary conditions, this is a \textit{Sturm–Liouville problem}. For given \( u \) and for every \( t \), there are only certain \( \lambda \) (called \textit{eigenvalues}) that admit solutions \( \psi \) (called \textit{eigenfunctions}). Furthermore, it turns out that (7.13) constitutes an implicit linearization of the KdV equation: the evolution of \( u(x, t) \) according to the KdV equation, forces \( \lambda(t) \) and \( \psi(x, t) \) to evolve in specific way, which will be calculated below.

### 7.3.2 The evolution equation

What happens when one substitutes (7.13) into the KdV equation? To simplify calculations, define

\[ R(x, t) := \psi_t(x, t) + u_x(x, t) \psi(x, t) - 2(u(x, t) + 2\lambda(t)) \psi_x(x, t) \]

and let

\[ W(f, g) := fg_x - f_x g \]

be the \textit{Wronskian} of two functions \( f \) and \( g \) (with respect to \( x \)). Then

\[
\begin{align*}
W(\psi, R) &= \psi^2 \left( \frac{R}{\psi} \right)_x \\
&= \psi^2 \left( \psi_t + u_x \psi - 2(2 + 2\lambda) \psi_x \right)_x \\
&= \psi_{tx} \psi - \psi_t \psi_x + u_{xx} \psi^2 + u_x \psi_x \psi - u_x \psi \psi_x - 2u_x \psi \psi_x \\
&- 2(u + 2\lambda) \psi_{xx} \psi + 2(u + 2\lambda) \psi_x^2 \\
&= \psi_{tx} \psi - \psi_t \psi_x + u_{xx} \psi^2 - 2u_x \psi \psi_x + 2(u + 2\lambda) \left( \psi_x^2 - \psi_{xx} \psi \right),
\end{align*}
\]
and

\[ \partial_x W(\psi, R) = \psi_{xx} \psi + \psi_{x} \psi_x - \psi_{xx} \psi_x - \psi_t \psi_{xx} + u_{xxx} \psi^2 + 2u_{xx} \psi \psi_x \]

\[ - 2u_{xx} \psi_x - 2u_x \psi_{xx} \psi - 2u_x \psi^2 + 2u \left( \psi^2 - \psi_{xx} \psi \right) + 2(u + 2\lambda) \left( 2\psi_x \psi_{xx} - \psi_{xxxx} \psi - \psi_{xx} \psi_x \right) \]

\[ = \psi^2 \left( \frac{\psi_{xx}}{\psi} \right)_t + u_{xxx} \psi^2 - 4u_x \psi_{xx} \psi + 2(u + 2\lambda) \left( \psi_{xx} \psi - \psi_{xxxx} \psi \right) \]

\[ = \psi^2 (u - \lambda)_x + u_{xxx} \psi^2 - 4u_x \psi^2 (u - \lambda) - 2(u + 2\lambda) \psi^2 \left( \frac{\psi_{xx}}{\psi} \right)_x \]

\[ = \psi^2 ((u - \lambda)_x + u_{xxx} - 4u_x (u - \lambda) - 2(u + 2\lambda) (u - \lambda)_x) \]

\[ = \psi^2 (u_t - \lambda_t + u_{xxx} - 6uu_x). \]

Thus, if (7.13) holds, then

\[ \psi^2 \partial_x \lambda = \partial_x W(R, \psi) \quad \text{exactly when} \quad u_t + u_{xxx} = 6uu_x. \]

In particular, if \( W(\psi, R) \) has decay, then we may integrate over \( \mathbb{R} \) to obtain that

\[ \lambda_t \int_{\mathbb{R}} \psi^2 \, dx = 0. \]

This is an unexpected and amazing result: if \( u(x, t) \) solves the KdV equation, then the eigenvalues \( \lambda \) of (7.13) are time-independent, although both \( u \) and \( \psi \) depend on time. Moreover, \( W(\psi, R) \) is a function of time only, and thus

\[ W(\psi, R) = W(\psi, R) \bigg|_{x=\infty} = 0, \]

if we assume, for example, that \( \psi(\cdot, t) \in \mathcal{S}(\mathbb{R}) \). Put differently,

\[ \psi^2 \partial_x \left( \frac{R}{\psi} \right) = 0, \]

so that \( R/\psi \), too, is a function of time only. Given that

\[ \frac{R}{\psi} \rightarrow C \quad \text{as} \quad x \rightarrow \infty, \]

with \( C \) independent of time, we can deduce the following evolution equation for \( \psi \):

\[ \psi_t + u_x \psi - 2(u + 2\lambda) \psi_x = C \psi. \]

(7.14)

Thus, at least formally, we have related the KdV equation in \( u \) to a linear problem in \( \psi \):

\[ \begin{align*}
\psi_{xx} + (\lambda - u) \psi &= 0, \\
\psi_t + (u_x - C) \psi - 2(u + 2\lambda) \psi_x &= 0.
\end{align*} \]

(7.15)
Given (7.15), one can show (do this!) that the compatibility condition

\[ \psi_{xx} = \psi_{xxt}, \]

is equivalent to the relation

\[ \psi (u_t - \lambda_t + u_{xxx} - 6uu_x) = 0, \]

found above. One sometimes says that the KdV equation is the compatibility condition for the system (7.15), meaning that \( \lambda = \text{const} \) in (7.15) if and only if \( u \) satisfies the KdV equation.

### 7.3.3 The relation to Lax pairs

The formulation (7.15) can be used to solve the KdV equation by means of the inverse scattering transform. The idea is basically that one starts with the initial datum \( u_0 \), and calculates the corresponding eigenvalues \( \lambda \) and eigenfunctions \( \psi \). One then let these evolve in time, and at a later point in time tries to recover \( u \) from \( \lambda \) and \( \psi \). Note, however, that the evolution equation for \( \psi \) involves \( u \). It is therefore not feasible to keep track of the actual eigenfunctions. Moreover, for eigenvalue problems such as the one above, the spectrum does not only consist of eigenvalues, but also of so called continuous spectrum. It turns out, however, that the function \( u \) in the scattering problem can be determined from the eigenvalues and some additional constants, whose evolution is also known. For an introduction to this beautiful subject we refer to [5]. The scattering/inverse scattering approach to the KdV equation seems like magic, and it was for some time unclear if it applied to other equations. In 1968, Peter Lax discovered an abstract structure behind the calculations above. This structure is called a Lax pair in his honour. There are by now many examples of equations having Lax pairs.

One first notes that (7.15) can be reformulated as

\[
\begin{align*}
L \psi &= \lambda \psi, \\
\psi_t &= B \psi,
\end{align*}
\]  

(7.16)

where \( L \) is the linear and symmetric operator in the scattering problem, and \( B \) is the corresponding evolution operator. Now, assume that \( \partial_t \lambda = 0 \). Taking the time-derivative of \( L \psi = \lambda \psi \), we then get that

\[
\begin{align*}
0 &= L_t \psi + L \psi_t - \lambda \psi_t \\
&= L_t \psi + LB \psi - \lambda B \psi \\
&= L_t \psi + LB \psi - B \lambda \psi \\
&= (L_t + LB - BL) \psi.
\end{align*}
\]

Denoting \( BL - LB \) by \( [B, L] \), we see that

\[ L_t = [B, L], \]

for all \( \psi \) which solve (7.16).
Contrariwise, suppose that
\[ L_t = [B, L], \]
where \( L \) and \( B \) are spatial, but time-dependent, operators on some Hilbert space \( X \), and \( L \) is linear and symmetric, i.e.
\[ (L\psi, \varphi)_X = (\psi, L\varphi)_X, \quad \psi, \varphi \in X. \]
One may then consider the eigenvalue problem
\[ L\psi = \lambda \psi, \quad 0 \neq \psi \in X. \]
We assume also that \( L, B \) and the eigenfunctions of \( L \) are continuously differentiable with respect to \( t \). Just as above, we find that
\[ L_t \psi + L\psi_t = \lambda_t \psi + \lambda \psi_t, \]
whence
\[ \lambda_t \psi = (L - \lambda) \psi_t + [B, L] \psi \]
\[ = (L - \lambda) \psi_t + B\lambda \psi - LB\psi \]
\[ = (L - \lambda) \psi_t + (\lambda - L)B\psi \]
\[ = (L - \lambda)(\psi_t - B\psi), \quad (7.17) \]
in view of that \( L_t = [B, L] \) and \( L\psi = \lambda \psi \). By using the symmetry of \( L \) it follows that
\[ \lambda_t(\psi, \psi)_X = (\psi, (L - \lambda)(\psi_t - B\psi))_X \]
\[ = ((L - \lambda)\psi, (\psi_t - B\psi))_X \]
\[ = (0, (\psi_t - B\psi))_X = 0. \]
Thus \( \lambda_t = 0 \), and we infer from (7.17) that
\[ (L - \lambda)(\psi_t - B\psi) = 0, \]
i.e. \( \psi_t - B\psi \) is an eigenfunction of \( L \) for the eigenvalue \( \lambda \). In particular, when the eigenspace corresponding to \( \lambda \) is one-dimensional, then
\[ \psi_t - B\psi = f(t) \psi, \]
for some function \( f \) depending only on time. Since \( f \) commutes with the spatial operator \( L \), we find that \( \tilde{B} := B + f \) satisfies both
\[ \psi_t = \tilde{B}\psi \quad \text{and} \quad L_t = [\tilde{B}, L]. \]
7.3.4 A Lax-pair formulation for the KdV equation

**Theorem 7.6.** The KdV equation can be written as the Lax equation \( L_t = [B, L] \) for the Lax pair

\[
L := -\partial_x^2 + u, \quad \text{and} \quad B := -4\partial_x^3 + 6u\partial_x + 3u_x + C, \quad C \in \mathbb{R}.
\]

**Remark 7.7.** Notice that this Lax pair appears already in the linear formulation (7.15) of the KdV equation. This follows, since the scattering problem

\[
\lambda \psi = L\psi = (u - \partial_x^2) \psi,
\]

implies that

\[
\psi_t = -(u_x - C) \psi + 2(u + 2\lambda) \psi_x,
\]

\[
= [-u_x + C + 2u\partial_x + 4\partial_x\lambda] \psi
\]

\[
= [-u_x + C + 2u\partial_x + 4\partial_x(u - \partial_x^2)] \psi
\]

\[
= [-u_x + C + 2u\partial_x + 4(u_x + u\partial_x - \partial_x^3)] \psi
\]

\[
= [-4\partial_x^3 + 6u\partial_x + 3u_x + C] \psi,
\]

and \( C \) is an arbitrary constant.

**Proof.** Elementary calculations yield that

\[
LB = 4\partial_x^5 - 4u\partial_x^3 - 6(u_{xx}\partial_x + 2u_x\partial_x^2 + u\partial_x^3) + 6u^2\partial_x
\]

\[-3(u_{xxx} + 2u_{xx}\partial_x + u_x\partial_x^2) + 3uu_x + CL
\]

\[= 4\partial_x^5 - 10u\partial_x^3 - 15u_x\partial_x^2 - 12u_{xx}\partial_x + 6u^2\partial_x - 3u_{xxx} + 3uu_x + CL.
\]

and

\[
BL = 4\partial_x^5 - 4(u_{xxx} + 3u_{xx}\partial_x + 3u_x\partial_x^2 + u\partial_x^3)
\]

\[-6u\partial_x^3 + 6u(u_x + u\partial_x) - 3u_x\partial_x^2 + 3uu_x + CL
\]

\[= 4\partial_x^5 - 10u\partial_x^3 - 15u_x\partial_x^2 - 12u_{xx}\partial_x + 6u^2\partial_x - 4u_{xxx} + 9uu_x + CL.
\]

We see that

\[ L_t = u_t, \quad \text{and} \quad [B, L] = -u_{xxx} + 6uu_x, \]

whence the Lax equation \( L_t = [B, L] \) is just the KdV equation itself. This explains why the eigenvalues are constant in time, as also explicitly calculated above. \( \square \)
Appendix A

Functional analysis

The basic idea of functional analysis is to consider functions (or other objects) as points or vectors in a vector space (also called a linear space). The space should be equipped with some topology, so that one can discuss things such as convergence and continuity. By proving results based on the structure of the vector space and not on the specific objects under consideration, one can deduce results which apply in many different concrete cases. In most cases discussed in these notes, the topology is defined in terms of a norm. We assume that the reader is familiar with the definition of a vector space. Throughout the chapter, if nothing else is said, $X$ defines a vector space over the real or complex numbers. The letter $K$ will denote $\mathbb{R}$ or $\mathbb{C}$, depending on whether $X$ is real or complex. Since the idea is to give a brief overview, we will not give complete proofs of all the statements. The reader will find proofs in any book on functional analysis, e.g. [16].

A.1 Banach spaces

Definition A.1. A norm on $X$ is a function $\| \cdot \| : X \to [0, \infty)$ such that

1. $\| \lambda x \| = |\lambda| \|x\|$ (positive homogeneity);
2. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality);
3. $\|x\| = 0 \iff x = 0$ (positive definiteness);

for all $x, y \in X$ and $\lambda \in K$. The pair $(X, \| \cdot \|)$ is called a normed vector space.

We will sometimes also make use of functions $\| \cdot \| : X \to [0, \infty)$ which satisfy conditions (1) and (2) of the above definition, but not necessarily (3). Such a function is called a semi-norm. Whenever we have a semi-norm $\| \cdot \|_X$ on $X$, we can however obtain a normed vector space by introducing an equivalence relation. We say that $x \sim y$ if $\|x - y\|_X = 0$. The triangle inequality shows that $\sim$ really defines an equivalence relation. By the reverse triangle inequality, we find that

$$\|\|x\|_X - \|y\|_X \| \leq \|x - y\|_X = 0$$
if \( x \sim y \). We can therefore define a norm \( \| \cdot \|_Y \) by setting \( \|[x]\|_Y = \|x\|_Y \) for the equivalence class \([x] \in Y\) containing the element \( x \in X \) (the above argument shows that it is independent of the choice of representative). We leave it to the reader to verify that \( \| \cdot \|_Y \) really is a norm on \( Y \).

Example A.2.

\[
\|u\| = \sup_{0 \leq x \leq 1} |u(x)|
\]

defines a norm on \( C[0, 1] \), the space of continuous function on the interval \([0, 1]\). It only defines a semi-norm on \( C(\mathbb{R}) \), since any function \( u \) which vanishes outside of the interval \([0, 1]\) satisfies \( \|u\| = 0 \).

From now on we suppose that \( X \) is endowed with a norm \( \| \cdot \| \). Using the norm one can define the open ball \( B_r(x) \) of radius \( r > 0 \) around a point \( x \in X \) as \( B_r(x) = \{ y \in X : \|x - y\| < r \} \). Using these open balls one can define open and closed sets and continuous functions just as in \( \mathbb{R}^d \).

Note that any two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) on a finite-dimensional space are equivalent, in the sense that there exist constants \( C_1, C_2 > 0 \), such that \( C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1 \). This implies that the norms define the same topology, i.e. the same notion of open sets.

Of particular interest are linear operators between normed vector spaces. We have the following basic result.

**Proposition A.3.** Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be normed vector spaces and let \( T : X \to Y \) be a linear operator. The following are equivalent.

1. \( T \) is continuous at some point in \( X \);
2. \( T \) is continuous at every point in \( X \);
3. there exists a constant \( C \geq 0 \) such that
\[
\|Tx\|_Y \leq C \|x\|_X
\]

for all \( x \in X \).

Due to (3), a continuous linear operator \( X \to Y \) is also called a **bounded linear operator**. The smallest possible constant in (3),
\[
\|T\| := \sup_{\|x\|=1} \|Tx\|_Y = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X},
\]
is called the **operator norm** of \( T \). By definition, we have \( \|Tx\|_Y \leq \|T\| \|x\|_X \). Note that the set \( B(X, Y) \) of bounded linear operators \( X \to Y \) is a normed vector space when endowed with the operator norm.
Every linear operator defined on a finite-dimensional vector space is bounded. If $X$ is infinite-dimensional there always exist unbounded operators on $X$, although they can’t be constructed explicitly. The simplest kind of linear operators are the linear functionals. These are operators $X \to K$.

**Definition A.4.** The dual of $X$ is the space of bounded linear functionals on $X$. It is denoted $X^*$. The dual of a finite-dimensional space $X$ can be identified with itself. Indeed, given a basis \( \{e_1, \ldots, e_n\} \) for $X$, any linear functional $\ell$ on $X$ can be written

$$\ell(x_1 e_1 + \cdots + x_n e_n) = c_1 x_1 + \cdots + c_n x_n,$$

where $c_j = \ell(e_j)$. This defines a linear isomorphism $X^* \to \mathbb{K}^n$.

The linear functionals on $X$ induce a topology on $X$ which is different from the one induced by the norm.

**Definition A.5.** We say that a sequence $\{x_n\}_{n=1}^\infty$ in $X$ converges weakly to $x \in X$ if $\ell(x_n) \to \ell(x)$ for every $\ell \in X^*$. We then write $x_n \rightharpoonup x$. The topology defined by weak convergence is called the weak topology on $X$.

This topology is important since it is often much easier to prove weak convergence than strong convergence (convergence in norm). There are many cases where it is enough to have a weak limit.

An important property which separates $\mathbb{R}$ from $\mathbb{Q}$ is its completeness. This can be expressed in many ways, e.g. as the property that an increasing sequence which is bounded from above always has a limit in $\mathbb{R}$ (this is not true in $\mathbb{Q}$ as can be seen e.g. by looking at the decimal expansion of any irrational number). An equivalent formulation of this is that any Cauchy sequence in $\mathbb{R}$ is convergent. This turns out to be the correct generalization. Recall that a Cauchy sequence is a sequence $\{x_n\}$ satisfying $x_n - x_m \to 0$ as $n, m \to \infty$.

**Definition A.6.** $X$ is said to be complete if every Cauchy sequence in $X$ is convergent. A complete normed vector space is also called a Banach space.

**Proposition A.7.** A closed subspace of a Banach space is a Banach space.

The basic example of a function space which a Banach space is the space $C(I)$ of continuous real-valued functions on a compact interval $I$, endowed with the supremum norm. The completeness property basically follows from the fact that $\mathbb{R}$ is complete and that uniform limits of continuous functions are continuous. If $I$ is not compact, one merely has to add the condition $\sup_{t \in I} \|u(t)\|_X < \infty$ to the definition. We denote the space of bounded continuous real-valued functions on $I$ by $C_b(I)$. If $I$ is compact we have that $C_b(I) = C(I)$. This example can be generalized in many ways. One example which we use repeatedly in the notes is the following.
Example A.8. Let $X$ be a Banach space and $I$ a compact interval. Then the space $C(I; X)$ consisting of continuous functions $u : I \to X$ is a Banach space, with norm given by

$$\|u\|_{C(I; X)} = \sup_{t \in I} \|u(t)\|_X.$$  

The proof of completeness is almost word for word the same as in case of real-valued functions. One only has to use the completeness of $X$ instead of the completeness of $\mathbb{R}$. Similarly, for any $k \geq 0$, we introduce the space $C^k(I; X)$ consisting of $k$ times continuously differentiable functions from $I$ to $X$. Here, differentiability of $u$ at $t$ means that there exists an element $v \in X$ such that

$$\lim_{h \to 0} \frac{u(t + h) - u(t)}{h} = v$$

in $X$. $C^k(I; X)$ is also a Banach space when endowed with the norm

$$\|u\|_{C^k(I; X)} = \max_{0 \leq j \leq k} \sup_{t \in I} \|u^{(j)}(t)\|_X.$$  

A.2 Hilbert spaces

A norm often comes from an inner product. We recall the definition.

Definition A.9. An inner product on $X$ is a function $(\cdot, \cdot) : X \times X \to \mathbb{K}$, such that

1. $(\lambda x + \mu y, z) = \lambda (x, y) + \mu (y, z)$ (linearity in the first argument);
2. $(x, y) = \overline{(y, x)}$ (conjugate symmetry);
3. $(x, x) \geq 0$ with equality iff $x = 0$ (positive definiteness);

for every $x, y, z \in X$ and $\lambda, \mu \in \mathbb{K}$. The pair $(X, (\cdot, \cdot))$ is called an inner product space.

Using the conjugate symmetry one finds that $(x, \lambda y + \mu z) = \overline{(x, y)} + \overline{(x, z)}$. One says that the inner product is sesquilinear. If $\mathbb{K} = \mathbb{R}$, then (2) simply means that $(x, y) = (y, x)$ and the inner product is then bilinear. The norm associated with the inner product is $\|x\| = \sqrt{(x, x)}$.

We recall the Cauchy-Schwarz inequality: $|(x, y)| \leq \|x\| \|y\|$, with equality if and only if $x$ and $y$ are linearly dependent.

Definition A.10. A complete inner product space is called a Hilbert space.

Example A.11. The expression $(u, v)_{L^2} = \int_{-1}^{1} u(x)v(x) \, dx$ defines an inner product on $C[-1, 1]$. $C[-1, 1]$ is not a Hilbert space with this inner product. The sequence $u_n(x) = nx/(n|x| + 1)$ is an example of a Cauchy sequence which does not converge to an element of $C[-1, 1]$. Indeed, if $v(x) = 1$ for $x > 0$ and $v(x) = -1$ for $x < 0$, we find that

$$\|u_n - v\|_{L^2}^2 = 2 \int_{0}^{1} \frac{dx}{(nx + 1)^2} = \frac{2}{n + 1} \to 0$$

as $n \to \infty$. It follows that $\|u_n - u_m\|_{L^2} \leq \|u_n - v\|_{L^2} + \|v - u_m\| \to 0$ as $n, m \to \infty$, so that the \{u_n\}_n is a Cauchy sequence. On the other hand the sequence can’t have a limit $u \in C[-1, 1]$ since we would then obtain that $\int_{-1}^{1} |u(x) - v(x)|^2 \, dx = 0$, which would imply that $u(x) = 1$ for $x > 0$ and $u(x) = -1$ for $x < 0$. 

We saw earlier that the dual of a finite-dimensional space can be identified with the space itself. Generally, this is not the case in infinite dimensions. However, for Hilbert spaces it is true. Note that any \( y \in X \) defines a bounded linear functional \( \ell_y \) through the formula
\[
\ell_y(x) = (x, y).
\]
The fact that this a linear functional is obvious; the fact that it is bounded with norm \( \|y\| \) follows from the Cauchy-Schwarz inequality. Riesz' representation theorem asserts that the converse is also true.

**Theorem A.12** (Riesz' representation theorem). Let \((X, \langle \cdot, \cdot \rangle)\) be a Hilbert space. Given any bounded linear functional \( \ell \in X^* \), there exists a unique element \( y = y_\ell \in X \) such that
\[
\ell(x) = (x, y)
\]
for every \( x \in X \). The map \( \Phi : X^* \to X, \ell \mapsto y_\ell \) is an isometric anti-isomorphism, meaning that
1. \( \Phi \) is bijective;
2. \( \|\Phi(\ell)\| = \|\ell\| \);
3. \( \Phi(\lambda \ell_1 + \mu \ell_2) = \lambda \Phi(\ell_1) + \mu \Phi(\ell_2) \);
for all \( \ell, \ell_1, \ell_2 \in X^* \) and \( \mu, \lambda \in \mathbb{K} \).

**A.3 Metric spaces and Banach's fixed point theorem**

When solving nonlinear equations, the most important tool is Banach’s fixed point theorem (also called the contraction principle). This is e.g. the main ingredient in the implicit function theorem and the inverse function theorem. It can also be used to prove the Picard-Lindelöf theorem on existence and uniqueness of solutions of the initial-value problem for ordinary differential equations. Although we will always use it in Banach spaces, it is more natural to present it in the setting of metric spaces.

**Definition A.13.** Let \( X \) be a set. A map \( \rho : X \times X \to [0, \infty) \) is called a metric if
1. \( \rho(x, y) = 0 \iff x = y \);
2. \( \rho(x, y) = \rho(y, x) \);
3. \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) \);
for every \( x, y, z \in X \). \((X, \rho)\) is called a metric space.

Given a metric, one can again define a topology by introducing the open balls \( B_r(x) = \{ y \in X : \rho(x, y) < r \} \). Any norm induces a metric \( \rho(x, y) = \|x - y\| \). Note that we can still define Cauchy sequences in metric spaces by requiring that \( \rho(x_n, x_m) \to 0 \) as \( m, n \to \infty \). A metric space is said to be complete if every Cauchy sequence converges.
**Definition A.14.** Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be metric spaces. A function \(f: X \to Y\) is called a **contraction** if there exists a constant \(\theta \in [0, 1)\) such that

\[
\rho_Y(f(x), f(y)) \leq \theta \rho_X(x, y)
\]

for every \(x, y \in X\).

**Theorem A.15** (Banach’s fixed point theorem). Let \((X, \rho_X)\) be a complete metric space. If \(f: X \to X\) is a contraction, then the fixed point equation

\[
f(x) = x
\]

has a unique solution.

**Proof.** If \(x, y\) are solutions, then

\[
\rho(x, y) = \rho(f(x), f(y)) \leq \theta \rho(x, y) \Rightarrow \rho(x, y) = 0 \Rightarrow x = y.
\]

This proves uniqueness.

To prove the existence of a solution, we pick an arbitrary element \(x_0 \in X\) and set \(x_1 = f(x_0), x_2 = f(x_1), x_3 = f(x_2), \ldots\). We have

\[
\rho(x_{n+1}, x_n) = \rho(f(x_n), f(x_{n-1})) \leq \theta \rho(x_n, x_{n-1})
\]

and hence \(\rho(x_{n+1}, x_n) \leq \theta^n \rho(x_1, x_0)\). If \(m > n\) it follows that

\[
\rho(x_m, x_n) \leq \rho(x_m, x_{m-1}) + \cdots + \rho(x_{n+1}, x_n)
\]

\[
\leq (\theta^{m-1} + \cdots + \theta^n) \rho(x_1, x_0)
\]

\[
\leq \sum_{k=n}^{\infty} \theta^k \rho(x_1, x_0)
\]

\[
= \frac{\theta^n}{1-\theta} \rho(x_1, x_0)
\]

\[\to 0\]

as \(n \to \infty\). It follows that \(\{x_n\}\) is a Cauchy sequence and thus has a limit \(x\) since \(X\) is complete. The fact that \(x\) is a solution of the fixed point problem follows by passing to the limit in the relation \(x_{n+1} = f(x_n)\) (note that a contraction is always continuous).

Note that any closed subset of a Banach space \(X\) is a complete metric space with the metric induced by the norm on \(X\). This is the setting in which we will use the above theorem.

### A.4 Completions

A subset of \(U\) of a metric space \((X, \rho)\) is said to be **dense** in \(X\) if for every \(x \in X\) there exists a sequence \(\{x_n\}\) in \(U\) such that \(x_n \to x\). An example is that \(\mathbb{Q}\) is dense in \(\mathbb{R}\). Whenever \(U\) is a subset of \(X\), \(U\) will be dense in the closure \(\overline{U}\), defined as the smallest closed subset of \(X\) containing \(U\) (that such a set exists follows from the fact that intersections of closed sets are closed).
For many purposes in functional analysis it is important to work in a complete space. As an example we mention Banach’s fixed point theorem above. The question is what to do if the space is not complete. There is an abstract answer to this question which is contained in the following proposition. An isometry between to metric spaces is a map which preserves distances. Note that the inverse of a bijective isometry also is an isometry.

**Proposition A.16.** Given any metric space \((X, \rho)\), one can find a complete metric space \((\bar{X}, \bar{\rho})\) and a map \(\Phi: X \to \bar{X}\) which is a bijective isometry onto a dense subset of \(\bar{X}\).

**Proof.** We give a sketch of the proof. The idea is to let \(\bar{X}\) consist of Cauchy sequences in \(X\) and to define the distance as the limit of the distance between elements of two such sequences. However, two sequences with the same limit will have distance 0 unless one introduces equivalence classes. Two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) are defined to be equivalent if \(x_n - y_n \to 0\). The space \(\bar{X}\) consists of all equivalence classes of Cauchy sequences in \(X\) and the metric \(\bar{\rho}\) is defined as

\[
\bar{\rho}([\{x_n\}], [\{y_n\}]) = \lim_{n \to \infty} \rho(x_n, y_n),
\]

where \([\{x_n\}]\) denotes the equivalence class containing the sequence \(\{x_n\}\). The limit on the right exists since

\[
|\rho(x_m, y_m) - \rho(x_n, y_n)| \leq |\rho(x_m, y_m) - \rho(x_n, y_m)| + |\rho(x_n, y_m) - \rho(x_n, y_n)|
\]

and \(\mathbb{R}\) is complete. It is independent of the choice of representative by a similar calculation. The map \(\Phi\) is defined by letting \(\Phi(x)\) be the constant sequence \(x, x, \ldots\) (or rather the corresponding equivalence class). This map is clearly an isometry. \(\Phi(X)\) is dense since we can approximate \(x = [\{x_n\}] \in \bar{X}\) by \(x(m) = \Phi(x_m) = [\{x_m, x_m, \ldots\}]\). Indeed, \(\bar{\rho}(x, x(m)) = \lim_{n \to \infty} \rho(x_n, x_m)\) which converges to 0 as \(m \to \infty\). The only difficult part is the proof of completeness. The trick is to approximate a Cauchy sequence in \(\bar{X}\) by one in \(\Phi(X)\) (we proved that \(\Phi(X)\) is dense in \(\bar{X}\)). On the other hand, we know that a Cauchy sequence \(\Phi(x_n)\) in \(\Phi(X)\) has a limit in \(\bar{X}\), namely \([\{x_n\}]\).

More generally, one says that a complete metric space \((Y, \rho_Y)\) is a completion of \((X, \rho_X)\) if there exists a map \(\Phi: X \to Y\) which is a bijective isometry onto a dense subset of \(Y\). One usually identifies \(X\) with the set \(\Phi(X)\).

The following properties of the completion are important.

**Proposition A.17.** If \(X\) is a normed space then the completion is a Banach space (with the norm corresponding to the metric). Similarly, if \(X\) is an inner product space then the completion is a Hilbert space with the natural inner product. Moreover, the map \(\Phi\) is an isometric isomorphism, that is, a linear bijection which preserves norms.

Consider the normed vector space from Example (A.11). The completion of this space is a Hilbert space. In Appendix B, we show that the completion can be identified with the space of square-integrable functions in the sense of Lebesgue, \(L^2(-1, 1)\). This is clearly preferable to working with equivalence classes of Cauchy sequences.
Appendix B

Integration

The Riemann integral is insufficient for many purposes in analysis. Specifically it doesn’t behave well with respect to limits. Suppose that \( \{f_n\} \) is an increasing sequence of non-negative Riemann integrable functions. Then both \( f(x) = \lim_{n \to \infty} f_n(x) \) and \( \lim_{n \to \infty} \int f_n \, dx \) exists and it would be natural to expect that \( \int f \, dx = \lim_{n \to \infty} \int f_n \, dx \). However, \( f \) might not be Riemann integrable, as the following example illustrates.

Example B.1. Let \( \{x_n\}_{n=1}^\infty \) be an enumeration of the rational numbers in the interval \([0, 1]\). Define the functions \( f_n: [0, 1] \to \mathbb{R} \) by setting \( f_n(x) = 1 \) if \( x \in \{x_1, \ldots, x_n\} \) and \( f_n(x) = 0 \) otherwise. Each \( f_n \) is 0 except at finitely many points, so it is Riemann integrable with \( \int f_n \, dx = 0 \). The sequence \( \{f_n(x)\} \) is increasing and bounded from above for each \( x \), so it has a limit \( f(x) \). In fact, it is clear that \( f(x) \) is 1 at the rational numbers in \([0, 1]\) and zero everywhere else. The function \( f \) is not Riemann integrable.

As you know from calculus, integrals measure some form of area or volume. It is therefore natural to start by discussing measure, that is, the generalization of length, area and volume to arbitrary dimensions. By assigning a measure to the set of rational points in \([0, 1]\), one can integrate the function \( f \) in the previous example. There are other ways of introducing the Lebesgue integral on \( \mathbb{R}^d \), e.g. by approximation with continuous functions of compact support.

The main purpose of this appendix is to describe the construction of the Lebesgue integral and to collect some results which are used throughout the text. Proofs of the theorems can be found in any book on measure theory and real analysis, e.g. [8].

B.1 Lebesgue measure

The measure of a set should be a function \( \mu \) which assigns a non-negative number \( \mu(E) \) (the measure of \( E \)) to sets \( E \subset \mathbb{R}^d \). There are two things to note here. First of all, it is natural to allow \( \mu(E) \) to take the value \(+\infty\). We let \([-\infty, \infty]\) be extended real line. Any operation involving \( \pm \infty \) which makes sense in an intuitive way, e.g. \( x + \infty = \infty \) if \( x \in \mathbb{R} \), is defined in this way. We define \( 0 \cdot (\pm \infty) = 0 \). Otherwise, any operation which can’t be defined in an intuitive
way, such as $\infty - \infty$, is undefined. Secondly, we will not be able to measure all subset of $\mathbb{R}^d$. This is not a big problem, since the class of sets we can measure is very large (e.g. it contains all open sets and all closed sets).

**Definition B.2 (Sigma algebras).** A collection $\Sigma$ of subsets of $\mathbb{R}^d$ is called a $\sigma$-algebra if the following conditions hold

1. $\mathbb{R}^d \in \Sigma$;
2. if $A \in \mathbb{R}^d$, then $A^c \in \mathbb{R}^d$;
3. if $A_j \in \Sigma$, $j = 1, 2, \ldots$, then $\bigcup_{j=1}^{\infty} A_j \in \Sigma$.

It follows from the definition that
- $\emptyset = (\mathbb{R}^d)^c \in \Sigma$;
- $\bigcap_{j=1}^{\infty} A_j = (\bigcup_{j=1}^{\infty} A_j^c)^c \in \Sigma$ if $A_j \in \Sigma$ for all $j$;
- $A \setminus B = A \cap B^c \in \Sigma$ if $A, B \in \Sigma$.

**Definition B.3 (Measures).** A measure on a $\sigma$-algebra $\Sigma$ is a function $\mu : \Sigma \to [0, \infty]$ which is countably additive, that is,

$$\mu \left( \bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} \mu(A_j)$$

whenever $A_j \in \Sigma$ for all $j$ and the sets $A_j$ are pairwise disjoint ($A_j \cap A_k = \emptyset$ if $j \neq k$), and satisfies $\mu(\emptyset) = 0$.

One can show from the definition that
- $\mu(A) \leq \mu(B)$ if $A, B \in \Sigma$ and $A \subseteq B$;
- $\mu\left( \bigcup_{j=1}^{\infty} A_j \right) \leq \sum_{j=1}^{\infty} \mu(A_j)$ if $A_j \in \Sigma$ for all $j$;
- $\mu\left( \bigcup_{j=1}^{\infty} A_j \right) = \lim_{j \to \infty} \mu(A_j)$ if $A_j \in \Sigma$ and $A_j \subseteq A_{j+1}$ for all $j$.

**Theorem B.4 (Lebesgue measure).** There exists a $\sigma$-algebra of subsets of $\mathbb{R}^d$ and a measure $\mu$ on $\Sigma$ such that

1. every open set (and thus every closed set) in $\mathbb{R}^d$ belongs to $\Sigma$;
2. if $A \subseteq B$ and $B \in \Sigma$ with $\mu(B) = 0$, then $A \in \Sigma$ with $\mu(A) = 0$;
3. if $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, then $\mu(A) = \prod_{j=1}^{d} (b_j - a_j)$;

We denote this measure $|\cdot|$ and call it the Lebesgue measure on $\mathbb{R}^d$. A set in the above $\sigma$-algebra is called Lebesgue measurable. Since we will only deal with the Lebesgue measure, we usually drop the word ‘Lebesgue’ in what follows. The Lebesgue measure is a generalization of the intuitive idea of lengths in $\mathbb{R}$, areas in $\mathbb{R}^2$ and volumes in $\mathbb{R}^3$. It gives the usual value to sets such as polygons, balls etc, but it also allows us to measure ‘wilder’ sets and sets in higher dimensions than 3.
Definition B.5. A set of measure zero is called a null set. A property which holds everywhere on $\mathbb{R}^d$, except on a null set, is said to hold almost everywhere, abbreviated a.e.

Any countable set is a null set, in particular the set $[0,1] \cap \mathbb{Q}$ which appeared in Example B.1.

### B.2 Lebesgue integral

Just as in the previous section, we will only be able to define the integral for certain functions.

**Definition B.6.** Let $A$ be a measurable set. A function $f : A \to [\mathbb{R}, \infty]$ is said to be measurable if the set

$$\{x : f(x) > a\}$$

is measurable for each $a \in \mathbb{R}$.

**Proposition B.7.**

1. If $f$ is measurable, then so is $|f|$;
2. If $f$ and $g$ are measurable and real-valued, then so are $f + g$ and $fg$;
3. If $\{f_j\}$ is a sequence of measurable functions, then $\sup_j f_j$, $\inf_j f_j$, $\limsup_{j \to \infty} f_j$ and $\liminf_{j \to \infty} f_j$ are measurable;
4. A continuous function defined on a measurable set is measurable;
5. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g$ is measurable and real-valued, then the composition $f \circ g$ is measurable.

Recall that the characteristic function $\chi_A$ of a set $A \subset \mathbb{R}^d$ is defined by

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

**Definition B.8.** A simple function $s$ on $\mathbb{R}^d$ is a finite linear combination of characteristic functions of measurable subsets of $\mathbb{R}^d$, that is,

$$s(x) = \sum_{j=1}^{n} a_j \chi_{A_j}(x),$$

where $a_j \in \mathbb{R}$ for all $j$.

**Definition B.9.** The Lebesgue integral of a non-negative simple function $s = \sum_{j=1}^{n} a_j \chi_{A_j}$, $a_j \geq 0$, is defined as

$$\int_{\mathbb{R}^d} s(x) \, dx = \sum_{j=1}^{n} a_j |A_j|.$$
The Lebesgue integral of a non-negative measurable function \( f : \mathbb{R}^d \to [0, \infty] \) is defined as

\[
\int_{\mathbb{R}^d} f(x) \, dx = \sup \int_{\mathbb{R}^d} s(x) \, dx,
\]

where the supremum is taken over all simple functions \( s \) satisfying \( 0 \leq s(x) \leq f(x) \) for all \( x \).

Note that the integral of a non-negative function can have the value \(+\infty\). If it is finite, we say that the function is integrable. We allowed \( f \) to take the value \(+\infty\), but if it is integrable this cannot happen very often.

**Proposition B.10.** If \( f : \mathbb{R}^d \to [0, \infty] \) is integrable, then, \( f(x) < \infty \) a.e.

If \( f \) is measurable and real-valued, we set \( f = f^+ - f^- \), where \( f^+(x) = \max\{f(x), 0\} \) and \( f^-(x) = -\min\{f(x), 0\} \), are both measurable and real-valued. We define

\[
\int_{\mathbb{R}^d} f(x) \, dx = \int_{\mathbb{R}^d} f^+(x) \, dx - \int_{\mathbb{R}^d} f^-(x) \, dx,
\]

provided that at least one of \( f^+ \) and \( f^- \) are integrable. If both \( f^+ \) and \( f^- \) are integrable, we say that \( f \) is integrable (some authors use the word summable). Note that this is equivalent to

\[
\int_{\mathbb{R}^d} |f(x)| \, dx < \infty,
\]

so Lebesgue integrals are always absolutely convergent. The set of integrable functions on \( \mathbb{R}^d \) is denoted \( L^1(\mathbb{R}^d) \). This set is closed under linear combinations and hence a vector space with the usual pointwise operations.

We can also define integrals over a measurable subset \( A \subset \mathbb{R}^d \). If \( f \) is a measurable function on \( A \), then the extension \( \tilde{f}(x) = f(x), \quad x \in A, \quad f(x) = 0, \quad x \in A^c \) is also measurable and we set

\[
\int_A f(x) \, dx = \int_{\mathbb{R}^d} \tilde{f}(x) \, dx
\]

when the right hand side is well-defined. Similarly, if \( f \) is a measurable function on \( \mathbb{R}^d \), we define the integral of \( f \) over \( A \) as the integral over \( \mathbb{R}^d \) of the function \( \chi_A f \) (when this is possible). Complex-valued functions are integrated by separating into real and imaginary parts; vector-valued functions are integrated componentwise.

Note that all the usual rules that you have learned from Riemann integration also apply to Lebesgue integrals, e.g. \( \int (f + g) \, dx = \int f \, dx + \int g \, dx \) and \( \left| \int f \, dx \right| \leq \int |f| \, dx \). Moreover, any Riemann integrable function is also Lebesgue integrable and the values of the two integrals coincide. The only thing one has to watch out for is generalized integrals. The Lebesgue integral automatically handles absolutely convergent generalized integrals, but one still needs a limiting procedure to take care of conditionally convergent generalized integrals, such as \( \int_{\mathbb{R}} \frac{\sin x}{x} \, dx \). One thing which is much easier for the Lebesgue integral than for the Riemann integral is iterated integration.
Theorem B.11 (Fubini-Tonelli theorem). Let $f$ be a measurable function on $\mathbb{R}^{n+d}$ and suppose that at least one of the integrals

\[
I_1 = \int_{\mathbb{R}^{n+d}} |f(x,y)| \, dx \, dy, \\
I_2 = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^n} |f(x,y)| \, dx \right) \, dy, \\
I_3 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^d} |f(x,y)| \, dy \right) \, dx
\]

exists and is finite. For $I_2$ and $I_3$ this means that the inner integral exists a.e. and defines an integrable function. Then all the integrals exist and $I_1 = I_2 = I_3$. Moreover,

1. $f(\cdot,y) \in L^1(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^d$ and $\int_{\mathbb{R}^n} f(x,y) \, dx \in L^1(\mathbb{R}^d)$;
2. $f(x,\cdot) \in L^1(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^d} f(x,y) \, dy \in L^1(\mathbb{R}^n)$;
3. $\int_{\mathbb{R}^{n+d}} f(x,y) \, dx \, dy = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^n} f(x,y) \, dx \right) \, dy = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^d} f(x,y) \, dy \right) \, dx$.

B.3 Convergence properties

One of the most useful aspects of the Lebesgue integral is that it behaves well with respect to limits. More precisely, we have the following convergence results.

Theorem B.12 (Fatou’s lemma). Assume that $\{f_n\}$ is a sequence of non-negative, measurable functions. Then

\[
\int_{\mathbb{R}^d} \liminf_{n \to \infty} f_n \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} f_n \, dx.
\]

Theorem B.13 (Monotone convergence theorem). Assume that $\{f_n\}$ is an increasing sequence of non-negative measurable functions. Then

\[
\int_{\mathbb{R}^d} \lim_{n \to \infty} f_n \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} f_n \, dx.
\]

Theorem B.14 (Dominated convergence theorem). Assume that $\{f_n\}$ is a sequence of complex-valued integrable functions and that

\[
f_n \to f \text{ a.e.}
\]

If

\[
|f_n| \leq g \text{ a.e.}
\]
for some nonnegative integrable function \(g\), then

\[
\int_{\mathbb{R}^d} f_n \, dx \to \int_{\mathbb{R}^d} f \, dx.
\]

Using the dominated convergence theorem one can prove the following important results about the dependence of integrals on parameters.

**Theorem B.15.** Let \(\Omega\) be an open subset of \(\mathbb{R}^n\). Suppose that \(f : \Omega \times \mathbb{R}^d \to \mathbb{C}\) and that \(f(x, \cdot) : \mathbb{R}^d \to \mathbb{C}\) is integrable for each \(x \in \Omega\). Let \(F(x) = \int_{\mathbb{R}^d} f(x, y) \, dy\).

1. Suppose that there exists \(g \in L^1(\mathbb{R}^d)\) such that \(|f(x, y)| \leq g(y)\) for all \(x, y\). If \(\lim_{x \to x_0} f(x, y) = f(x_0, y)\) for every \(y\), then \(\lim_{x \to x_0} F(x) = F(x_0)\). In particular, if \(f(\cdot, y)\) is continuous for each \(y\), then \(F\) is continuous.

2. Suppose that \(\partial_{x_j} f(x, y)\) exists for all \(x, y\) and that there exists \(g \in L^1(\mathbb{R}^d)\) such that \(|\partial_{x_j} f(x, y)| \leq g(y)\) for all \(x\). Then \(\partial_{x_j} F(x)\) exists and is given by

\[
\partial_{x_j} F(x) = \int_{\mathbb{R}^d} \partial_{x_j} f(x, y) \, dy.
\]

**B.4 \(L^p\) spaces**

If \(p \in [1, \infty)\) and \(f\) is a measurable complex-valued function on \(\mathbb{R}^d\), we define

\[
\|f\|_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |f|^p \, dx \right)^{1/p}.
\]

We will also write \(\|f\|_{L^p}\) if the dimension \(d\) is clear from the context. We also define

\[
\|f\|_{L^\infty(\mathbb{R}^d)} = \operatorname{ess sup} |f|,
\]

where the *essential supremum* of a measurable function \(g\) is defined as the minimal \(c \in [-\infty, \infty]\) such that \(g(x) \leq c\) a.e., that is,

\[
\operatorname{ess sup} g = \inf \{c \in [-\infty, \infty] : |\{x : g(x) > c\}| = 0\}
\]

We say that \(f \in L^p(\mathbb{R}^d)\) if \(\|f\|_{L^p(\mathbb{R}^d)} < \infty\).

**Theorem B.16** (Hölder’s inequality). Suppose that \(1 \leq p \leq \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). If \(f\) and \(g\) are measurable functions on \(\mathbb{R}^d\), then

\[
\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.
\]

In particular, \(fg \in L^1\) if \(f \in L^p\) and \(g \in L^q\).
Theorem B.17 (Minkowski’s inequality). If \( 1 \leq p \leq \infty \) and \( f, g \in L^p(\mathbb{R}^d) \), then
\[
\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.
\]

Minkowski’s inequality shows that \( \| \cdot \|_{L^p} \) defines a semi-norm on \( L^p \) for \( 1 \leq p \leq \infty \). It is not a norm, however, since \( \|f\|_{L^p} = 0 \) when \( f = 0 \) a.e. This is easy to fix.

Definition B.18. We introduce an equivalence class on measurable functions by defining two functions as equivalent if they are equal a.e. For \( 1 \leq p \leq \infty \), the space \( L^p(\mathbb{R}^d) \) is defined as \( L^p(\mathbb{R}^d) \) modulo this equivalence relation.

The definition means that in contrast to \( L^p \), an element of \( L^p \) is not a function; it is an equivalence class of functions which are equal a.e. However, in practice little confusion is caused by identifying equivalence classes with representatives. From now on, we shall make no difference between \( L^p \) and \( L^p \).

Minkowski’s inequality can be iterated to give an inequality for a sum of finitely many functions, or even a series. Since an integral can be thought of as a continuous sum, it is natural to expect the following result.

Theorem B.19 (Minkowski’s inequality for integrals). Let \( 1 \leq p < \infty \). Suppose that \( f \) is measurable on \( \mathbb{R}^n \times \mathbb{R}^d \), that \( f(\cdot, y) \in L^p(\mathbb{R}^n) \) for almost every \( y \in \mathbb{R}^d \), and that the function \( y \mapsto \|f(\cdot, y)\|_{L^p(\mathbb{R}^n)} \) belongs to \( L^1(\mathbb{R}^d) \). Then the function \( x \mapsto \int_{\mathbb{R}^d} f(x, y) \, dy \) belongs to \( L^p(\mathbb{R}^d) \) and
\[
\left\| \int_{\mathbb{R}^d} f(x, y) \, dy \right\|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^d} \|f(\cdot, y)\|_{L^p(\mathbb{R}^n)} \, dy.
\]

Proof. After splitting the function \( f \) into real and imaginary parts, and then splitting these into positive and negative parts, we can assume that \( f \geq 0 \). For \( p = 1 \) the result holds with equality by Fubini’s theorem. If \( p > 1 \) we write
\[
\left( \int_{\mathbb{R}^d} f(x, y) \, dy \right)^p = \left( \int_{\mathbb{R}^d} f(x, y) \, dy \right) w(x),
\]
where \( w(x) = (\int_{\mathbb{R}^d} f(x, y) \, dy)^{p-1} \). By Fubini’s theorem and Hölder’s inequality, it follows that
\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^d} f(x, y) \, dy \right) w(x) \, dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^n} f(x, y) w(x) \, dx \right) \, dy
\]
\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^n} f(x, y)^p \, dx \right)^{1/p} \left( \int_{\mathbb{R}^n} w(x)^q \, dx \right)^{1/q} \, dy
\]
\[
= \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^n} f(x, y)^p \, dx \right)^{1/p} \, dy \right) \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^d} f(x, y) \, dy \right)^p \, dx \right)^{1/q}
\]
where \( 1/q = 1 - 1/p \). Hence,
\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^d} f(x, y) \, dy \right)^p \, dx \leq \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^n} f(x, y)^p \, dx \right)^{1/p} \, dy \right) \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^d} f(x, y) \, dy \right)^p \, dx \right)^{1/q}.
\]
The result follows after dividing with the second factor in the right hand side. \( \square \)
As already mentioned in Appendix A, the great advantage of working with $L^p$ spaces is that they are complete.

**Theorem B.20.** $L^p(\mathbb{R}^d)$ is a Banach space for $1 \leq p \leq \infty$. For $p = 2$ it is a Hilbert space.

Recall that the support of a continuous function $f$ is defined as $\text{supp } f = \{ x \in \mathbb{R}^d : f(x) \neq 0 \}$. We denote by $C_0(\mathbb{R}^d)$ the set of continuous functions on $\mathbb{R}^d$ with compact support (the support is always closed, so the only condition is that it should be bounded). If $f$ is only measurable, one instead defines the essential support, $\text{ess supp } f$, as the smallest closed set outside of which $f = 0$ a.e. We will use the notation $\text{supp } f$ for $\text{ess supp } f$ when $f$ is not continuous. The fact that $L^p(\mathbb{R}^d)$ is the completion of $C_0(\mathbb{R}^d)$ now follows from the following theorem.

**Theorem B.21.** $C_0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

Note that Hölder’s inequality implies that a function $g \in L^q$ defines a bounded linear functional on $L^p$, 

$$\ell_g(f) = \int_{\mathbb{R}^d} f g \, dx,$$  

where $p$ and $q$ are conjugate indices ($\frac{1}{p} + \frac{1}{q} = 1$). Moreover, $\| \ell_g \| \leq \| g \|_q$.

**Theorem B.22.** For $1 \leq p < \infty$, the dual of $L^p(\mathbb{R}^d)$ can be identified with $L^q(\mathbb{R}^d)$, where $p$ and $q$ are conjugate indices. That is, the map $g \mapsto \ell_g$ from $L^q(\mathbb{R}^d) \to (L^q(\mathbb{R}^d))^*$ defined by (B.1) is an isometric isomorphism.

In the case $p = 2$ this is a special case of Riesz’ representation theorem, at least if one replaces $g$ by $g$.

We can also define $L^p$ on a measurable subset $A$ of $\mathbb{R}^d$ by replacing the domain of integration by $A$. Recall that the function $|x|^a$ is integrable on $B_1(0) \subset \mathbb{R}^d$ if and only if $a > -d$, whereas it is integrable on $\mathbb{R}^d \setminus B_1(0)$ if and only if $a < -d$. If we are only interested in local integrability properties and not in the decay at infinity, the following spaces are useful.

**Definition B.23.** We say that $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ if $f \in L^p(K)$ for any compact subset $K \subset \mathbb{R}^d$.

### B.5 Convolutions and approximation with smooth functions

The convolution $f * g$ of two measurable functions $f$ and $g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y) g(y) \, dy$$

whenever this makes sense. From the definition it follows that $(f * g)(x) = (g * f)(x)$ a.e. By Hölder’s inequality, we obtain that $f * g \in L^\infty$ if $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$. Moreover,

$$\| f * g \|_{L^\infty} \leq \| f \|_{L^p} \| g \|_{L^q}.$$ (B.2)
From Minkowski’s inequality for integrals, we obtain that
\[ \|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}, \quad 1 \leq p \leq \infty, \] (B.3)
so that \( f * g \in L^p \) if \( f \in L^p \) and \( g \in L^1 \). If \( f \) has compact support, it also follows from the definition that
\[ \text{supp}(f * g) \subset \text{supp} f + \text{supp} g. \]
Finally, a very important property of convolutions is the following. If \( g \in L^1(\mathbb{R}^d) \) and \( f \in C^1_b(\mathbb{R}^d) \), then \( f * g \in C^1 \) with
\[ \partial_x (f * g) = \partial_x f * g. \]
This follows from the definition of the convolution and Theorem B.15.

In many situations it is useful to approximate functions in \( L^p \) by nicer functions. E.g. a part of a proof might only make sense for \( C^\infty \) functions, whereas the outcome of the proof makes sense in \( L^p \). One can then prove the result in \( L^p \) by approximating with smooth functions. We already saw that \( C_0(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d) \), but we shall now show that much more is true.

We begin by proving that there exist smooth functions with compact support.

**Lemma B.24** (Existence of functions in \( C^\infty_0(\mathbb{R}^d) \)). There exists a non-negative function \( \varphi \in C^\infty(\mathbb{R}^d) \) with \( \text{supp} \varphi = B_1(0) \).

**Proof.** Note first that the function
\[ f(t) = \begin{cases} \epsilon^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases} \]
is \( C^\infty \) on \( \mathbb{R} \) with support \([0, \infty)\). Indeed, for \( t > 0 \) all the derivatives are polynomials in \( 1/t \) times \( \epsilon^{-1/t} \) and \( t^a \epsilon^{-1/t} \to 0 \) as \( t \to 0^+ \) for any \( a \in \mathbb{R} \). The function \( \varphi \) can e.g. be defined as
\[ \varphi(x) = f(1 - |x|^2). \]

We can now construct a smooth ‘approximation of the identity’ as follows. After dividing \( \varphi \) in Lemma B.24 by \( \int_{\mathbb{R}^d} \varphi \, dx = 1 \), we can assume that \( \int_{\mathbb{R}^d} \varphi \, dx = 1 \). We set
\[ J_\epsilon(x) = \epsilon^{-d} \varphi(\epsilon^{-1} x), \quad \epsilon > 0. \]
Then \( J_\epsilon \in C^\infty_0(\mathbb{R}^d) \) with \( \text{supp} J_\epsilon = B_\epsilon(0) \) and \( \int_{\mathbb{R}^d} J_\epsilon(x) \, dx = 1 \) for all \( \epsilon > 0 \) (change of variables). One can think of \( J_\epsilon \) as converging to a ‘unit mass’ at the origin. This can be made sense of using distribution theory.

Generally, we call a function \( J \in C^\infty_0(\mathbb{R}^d) \) with \( J \geq 0 \) and \( \int_{\mathbb{R}^d} J \, dx = 1 \) a *mollifier*. We set \( J_\epsilon(x) = \epsilon^{-d} J(\epsilon^{-1} x) \). The convolution
\[ (J_\epsilon * f)(x) = \int_{\mathbb{R}^d} J_\epsilon(x-y) f(y) \, dy = \int_{\mathbb{R}^d} J_\epsilon(y) f(x - y) \, dy \]
is defined for \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) and is called the *mollification* or *regularization* of \( f \). \( (J_\epsilon * f)(x) \) can be thought of as an average of \( f \) over a small neighborhood of \( x \).
Theorem B.25. Let \( J \) be a mollifier.

1. If \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \), then \( J_\varepsilon \ast f \in C^\infty(\mathbb{R}^d) \).
2. If \( f \in L^1(\mathbb{R}^d) \) with compact support, then \( J_\varepsilon \ast f \in C^0_0(\mathbb{R}^d) \).
3. If \( f \in C(\mathbb{R}^d) \), then \( J_\varepsilon \ast f \to f \) uniformly on compact subset of \( \mathbb{R}^d \).
4. If \( f \in L^p(\mathbb{R}^d) \), where \( 1 \leq p < \infty \), then \( \| J_\varepsilon \ast f \|_{L^p} \leq \| f \|_{L^p} \) and \( \lim_{\varepsilon \to 0^+} \| J_\varepsilon \ast f - f \|_{L^p} \).

Proof. The first statement follows by repeated differentiation under the integral sign.

The second statement follows since \( \text{supp} J_\varepsilon \subset \text{supp} f + \text{supp} J_\varepsilon \) and both \( f \) and \( J_\varepsilon \) have compact supports.

We have that
\[
(J_\varepsilon \ast f)(x) - f(x) = \int_{\mathbb{R}^d} J_\varepsilon(y)(f(x - y) - f(x)) \, dy = \int_{\text{supp} J} J(y)(f(x - \varepsilon y) - f(x)) \, dy
\]

Since \( f \) is uniformly continuous on a compact subset \( K \subset \mathbb{R}^d \), it follows that \( \sup_{(x,y) \in K \times \text{supp} J} |f(x - \varepsilon y) - f(x)| \to 0 \) as \( \varepsilon \to 0^+ \).

The first part of (4) is a consequence of (B.2) Using Theorem B.21 we can find \( g \in C_0(\mathbb{R}^d) \) such that \( \| f - g \|_{L^p} < \delta \) and it follows from (2) that \( J_\varepsilon \ast g - g \) has support in a compact set which is independent of \( \varepsilon \) (sufficiently small) and hence from (3) that \( J_\varepsilon \ast g \to g \) uniformly. It follows that \( \| J_\varepsilon \ast g - g \|_{L^p} < \delta \) if \( \varepsilon \) is sufficiently small. For \( \varepsilon \) sufficiently small, we therefore have
\[
\| J_\varepsilon \ast f - f \|_{L^p} \leq \| J_\varepsilon \ast f - J_\varepsilon \ast g \|_{L^p} + \| J_\varepsilon \ast g - g \|_{L^p} + \| g - f \|_{L^p} \\
\leq 2\| g - f \|_{L^p} + \| J_\varepsilon \ast g - g \|_{L^p} < 3\delta.
\]

This proves (4). \( \qed \)

Corollary B.26. \( C^\infty_0(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d) \) for \( 1 \leq p < \infty \).

Proof. This follows by approximating \( f \in L^p(\mathbb{R}^d) \) with a function \( g \in C_0(\mathbb{R}^d) \) using Theorem B.21 and then approximating \( g \) with a function in \( C^\infty_0(\mathbb{R}^d) \) using Theorem B.25. \( \qed \)

We also prove the following variant which has many applications.

Theorem B.27. Suppose that \( K \in L^1(\mathbb{R}^d) \) and that \( \int_{\mathbb{R}^d} K(x) \, dx = 1 \). Define \( K_\varepsilon(x) = \varepsilon^{-d} K(\varepsilon^{-1} x) \), \( \varepsilon > 0 \).

1. If \( f \in C_0(\mathbb{R}^d) \), then \( \lim_{\varepsilon \to 0^+} K_\varepsilon \ast f = f \) uniformly on compact subsets of \( \mathbb{R}^d \). If \( f \) is uniformly continuous, the convergence is uniform on \( \mathbb{R}^d \).

2. If \( f \in L^1(\mathbb{R}^d) \) with compact support, then \( K_\varepsilon \ast f \in C^\infty(\mathbb{R}^d) \).

3. If \( f \in C(\mathbb{R}^d) \), then \( K_\varepsilon \ast f \to f \) uniformly on compact subset of \( \mathbb{R}^d \).

4. If \( f \in L^p(\mathbb{R}^d) \), where \( 1 \leq p < \infty \), then \( \| K_\varepsilon \ast f \|_{L^p} \leq \| f \|_{L^p} \) and \( \lim_{\varepsilon \to 0^+} \| K_\varepsilon \ast f - f \|_{L^p} \).

Proof. The proof follows the same lines as Theorem B.25.
(2) If \( f \in L^p(\mathbb{R}^d) \), \( 1 \leq p < \infty \), then \( K_\varepsilon * f \to f \) in \( L^p(\mathbb{R}^d) \).

**Proof.** (1) Write
\[
K_\varepsilon * f = f = \int_{\mathbb{R}^d} K(y) (f(x-\varepsilon y) - f(x)) \, dy
\]
\[
= \int_{|y| \leq R} K(y) (f(x-\varepsilon y) - f(x)) \, dy + \int_{|y| \geq R} K(y) (f(x-\varepsilon y) - f(x)) \, dy.
\]
The second term is estimated as follows:
\[
\left| \int_{|y| \geq R} K(y) (f(x-\varepsilon y) - f(x)) \, dy \right| \leq 2 \sup_{y \in \mathbb{R}^d} |f(x)| \int_{|y| \geq R} |K(y)| \, dy \to 0
\]
as \( R \to \infty \). To estimate the first term, we note that \( \lim_{\varepsilon \to 0^+} (f(x-\varepsilon y) - f(x)) = 0 \) uniformly for \( |y| \leq R \) and \( x \) in a compact subset of \( \mathbb{R}^d \). It therefore follows that the first integral converges to 0 locally uniformly in \( x \) for fixed \( R \). The result is obtained by combining these two estimates. The second part is obvious because the estimate for the first term is now uniform over \( \mathbb{R}^d \).

(2) Let \( \delta > 0 \). Approximate \( f \) by \( g \in C_0(\mathbb{R}^d) \) with \( \| f - g \|_{L^p} < \delta \) and \( K \) by \( J \in C_0(\mathbb{R}^d) \) with \( \| K - J \|_{L^1} < \delta \). We then have
\[
\| K_\varepsilon * f - f \|_{L^p} \leq \| K_\varepsilon * f - K_\varepsilon * g \|_{L^p} + \| K_\varepsilon * g - J_\varepsilon * g \|_{L^p} + \| J_\varepsilon * g - g \|_{L^p} + \| g - f \|_{L^p}.
\]
We have that
\[
\| K_\varepsilon * f - K_\varepsilon * g \|_{L^p} \leq \| K_\varepsilon \|_{L^1} \| f - g \|_{L^p} \leq \delta \| K \|_{L^1},
\]
\[
\| K_\varepsilon * g - J_\varepsilon * g \|_{L^p} \leq \| K_\varepsilon - J_\varepsilon \|_{L^1} \| g \|_{L^p}
\]
\[
\quad \leq \| K - J \|_{L^1} (\| f - g \|_{L^p} + \| f \|_{L^p})
\]
\[
\leq \delta (\| f \|_{L^p}).
\]
\( J_\varepsilon * g - g \) converges to 0 uniformly as \( \varepsilon \to 0^+ \) and has support contained in a fixed compact set since \( J_\varepsilon \) and \( g \) both do. It follows that
\[
\| J_\varepsilon * g - g \|_{L^p} \to 0
\]
as \( \varepsilon \to 0^+ \) for fixed \( \delta \). Since \( \delta \) is arbitrary, the result follows.

\[ \square \]

## B.6 Interpolation

If \( E \) has finite measure, it follows from Hölder’s inequality that \( L^{p_2}(E) \subset L^{p_1}(E) \) if \( p_1 < p_2 \). This means that \( L^{p_2}_{lo} (\mathbb{R}^d) \subset L^{p_1}_{lo} (\mathbb{R}^d) \) (a higher exponent means less singular behavior locally). However, at infinity the inclusion is reversed in the sense that \( L^{\infty} (\mathbb{R}^d) \cap L^{p_1} (\mathbb{R}^d) \subset L^{p_2} (\mathbb{R}^d) \) (a
smaller exponent means more decay). In fact, more generally, we have that \( f \in L^{p_1} \cap L^{p_2} \Rightarrow f \in L^p \) for any \( p \in (p_1, p_2) \). This follows from Hölder’s inequality:

\[
\|f\|_{L^p}^p = \int_{\mathbb{R}^d} |f|^p \, dx = \left( \int_{\mathbb{R}^d} |f|^{\theta p_1} |f|^{(1-\theta)p_2} \, dx \right)^{1-\theta} \leq \left( \int_{\mathbb{R}^d} |f|^{p_1} \, dx \right)^\theta \left( \int_{\mathbb{R}^d} |f|^{p_2} \, dx \right)^{1-\theta} = \|f\|_{L^{p_1}}^{p \theta} \|f\|_{L^{p_2}}^{p(1-\theta)},
\]

where \( p = \theta p_1 + (1-\theta)p_2, \theta \in (0, 1) \).

Suppose that \( T \) is a linear map defined on a dense subspace \( X \subset L^{p_1} \). Suppose also that \( X \) is dense in \( L^{p_2} \) and that \( T \) is bounded from \( X \subset L^{p_2} \) to \( L^{q_2} \) in the sense that \( T f \in L^{q_2} \) and

\[
\|T f\|_{L^{q_2}} \leq C_j \|f\|_{L^{p_1}}, \quad f \in X.
\]

Note that this means that \( T \) extends uniquely to a bounded operator \( T: L^{p_1} \to L^{q_2} \) (we abuse notation by using the same letter for the extension). Indeed, if \( x_n \to x \in L^{p_1} \) then \( \{T x_n\} \) is a Cauchy sequence in \( L^{q_2} \). Since \( L^{q_2} \) is complete we can define \( T x = \lim_{n \to \infty} T x_n \). This operator will be bounded (by the same constant) and this is the only bounded extension.

One can also extend \( T \) to any \( L^p \) with \( p \in (p_1, p_2) \). Indeed, let \( A = \{ x : |f(x)| > 1 \} \). Write \( f = f_1 + f_2 \), where \( f_1 = \chi_A f \) and \( f_2 = \chi_{\mathbb{R}^d \setminus A} f \). Then \( f_1 \in L^{p_1} \) since \( A \) has finite measure if \( f \in L^p \), and \( f_2 \in L^{p_2} \) since \( f_2 \in L^p \cap L^\infty \). Hence we can define

\[
T f = T f_1 + T f_2 \in L^{q_1} + L^{q_2}.
\]

The next results shows that in fact \( T f \in L^r \) for some \( r \) between \( r_1 \) and \( r_2 \).

**Theorem B.28** (Riesz-Thorin interpolation theorem). *Let \( T \) be as above and let

\[
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}
\]

and

\[
\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}
\]

for some \( \theta \in (0, 1) \). Then \( T: L^p \to L^r \) is bounded with

\[
\|T f\|_{L^r} \leq C_j \|f\|_{L^p}.
\]

**Theorem B.29** (Young’s inequality). *Suppose that \( p, q, r \in [1, \infty] \) and \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \). If \( f \in L^p \) and \( g \in L^q \), then \( f \ast g \in L^r \) with

\[
\|f \ast g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.
\]

*Proof.* Let \( q \) and \( g \) be fixed. The special case \( p = 1, r = q \) follows from Minkowski’s inequality for integrals. The case \( p = q/(q-1), r = \infty \) follows from Hölder’s inequality. The general case now follows from the Riesz-Thorin theorem. \( \square \)
Bibliography


