Branching random walk in random environment:

fully quenched case

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Abstract

The purpose of this short report is to introduce a branching random walk in random environment on $\mathbb{Z}^d$ where particles perform independent simple random walks and branch, according to a given law, which is obtained by fixing branching numbers at each point in the space. These numbers represent a realization of an integer-valued random field on $\mathbb{Z}^d$ with the value at each point being independent of all other points.

With just one particle starting at the origin, we identify the conditions which separate transience and recurrence, i.e., the progeny hits the origin with probability $< 1$ and resp. $1$, in the same manner as it is done by F. den Hollander, M.V. Menshikov and S.Yu. Popov in [1], and previously by M.V. Menshikov and S.V. Volkov in [3].

1 Fully quenched problem

The results obtained in this paper are based on the facts from [1].

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The ‘fully quenched’ problem is defined in the following manner. Suppose that for every \( x \in \mathbb{Z}^d, \ d \geq 3 \), there exists a sequence of non-negative functions \( g_k(x), \ k = 0, 1, 2, \ldots \) such that

\[
\sum_{k=0}^{\infty} g_k(x) = 1
\]

so that \( g(x) \) represents the distribution function of a non-negative integer random variable \( k(x) \). We assume that \( \{k(x), \ x \in \mathbb{Z}^d\} \) is the set of independent but not identically distributed random variables. We fix a realization of a random field \( \{k(x)\} \) and define branching random walk (BRW) as it is defined in [1] and [3]. Namely, there is only one particle starting from the origin which performs a simple random walk (SRW). Once it hits a point \( x \) where \( k(x) \neq 1 \) it splits into \( k(x) \) new particles, and each of them performs a SRW independent from another particles, obeying the rules of the initial particle (except it starts from \( x \)).

The difference between the results obtained here and [1] and [3] consists in the following: in the first paper the annealed problem was considered with the fields \( K = \{k(x)\} \) being updated after each step of a random walk. The second paper analyzed the situation when the only points where \( k(x) \) was allowed to differ from 1 were fixed (and called green) before BRW started, yet \( k(x) \) was updated each time a particle visited this point (‘quenched’ problem). In the current setting, we fix both, which gives rise to the ‘fully quenched’ problem.

The question we address is transience vs. recurrence of such process defined in the same way it is defined in the two papers mentioned above (see the abstract).

To make the problem non-trivial we will assume that \( \mathbb{P}(k(x) = 0) = 0 \) for all \( x \). We will also suppose that \( \mathbb{P}(k(x) = 1) \to 1 \). Also, we note that by Kolmogorov’s zero-one law, recurrence in the fully quenched problem holds either for almost all or almost no realizations of \( K \) (by the same arguments as in the quenched problem).
2 Results

Let $\mu_c$ be the reciprocal of the probability that simple random walk (SRW) returns to the origin and let $m = \lfloor \mu_c \rfloor$ be the largest integer smaller than $\mu_c$. All the following arguments are conducted under the assumption that $m \neq \mu_c$ (i.e. $\mu_c$ is not an integer), however, they can be adapted them for the case $m = \mu_c$ though with less generality.

Denote

$$p_u(x) = \sum_{i=2}^{m} g_i(x),$$

$$p_s(x) = \sum_{i=m+1}^{\infty} g_i(x)$$

and let $p(x) = p_u(x) + p_s(x)$. Since $P(k(x) = 1) \to 1$ it follows that $p(x) \to 0$. Further we will need a certain spherical symmetry of these functions.

**Assumption 1** There exists a $C_\alpha > 1$ for which

$$\sup_x \sup_{y \leq \alpha|z|} \frac{p_u(x)}{p_u(y)} < \infty$$

**Assumption 2** There exists a $C_\alpha > 1$ for which

$$\sup_x \sup_{y \leq \alpha|z|} \frac{p_s(x)}{p_s(y)} < \infty$$

**Assumption 3** There exists a $C > 1$ for which

$$\sup_x \sup_{|y| \leq \alpha|z|} \frac{p(x)}{p(y)} < \infty$$

Notice that Assumption 3 follows from the first two.

**Theorem 1** Under the Assumption 2 the condition

$$\sum_{x \neq 0} \frac{p_u(x)}{|x|^{1-2}} = \infty$$

(2.1)
is sufficient for the recurrence of BRW in a fully quenched environment. Together with Assumptions 3, the condition

\[ \sum_{x \neq 0} \frac{p(x)}{|x|^{d-2}} < \infty \]  \tag{2.2}

is sufficient for the transience of BRW.

Proof. Assumption 1 and (2.1) guarantee (see [4]) that the set \{x : k(x) > \mu_c\} is ‘trapping’ in the sense of [2] almost sure. Hence the proof of Theorem 1 from [1] applies. On the other hand, if (2.2) holds, the particle does not hit the set \{x : k(x) > 1\} with positive probability and does not branch at all. So transience in this case is equivalent to the transience of SRW on \(Z^d, d \geq 3\). \(\square\)

This statement leaves a gap between the transience and recurrence criteria. We will address this below.

**Theorem 2** Assume that \(p_x(x) > 0\) only for finitely many \(x\) and Assumption 1 holds. Then there exist constants \(\alpha_1\) and \(\alpha_2\) such that BRW is

transient, if \(\lim_{|x| \to \infty} \sup |x|^d p(x) < \alpha_1\),

recurrent, if \(\lim_{|x| \to \infty} \inf |x|^d p(x) > \alpha_2\).

Proof. The idea here is to compare the fully quenched BRW with its quenched counterpart and employ Theorem 2 from [1]. Consider BRW in the quenched problem with \(k(x) \equiv m\) (so that the average number of new-born particles \(\mu(x)\) coincides with this number) and let the probability of having a green point at \(x\) be \(p(x)\). (Recall that green points, those where \(k(x) \neq 1\), will be fixed.) Then there exist some constant \(\alpha_1\) for which quenched BRW is transient whenever \(\lim_{|x| \to \infty} \sup |x|^d p(x) < \alpha_1\). However, from coupling arguments it follows that fully quenched BRW with this condition on \(p(x)\) is not more recurrent than the above described quenched BRW. Consequently, the first statement is proven.

To show the second part, we notice that there exists a constant \(\alpha_2\) such that quenched BRW with \(k(x) \equiv 2\) at green points is recurrent as soon as
\[
\lim_{|x| \to \infty} \inf |x|^2 p(x) > \alpha_2. \quad \text{The observation that this quenched BRW is not more recurrent that the fully quenched one completes the proof.} \quad \square
\]

Now we consider the situation when \( p_\epsilon(x) \) is non-trivial with

\[
\sum_{x \neq 0} \frac{p_\epsilon(x)}{|x|^{2-\nu}} < \infty = \sum_{x \neq 0} \frac{p(x)}{|x|^{2-\nu}}
\]

(whence \( \sum_{x \neq 0} \frac{p_\epsilon(x)}{|x|^{2-\nu}} = \infty \)). Suppose that

\[
\lim_{|x| \to \infty} |x|^2 p_\epsilon(x) = \alpha. \quad (2.3)
\]

\[
\lim_{|x| \to \infty} |x|^{2+\varepsilon} p_\epsilon(x) = \beta \quad (2.4)
\]

and both Assumptions 1 and 2 are fulfilled.

**Theorem 3** If \( \alpha > \alpha_2 \) then BRW is recurrent. If \( \alpha < \alpha_1^* \) for some \( \alpha_1^* \leq \alpha_1 \) then two cases are possible:

1) if \( \varepsilon < \varepsilon_1(\alpha, \beta) \) then BRW is recurrent;

2) if there exists \( m_1 > m \) such that \( P(k(x) > m_1) = 0 \) for all but finitely many \( x \) and \( \varepsilon > \varepsilon_1(\alpha, \beta, m_1) \) then BRW is transient.

**Proof.** The case \( \alpha > \alpha_2 \) follows from Theorem 2. To show (1) and (2) we again compare a fully quenched problem with a quenched one and use Theorem 3 from [1]. For (1) we consider a quenched problem with two colors. The probability of having a green point is \( p_\epsilon(x) \) and branching at it satisfies \( k(x) \equiv 2 \). The probability of a blue point is \( p_\epsilon(x) \) and branching at it satisfies \( k(x) \equiv m + 1 \). Then a constant \( \varepsilon_1 \) which guarantees recurrence of quenched BRW is provided by Theorem 3 in [1]. However, non-recurrency would be more difficult in the fully quenched problem, so (1) is shown.

To justify (2) we compare a fully quenched BRW with a quenched one with the same probabilities of green and blue points as above, while \( k(x) \equiv m \) at green points and \( k(x) \equiv m_1 \) at blue points. The fully quenched BRW is ‘not less transient’ and the statement again follows from Theorem 3 in [1]. \( \square \)
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References


