A note on the lilypond model

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Abstract

We consider some generalizations of the germ-grain growing model studied by Daley, Mallows and Shepp (2000). In this model, a realization of a Poisson process on a line with points $X_i$ is fixed. At time zero simultaneously at each $X_i$ a circle (grain) starts growing at the same speed.

It grows until it touches another grain, and then it stops. The question is whether the point zero is eventually covered by some circle.

In our note we expand this model in the following three directions. We study: (1) a one-sided growth model with a fixed number of circles; (2) grain-growth model on a regular tree; and (3) grain-growth model on a line with non-Poisson distributed centers of the circles.

Keywords: lilypond model, Poisson point processes, non-overlapping spheres, germ-growth model.

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1 Introduction

The purpose of this note is to generalize the results obtained by Daley, Mallows and Shepp (2000) for a one-dimensional Poisson growth model with non-overlapping intervals. This model was conceived by Håggström

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and Meester (1999) for somewhat different purposes. The so-called *lilypond model* they suggested is as follows. Fix a realization of a Poisson process in $\mathbb{R}^d$, $d \geq 1$ (the rate of process is not important). Then, simultaneously at each node of the process a sphere centered at this node starts growing, with the same constant speed for each and every node, with the growth for any particular sphere ceasing as soon as the sphere touches any other sphere that may have ceased growing earlier or else ceases growth at the same time (meaning that unless the sphere centered at $X_i$ has stopped, at time $t$ its radius will be $vt$ where the fixed constant $v > 0$ is the same for all $i$’s).

Häggström and Meester (1999) showed that for any $d$ the model is well-defined and there is no infinite cluster of touching spheres, which they refer as to *grain-germs*.

Daley et al. (2000) considered the same model on a line, for $d = 1$, and in some aspects treated this model as a regenerative phenomenon in the sense of Kingman (1972). One of the main questions they answered is what the probability is that a randomly chosen point of a line is not eventually covered by some circle. The answer, which is slightly intriguing, is $e^{-1}$, and $e^{-1/2}$ if one considers this model only on a half-line $\mathbb{R}_+$ and asks what the probability is that the origin is covered.

Earlier results by Daley, Stoyan and Stoyan (1999) provided numerical estimates for the probability that a randomly chosen point in $\mathbb{R}^d$, $d \geq 1$ is covered by some sphere.

Since then, there were a few generalizations of Daley et al. (2000) results. Huffer (2002) considered a Poisson model on a line with random growth rate of different germs, also allowing the speed at which the germ grows to the left to be different from the speed at which it grows to the right, both being random. He found that many of the results of Daley et. al. (2000) still hold in this case. Some relevant results for directed annihilating growth can be found in O’Hely and Sudbury (2001) and Sudbury (2002).

So far there are no other theoretical results for similar questions about the lilypond model on $\mathbb{R}^d$, $d \geq 2$, though as mentioned above, some proven numerical estimates of the probability that a randomly chosen point is covered are available in Daley et. al. (1999).

In Section 2 we extend lilypond model to regular trees, and show that
if the lengths of the branches of a regular $b$-ary tree, $b \geq 2$, are iid exponentially distributed and the grains (spheres) grow only at the vertices of the tree, except the root, the probability that the root is not covered is $\left[1/2 + 1/(2b)\right]^{b/(b-1)}$. In Section 3 we investigate the lilypond model on $\mathbb{R}_+$. For a fixed number $k$ of Poisson-distributed centers of circles we explicitly calculate the probability that the origin is not covered by one of them. By letting $k \to \infty$ we get that this probability eventually converges to $e^{-1/2}$, thus providing an alternative method for Daley et al. (2000) result. In Sections 4 and 5 we omit the restriction that the centers of the circles are distributed according to the Poisson distribution, replacing this assumption by the condition that the distances between two neighboring centers are iid distributed according to an arbitrary continuous law with a finite mean. Using the theory of contraction mappings, we provide the method which allows, though not always explicitly, to calculate the probability that a randomly chosen point is not covered by any circle.

2 Liaries on regular trees

Consider a regular $b$-ary rooted tree with $b \geq 1$ and suppose that the lengths of the edges are iid exponentially($\lambda$) distributed. Denote the corresponding probability measure as $\mathbb{P}$. For any two vertices $v$ and $w$ sharing the same edge let $d(v, w)$ denote the distance between them; thus under $\mathbb{P}()$, $\{d(v, w) : (v, w) \text{ is an edge}\}$ is a collection of iid random variables.

At time $t = 0$ at each vertex except the root a circle starts growing with the same speed. Whenever two circles centered at adjacent vertices of the tree touch, both of them (or one of them if the other has already stopped) stop growing. We stress that a circle centered at $v$ can be stopped only by circles centered at the vertices adjacent to $v$. It is straightforward that all the circles on any finite subset of vertices of the tree eventually stop growing.

Let $v_0$ be the root of the tree, $v_1, v_2, \ldots, v_b$ be the vertices at the next level, and let $x = d(v_0, v_1)$.

We are interested in the probability that the root of the tree is not covered by any circle by the time the circles with centers in $v_1, v_2, \ldots, v_b$ have stopped growing.
Figure 1: Figure of the $b$-ary rooted tree.

**Theorem 1**

(1) If $b > 1$ then

$$P(\text{the root is not covered}) = \left( \frac{b + 1}{2b} \right) \frac{1}{2}$$  \hspace{1cm} (2.1)

(2) If $b = 1$ then

$$P(\text{the root is not covered}) = e^{-\frac{1}{2}}.$$

**Proof.** Without loss of generality we can set $\lambda = 1$. Label the direct descendants of $v_1$ by $w_1, w_2, \ldots, w_b$ and set $y := d(v_1, w_1)$. For $i = 1, 2, \ldots, b$ let $G_i$ be the tree consisting of the subtree rooted at $w_i$ and the bonds $\{v_0, v_i\}$ and $\{v_1, w_i\}$.

Fix an $x > 0$ and consider the model described before the theorem with the additional restriction that $d(v_0, v_1) = x$ is not random. Let $P_x$ be the probability measure corresponding to the model with this restriction and denote

$$G(x) = P_x(v_0 \text{ is covered by circle centered at } v_1).$$

For $i = 1, \ldots, b$ introduce the events

$$C_i = \{\text{the circle centered at } v_1 \text{ on } G_i \text{ eventually hits } v_0\}$$
and also let

\[ C = \{ v_0 \text{ is covered by the circle centered at } v_1 \} \]

Then the root \( v_0 \) will be covered by the circle with center \( v_1 \) if and only if this circle is not stopped by any of the circles centered at \( w_i, i = 1, 2, \ldots, b \) before it hits \( v_0 \). Hence,

\[ C = C_1 \cap C_2 \cap \cdots C_b. \]

Since \( \mathbb{P}_x(C_i) \) has the same value for all \( i \) and since \( C_i \)'s are independent under \( \mathbb{P}_x \), we have

\[ G(x) = \mathbb{P}_x(C) = \mathbb{P}_x(C_1)^b. \]

Let us calculate \( \mathbb{P}_x(C_1) \), using that the subtree rooted at \( v_1 \) is isomorphic to the original tree. If \( y < x \), the probability that the root \( v_0 \) is covered by the circle growing at \( v_1 \) for the lilypond model on the tree \( \mathcal{G}_1 \) is 0; if \( x < y < 2x \), this probability is \( 1 - G(y - x) \); and if \( y > 2x \), it is 1. Therefore

\[ \mathbb{P}_x(C_1) = \int_x^{2x} [1 - G(y - x)]e^{-y} dy + \int_{2x}^{\infty} e^{-y} dy. \quad (2.2) \]

Consequently, after re-arrangements,

\[ G(x) = \left( e^{-x} - \int_0^x G(u)e^{-u-x} du \right)^b. \]

Let

\[ Q(x) = 1 - \int_0^x G(u)e^{-u} du, \]

then (2.2) becomes \( G(x) = e^{-bQ(x)} \), which, due to the relationship between \( G(x) \) and \( Q(x) \), can be restated as

\[ -Q'(x)e^x = e^{-bQ(x)} \quad (2.3) \]

Let us consider the case \( b \geq 2 \) first. Solving the differential equation (2.3) with the obvious initial condition \( Q(0) = 1 \), since \( G(0) = 1 \) by definition, we get

\[ Q(x) = \left( \frac{b + 1}{2b - (b - 1)e^{-(b+1)x}} \right)^{\frac{1}{b+1}}. \]
Now recall that under the original model \( x = d(v_0, v_1) \) is itself exponentially distributed, whence

\[
\mathbb{P}(\text{the root is covered by circle with center } v_1) = \int_0^\infty \mathbb{P}_x(C)e^{-x} \, dx = \int_0^\infty G(x)e^{-x} \, dx = \lim_{x \to \infty} (1 - Q(x)) = 1 - \left(\frac{b + 1}{2b}\right) \int_0^1.
\]

Therefore,

\[
\mathbb{P}(\text{the root is not covered by the circle with center } v_1) = \left(\frac{b + 1}{2b}\right)^{\frac{1}{1 - t}},
\]

yielding

\[
\mathbb{P}(\text{the root is not covered}) = \left(\frac{b + 1}{2b}\right)^{\frac{1}{1 - t}}.
\]

The case \( b = 1 \) is equivalent to the model of by Daley, Mallows and Shepp in [2] on a half-line \( \mathbb{R}_+ \). Solving (2.3) with \( b = 1 \) and using the same arguments as in the case \( b \geq 2 \), we obtain that the probability that \( v_0 \) is not covered is \( e^{-\frac{x}{b}} \), which is the result of [2].

Note that the probability of interest for \( b = 1 \) can be obtained by formally letting \( b \to 1 \) in equation (2.1).

\[
\mathbb{P}(\text{the root is not covered}) = \left(\frac{b + 1}{2b}\right)^{\frac{1}{1 - t}}
\]

Theorem 1 yields that for \( b \geq 1 \) the probability that the root will not be covered decreases as \( b \) increases, and that this probability is bounded below by \( \frac{1}{2} \) and above by \( e^{-\frac{1}{2}} \).

### 3 Lilypond on a line with a fixed number of lilies

Here we investigate the model studied in [2]. The points which are centers of the circles are distributed on \( \mathbb{R} \) according to the Poisson point process on \( \mathbb{R} \) with intensity \( \lambda > 0 \). At time \( t = 0 \), circles start growing at a common constant rate 1. Whenever two circles touch, both those circles stop growing. Obviously, all circles situated on any fixed finite interval eventually stop growing. Daley et al. (2000) calculated the probability that at the time the
two circles to the left and to the right of some point \( v \in \mathbb{R} \) have stopped growing, \( v \) is not covered by either.

Because of the properties of the Poisson point process, this probability does not depend on the parameter \( \lambda \), yielding that it suffices to consider only the case \( \lambda = 1 \) and \( v = 0 \). We will refer to this model as the \textit{two-sided Poisson Model}.

The half-line version of the above problem was also studied in [2]. In this version, which we call a \textit{one-sided Poisson Model}, the centers of the circles are distributed as a Poisson point process on \( \mathbb{R}_+ \) with rate 1, and the probability that the point 0 is not covered by the circle to the right from it is calculated. Because of the memory-lack property of the exponential distribution, the probability that 0 is not covered in the two-sided model is the square of its counterpart in the one-sided version.

In this section we provide an alternative method to [2] to calculate this probability, by investigating it in the case of a fixed number of circles.

Let \( X_1, X_2, \ldots, X_n, \ldots \) be the points of the Poisson point process on the positive half-line and \( \Delta_1, \Delta_2, \ldots, \Delta_n, \ldots \) be the distances between consecutive nodes of the process, such that \( \Delta_1 = X_1 > 0, \Delta_2 = X_2 - X_1 > 0, \ldots, \Delta_n = X_n - X_{n-1} > 0, \ldots \) Now, let us fix a positive integer \( k \) and consider only the first \( k \) circles centered at \( X_1, \ldots, X_k \) and ignore all other circles. We are interested in the probability of the event \( \bar{A}_k \) that at the time all the \( k \) circles have stopped growing, 0 is not covered by the one centered at \( X_1 \).
We construct the following sequence of events:

\[
A_1 = \left\{ \Delta_1 > \frac{\Delta_2}{2} \right\}
\]

\[
A_2 = \left\{ \Delta_1 > \frac{\Delta_2}{2}, \Delta_2 - \Delta_1 > \frac{\Delta_3}{2} \right\}
\]

\[
A_3 = \left\{ \Delta_1 > \frac{\Delta_2}{2}, \Delta_2 - \Delta_1 > \frac{\Delta_3}{2}, \Delta_3 - \Delta_2 + \Delta_1 > \frac{\Delta_4}{2} \right\}
\]

\[\vdots\]

\[
A_n = \left\{ \Delta_1 > \frac{\Delta_2}{2}, \Delta_2 - \Delta_1 > \frac{\Delta_3}{2}, \Delta_3 - \Delta_2 + \Delta_1 > \frac{\Delta_4}{2}, \ldots, \Delta_n - \Delta_{n-1} + \Delta_1 > \frac{\Delta_{n+1}}{2} \right\}
\]

\[\vdots\]

For any two events, \(B\) and \(C\), we will write that \(B \cong C\), if \(B\) and \(C\) coincide up to the set of zero measure, that is the probability of the symmetric difference \(\mathbb{P}(B \triangle C) = 0\).

**Lemma 1** Let \(k \geq 2\).

(a) \(A_{k-1} \subset A_{k-2} \subset \cdots \subset A_2 \subset A_1\).

(b) The events \((A_1 \setminus A_2), (A_3 \setminus A_4), \ldots\) are disjoint.

(c)

\[
\tilde{A}_k \cong \begin{cases} 
(A_1 \setminus A_2) \cup (A_3 \setminus A_4) \cup \cdots \cup (A_{k-2} \setminus A_{k-1}) & \text{when } k \text{ is odd}; \\
(A_1 \setminus A_2) \cup (A_3 \setminus A_4) \cup \cdots \cup (A_{k-3} \setminus A_{k-2}) \cup A_{k-1} & \text{when } k \text{ is even.}
\end{cases}
\]

**Proof.** The statement in (a) immediately follows from the definition of the events \(A_i\) and (b) follows from (a).

Fix an \(i \in \{1, 2, \ldots, k\}\). Suppose for a moment that the centers of the circles are located at \(X_i, X_{i+1}, \ldots, X_k\) only (as if we have “removed” the first \(i - 1\) circles altogether) and let \(t_i\) be the time in this truncated model when the leftmost circle centered at \(X_i\) stops growing (set \(t_k = +\infty\) by default). Then \(t_i\) is a (piece-wise linear) function of \(\Delta_{i+1}, \ldots, \Delta_k\). Since in the situation when we have removed the first \(i - 1\) circles, the circle centered at \(X_i\) will eventually stop only when it hits the circle centered at \(X_{i+1}\), we
have

\[ t_{k-1} = \Delta_k / 2; \]
\[ t_{k-2} = \left\{ \begin{array}{ll} \Delta_{k-1}/2, & \text{if } \Delta_{k-1} < \Delta_k, \\ \Delta_{k-1} - \Delta_k/2, & \text{otherwise}; \end{array} \right. \]
\[ = \max(\Delta_{k-1}/2, \Delta_{k-1} - t_{k-1}), \]
\[ \vdots \]
\[ t_i = \max(\Delta_{i+1}/2, \Delta_{i+1} - t_{i+1}), \]
\[ \vdots \]

Consider the events

\[ E_i = \{ t_i < \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1 \}, \]
\[ B_i = \{ \Delta_{i+1}/2 < \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1 \}. \]

Note that since \( t_i \) does not depend on \( \Delta_j, j \leq i \), \( \mathbb{P}(t_i = \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1) = 0 \) and therefore

\[ E_i^c \cong \{ t_i > \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1 \}. \]

The point 0 is not covered when all first \( k \) circles are present only if \( t_1 < \Delta_1 \), hence \( \bar{A}_k \cong E_1 \). At the same time, for \( i = 1, 2, \ldots, k - 1 \),

\[ E_i = \{ \max(\Delta_{i+1}/2, \Delta_{i+1} - t_{i+1}) < \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1 \} \]
\[ = \{ \Delta_{i+1}/2 < \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1 \} \cap \{ \Delta_{i+1} - t_{i+1} < \Delta_i - \Delta_{i-1} + \cdots + \pm \Delta_1 \} \]
\[ \cong B_i E_{i+1}^c. \]

Plugging \( E_{i+1} \) into the formula for \( E_i \) recursively \( k - 1 \) times and taking into account that \( E_k = \emptyset \), we obtain that

\[ \bar{A}_k \cong \left\{ \begin{array}{ll} B_1 B_2^c \cup B_1 B_3 B_4^c \cup B_1 B_3 B_5 B_6^c \cup \cdots \cup B_1 B_3 \cdots B_{k-2} B_{k-1}^c, & k \text{ is odd}, \\ B_1 B_2^c \cup B_1 B_3 B_4^c \cup B_1 B_3 B_5 B_6^c \cup \cdots \cup B_1 B_3 \cdots B_{k-2} B_{k-1}, & k \text{ is even}. \end{array} \right. \]

For definiteness, suppose that \( k \) is odd. Then, using the fact that \( A^c \cap B = \)}
\(A_c \cup AB\) for any two events \(A\) and \(B\), we have

\[
\tilde{A}_k \equiv B_1 B_2^c \cup B_1 B_3 B_4^c \cup B_1 B_3 B_5 B_6^c \cup \cdots \cup B_1 B_3 \ldots B_{k-2} B_{k-1}^c \\
= B_1 \left[ B_2^c \cup (B_3 B_4^c \cup B_3 B_5 B_6^c \cup \cdots \cup B_3 \ldots B_{k-2} B_{k-1}^c) \right] \\
= B_1 B_2^c \cup B_2 B_3 B_4^c \cup B_2 B_3 B_5 B_6^c \cup \ldots \\
= B_1 B_2^c \cup B_1 B_2 B_3 \left[ B_4^c \cup (B_5 B_6^c \cup \ldots) \right] \\
= \cdots \equiv B_1 B_2^c \cup B_1 B_2 B_3 B_4^c \cup \ldots B_1 B_2 \ldots B_{k-3} B_{k-2} B_{k-1}^c.
\]

Since \(A_i = B_1 B_2 \ldots B_i\), we conclude that \(A_i \setminus A_{i+1} = B_1 B_2 \ldots B_i B_i^c\), and therefore

\[
\tilde{A}_k \equiv (A_1 \setminus A_2) \cup \cdots \cup (A_{k-2} \setminus A_{k-1}).
\]

The case when \(k\) is even can be handled similarly.

*Theorem 2* \(\mathbb{P}(\tilde{A}_k)\) is given by the formula

\[
\mathbb{P}(\tilde{A}_k) = \sum_{n=1}^{k-1} (-1)^{n-1} \mathbb{P}(A_n)
\]

and

\[
\mathbb{P}(A_n) = \frac{1}{2^{n-2}(n-1)! (2n-1)(2n-1)} \quad n \geq 1.
\]

**Proof.** By using the formula for \(\tilde{A}_k\) from the statement of Lemma 1, we obtain

\[
\mathbb{P}(\tilde{A}_k) = \sum_{n=1}^{k-1} (-1)^{n-1} \mathbb{P}(A_n).
\]

Now we calculate \(\mathbb{P}(A_n)\) for \(n \geq 1\). Set

\[
C_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 \geq \frac{x_2}{2} > 0, x_2 - x_1 > \frac{x_3}{2} > 0, \right. \\
\left. x_3 - x_2 + x_1 > \frac{x_4}{2} > 0, x_{n-1} - x_{n-2} + \cdots \pm x_1 > \frac{x_n}{2} > 0, \\
x_{n} - x_{n-1} + \cdots \pm x_1 > \frac{x_{n+1}}{2} > 0 \right\}
\]

Then, from the definition of \(A_n\), it follows that

\[
\mathbb{P}(A_n) = \int_{C_n} e^{-x_1 - x_2 - \cdots - x_{n+1}} \, dx_1 \, dx_2 \cdots \, dx_{n+1}. \tag{3.4}
\]
We make the following change of variables in the above integral:

\[
\begin{align*}
  x_1 &= z_1 + z_2 + \cdots + z_n + z_{n+1}, \\
  x_2 &= z_1 + 2z_2 + 2z_3 + \cdots + 2z_n + 2z_{n+1}, \\
  x_3 &= z_2 + 2z_3 + 2z_4 + \cdots + 2z_n + 2z_{n+1}, \\
  &\vdots \\
  x_n &= z_{n-1} + 2z_n + 2z_{n+1}, \\
  x_{n+1} &= z_n + 2z_{n+1}.
\end{align*}
\]

The initial conditions on \(x_1, x_2, \ldots, x_n\) imply that

\[
0 < z_1, z_2, \ldots, z_n < \infty, \quad -\frac{z_n}{2} < z_{n+1} < \infty.
\]

The Jacobian of the transformation is 1 and thus (3.4) becomes

\[
\mathbb{P}(A_n) = \int_0^\infty \cdots \int_0^\infty \int_0^{\frac{1}{2z_1}} e^{-(2n+1)z_{n+1}-2[nz_n+(n-1)z_{n-1}+\cdots+2z_n]} \, dz_1 \cdots dz_{n+1}
\]

Iteratively we obtain

\[
\begin{align*}
\mathbb{P}(A_n) &= \frac{1}{2n+1} \int_0^\infty \int_0^{z_1} \cdots \int_0^{z_{n-1}} e^{-\frac{2n+1}{2}z_{n+1}-2[(n-1)z_{n-1}+\cdots+2z_n]} \, dz_1 \cdots dz_n \\
&= \frac{1}{2n+1} \int_0^\infty e^{-\frac{2n+1}{2}z_1} \, dz_1 \int_0^\infty e^{-2(n-1)z_{n-1}} \, dz_{n-1} \cdots \int_0^\infty e^{-2z_n} \, dz_n \\
&= \frac{1}{2^{n-2}(n-1)! (2n+1)(2n-1)}.
\end{align*}
\]

**Lemma 2**

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}(n-1)! (2n-1)(2n+1)} = e^{-\frac{1}{2}}.
\]

**Proof.** For \(n \geq 2\), we have

\[
\frac{(-1)^{n-1}}{2^{n-2}(n-1)! (2n-1)(2n+1)} = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \left[ \frac{1}{2n-1} - \frac{1}{2n+1} \right]
\]

\[
= \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} + \frac{(-1)^{n}}{2^{n-1}(n-1)! (2n-1)(2n+1)}.
\]

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Therefore,
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}(n-1)! (2n-1)! (2n+1)} = \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1}}{2^{n-1}(n-1)! (2n-1)} + \frac{(-1)^n}{2^{n-1}(n-1)! (2n+1)!} \right]
\]
\[
= \left( \frac{2}{3} \right) - \frac{1}{6} + \cdots + \left( \frac{(-1)^n}{2^{n-1}(n-1)! (2n+1)!} + \frac{(-1)^n}{2^n(n-1)! n(2n+1)} \right) + \cdots
\]
\[
= 1 - \frac{1}{2} + \left( \frac{-1}{2} \right)^2 \frac{1}{2!} + \cdots + \left( \frac{-1}{2} \right)^n \frac{1}{n!} + \cdots
\]
\[
= \sum_{n=0}^{\infty} \left( \frac{-1}{2} \right)^n \frac{1}{n!}
\]
\[
= e^{-\frac{1}{2}}.
\]

The following statement proves the result of [2] using our alternative method.

**Corollary 1** Let \( \tilde{A} \) be the event \{0 is not covered\} when there are infinitely many centers of circles. Then
\[
\mathbb{P}(\tilde{A}) = \sum_{i=1}^{\infty} (-1)^{i-1} \mathbb{P}(A_i) = e^{-\frac{1}{2}}.
\]

**Proof.** For \( m = 1, 2, \ldots \) define \( R_m = \{ \Delta_{m-1} < \Delta_m/2 \} \) and
\[
\mathcal{R} = \limsup_{k \to \infty} R_{2k+1} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} R_{2k+1},
\]
the event that \( R_{2k+1} \) occur infinitely often. Since the events \( R_{2k+1} \) are independent for different positive integers \( k \), and each such an event has the same positive probability, by Borel-Cantelli lemma \( \mathbb{P}(\mathcal{R}) = 1 \).

On the other hand, \( R_m \) guarantees that the circle centered at \( X_{m-1} \) will hit the circle centered at \( X_{m-2} \) before it can possibly meet the one centered at \( X_m \). Hence, on \( R_m \), for any \( n \geq m \), the events \( \tilde{A} \) and \( \tilde{A}_n \) either both occur or both do not occur; formally
\[
\tilde{A} R_m = \tilde{A}_n R_m, \quad n \geq m.
\] (3.5)
Observe that by Lemma 1, part (c), the events $\tilde{A}_{2k+1}$, $k = 1, 2, \ldots$ form an increasing sequence of events; hence let

\[
\tilde{A}_{odd} = \lim_{k \to \infty} \tilde{A}_{2k+1} = \bigcup_{k=1}^{\infty} \tilde{A}_{2k+1}.
\]

Since $\mathbb{P}(\mathcal{R}) = 1$, $\mathbb{P}(\tilde{A}) = \mathbb{P}(\tilde{A}\mathcal{R})$, while using (3.5) we obtain

\[
\tilde{A}\mathcal{R} = \tilde{A} \bigcap \bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} R_{2k+1} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tilde{A}R_{2k+1} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tilde{A}_{2k+1}R_{2k+1} = \bigcap_{n=1}^{\infty} C_n,
\]

where $C_n := \bigcup_{k=n}^{\infty} \tilde{A}_{2k+1}R_{2k+1}$. The events $C_n$ form a decreasing sequence, therefore by continuity of probability

\[
\mathbb{P}(\tilde{A}) = \mathbb{P}(\tilde{A}\mathcal{R}) = \lim_{n \to \infty} \mathbb{P}(C_n). \tag{3.6}
\]

At the same time $\tilde{A}_{2n+1} \subseteq \tilde{A}_{2k+1} \subseteq \tilde{A}_{odd}$ for $k \geq n$, whence

\[
\tilde{A}_{2n+1} \left( \bigcup_{k=n}^{\infty} R_{2k+1} \right) \subseteq C_n \subseteq \tilde{A}_{odd} \left( \bigcup_{k=n}^{\infty} R_{2k+1} \right)
\]

and since $\mathcal{R} \subseteq \bigcup_{k=n}^{\infty} R_{2k+1}$ and $\mathbb{P}(\mathcal{R}) = 1$,

\[
\mathbb{P}(\tilde{A}_{2n+1}) = \mathbb{P} \left( \tilde{A}_{2n+1} \left( \bigcup_{k=n}^{\infty} R_{2k+1} \right) \right) \leq \mathbb{P}(C_n)
\]

\[
\leq \mathbb{P} \left( \tilde{A}_{odd} \left( \bigcup_{k=n}^{\infty} R_{2k+1} \right) \right) = \mathbb{P}(\tilde{A}_{odd}).
\]

Therefore, by Theorem 2,

\[
\sum_{i=1}^{2n} (-1)^{i-1} \mathbb{P}(A_i) = \mathbb{P}(\tilde{A}_{2n+1}) \leq P(C_n) \leq \mathbb{P}(\tilde{A}_{odd}) = \sum_{i=1}^{\infty} (-1)^{i-1} \mathbb{P}(A_i)
\]

where the last equality uses the continuity of probability applied to the increasing sequence $\tilde{A}_{2k+1}$. Letting $n \to \infty$ and combining this with (3.6) finishes the proof.

\[\blacksquare\]
4 The generalized model on a half line

In this section we remove the assumption that the distances between the centers of the circles are exponentially distributed (as it was the case in the Poisson model).

We start by defining this process on $\mathbb{R}_+$. Consider an ordinary renewal process on the positive real line such that the inter-arrival times are positive iid continuous random variables with common cdf $F(x)$ and the density function $f(x) = F'(x)$, $x > 0$.

Similarly to the Poisson model, at time $t = 0$ germs (circles) start growing around all the nodes with a common constant rate. Whenever two circles touch, both of them stop growing. Again it is clear that by some (random) finite time the circles on any finite interval eventually stop growing.

As before, we are interested in the probability that 0 is not covered by the time the circle with center in the closest to 0 node has stopped. We call this model the One-Sided Generalized Model.

One way to find this probability is to apply the method used in Section 3. Then, if $\bar{A}$ is the event \{0 is not covered\}, we have

$$\mathbb{P}(\bar{A}) = \sum_{n=1}^{\infty} (-1)^{n-1} \mathbb{P}(A_n),$$

(4.7)

where $\mathbb{P}(A_n)$ can be shown (see Theorem 2) to have the following representation:

$$\mathbb{P}(A_n) = \int_0^\infty \cdots \int_0^\infty \int_{-\delta_2}^\infty \int_{-\delta_2}^\infty f(z_1 + z_2 + \cdots + z_n + z_{n+1}) f(z_2 + z_3 + \cdots + 2z_n + 2z_{n+1}) \\
f(z_2 + 2z_3 + \cdots + 2z_n + 2z_{n+1}) \cdots f(z_n + 2z_{n+1}) dz_1 \cdots dz_{n+1}.$$  

We only have to prove now that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \mathbb{P}(A_n)$ is convergent. But, from the definition of $A_n$,

$$A_n \in \{\Delta_{n+1} < \Delta_n < \cdots < \Delta_2\} \subset \{\Delta_3 < \Delta_2, \Delta_5 < \Delta_4, \cdots \Delta_{2k+1} < \Delta_{2k} \cdots \}.$$  

Therefore, by the independence of the $\Delta_i$'s and using the above inclusions

$$\mathbb{P}(A_n) < \begin{cases} 2^{-\frac{2n}{2}} & \text{when } n \text{ odd;} \\ 2^{-\frac{n}{2}} & \text{when } n \text{ even.} \end{cases}$$

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Thus the series in (4.7) is (even absolutely) convergent. Yet it is still not possible to get an explicit formula out of it.

Further we adopt a different method to calculate the probability that 0 is not covered in the one-sided generalized model, and later we use this result for a renewal process defined on the whole real line. This method is similar to the one used in Section 2 and in [2].

Let $\Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$ be the inter-arrival times of the renewal process on the positive half-line. Again as in Section 2 consider a modified model: fix a constant $x > 0$, let $\Delta_1 = x$, while $\Delta_i$'s, $i = 2, 3, \ldots$ are iid with the cdf $F(x)$, and denote the probability measure corresponding to this model as $\mathbb{P}_x$. Set

$$G(x) = \mathbb{P}_x(0 \text{ is covered}).$$

Again, if $y := \Delta_2 < x$, the probability ($\mathbb{P}_x$) that 0 is covered is 0; if $x < y < 2x$, the probability is $1 - G(y - x)$, and if $y > 2x$, it is 1. Thus

$$G(x) = \int_x^{2x} [1 - G(y - x)] f(y) \, dy + \int_{2x}^{\infty} f(y) \, dy,$$

yielding

$$G(x) = \int_x^{\infty} f(u) \, du - \int_0^{x} G(u) f(u + x) \, du. \tag{4.8}$$

**Theorem 3** Equation (4.8) has a unique continuous solution such that $0 \leq G(x) \leq 1$.

**Proof.** Let $n = \inf \{u : F(u) > 0\}$ and $N = \sup \{u : F(u) < 1\}$, then $n$ ($N$ resp.) is the lower (upper resp.) bound of the support of $f(u)$. Below we will assume that $N < \infty$, and the case $N = \infty$ can be handled in a similar fashion. Fix $0 < A < N$ and consider the set of bounded continuous functions $\mathcal{X} \subseteq C[0, A]$ with the property that $\phi(x) \in \mathcal{X}$ if and only if $0 \leq \phi(x) \leq 1$, $0 \leq x \leq A$. Then $\mathcal{X}$ is a complete metric space with the distance defined as

$$d(\phi_1, \phi_2) = \sup_{x \in [0, A]} |\phi_1(x) - \phi_2(x)|$$

Let $K$ be a mapping on $\mathcal{X}$ such that

$$K : \phi \mapsto 1 - F(x) - \int_0^x \phi(u) f(u + x) \, du.$$
First, we prove that $K(\mathcal{X}) \subseteq \mathcal{X}$. Indeed, if $0 \leq \phi(x) \leq 1$, then
\[
1 \geq 1 - F(x) - \int_0^x \phi(u) f(u + x) \, du \geq 1 - F(x) - \int_0^x f(u + x) \, du \\
= 1 - F(2x) \geq 0,
\]
so $0 \leq K\phi(x) \leq 1$. Also, if $0 \leq \phi(x) \leq 1$ is continuous on $[0, A]$, it is uniformly continuous and therefore there exists $B(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ such that
\[
\sup_{x, y \in [0, A]: |x - y| \leq \varepsilon} |\phi(x) - \phi(y)| \leq B(\varepsilon).
\]
Therefore
\[
|K\phi(x + \varepsilon) - K\phi(x)| \leq \int_{x+\varepsilon}^{x+2\varepsilon} \phi(v - x - \varepsilon) f(v) \, dv - \int_{x+\varepsilon}^{x+2\varepsilon} \phi(v - x) f(v) \, dv \\
+ \int_{x+\varepsilon}^{x+2\varepsilon} \phi(v - x) f(v) \, dv - \int_x^{x+\varepsilon} \phi(v - x) f(v) \, dv + F(x + \varepsilon) - F(x) \\
\leq \int_{x+\varepsilon}^{x+2\varepsilon} |\phi(v - x - \varepsilon) - \phi(v - x)| f(v) \, dv + \int_{x+\varepsilon}^{x+2\varepsilon} \phi(v - x) f(v) \, dv \\
+ \int_{x+\varepsilon}^{x+2\varepsilon} \phi(v - x) f(v) \, dv + F(x + \varepsilon) - F(x) \\
\leq B(\varepsilon)[F(2x + 2\varepsilon) - F(x + \varepsilon)] + 2[F(x + \varepsilon) - F(x)] + |F(2x + 2\varepsilon) - F(2x)| \to 0
\]
when $\varepsilon \to 0$ since $F(\cdot)$ is a continuous function. Thus, $K\phi(x)$ is continuous and therefore $K(\mathcal{X}) \subseteq \mathcal{X}$. Note that the operator
\[
K_1 : \phi(x) \to \int_0^x \phi(u)(-f(u + x)) \, du
\]
is bounded ( $\sup|K_1\phi(x)| \leq \sup|\phi(x)| \int_0^A f(u) \, du \leq \sup|\phi(x)|$ ), and $K_1$ is a linear operator on a complete metric space; therefore $K_1$ is a continuous linear operator (see e.g. [9]). Since for any two functions $\phi'$ and $\phi''$ $K(\phi' - \phi'') = K_1(\phi' - \phi'')$, $K$ is a continuous mapping too. At the same time $\mathcal{X}$ is a compact convex subset of $C[0, A]$, consequently by Schauder fixed point theorem $K$ has a fixed point. Let us prove that this point is unique.

Suppose $K\phi'(x) = \phi'(x)$ and $K\phi''(x) = \phi''(x)$, and set $\phi(x) = \phi'(x) - \phi''(x)$. Then $K_1\phi(x) = \phi(x)$ and it suffices to show that $\phi \equiv 0$. Denote $\|\phi\|_{a,b} = \sup_{x \in [a,b]} |\phi(x)|$. Fix $\varepsilon > 0$ such that $n + \varepsilon < A$ whence $\gamma := F(n + \varepsilon) \in (0, 1)$. Then for $x \in [0, (n + \varepsilon)/2]
\[
|\phi(x)| \leq \int_0^x |\phi(u)|f(u + x) \, du \leq \|\phi\|_{0,(n+\varepsilon)/2}[F(2x) - F(x)] \leq \gamma\|\phi\|_{0,(n+\varepsilon)/2},
\]
but
Therefore, restricted on \( [0, (n+\varepsilon)/2] \) operator \( K_1 \) is contracting and therefore by contraction principle \( \phi(x) \equiv 0 \) for \( x \leq (n + \varepsilon)/2 \).

Next, for \( x \geq (n + \varepsilon)/2 \) we have (since \( \phi(x) = 0 \) for \( x \leq (n + \varepsilon)/2 \))
\[
|\phi(x)| \leq \int_0^x |\phi(u)| f(u + x) \, du = \int_{(n+\varepsilon)/2}^x |\phi(u)| f(u + x) \, du \\
\leq \| \phi \|_{(n+\varepsilon)/2, A} [F(2x) - F(x + (n + \varepsilon)/2)] \leq (1 - \gamma) \| \phi \|_{(n+\varepsilon)/2, A}
\]
and therefore \( K_1 \) is contracting on \( [(n + \varepsilon)/2, A] \) as well. Consequently \( \phi(x) \equiv 0 \) for all \( x \in [0, A] \), and the equation \( K \phi = \phi \) has on \( [0, A] \) a unique solution \( G_A(x) \), and so does \((4.8)\). Now choose \( A' \in (A, N) \) and consider the solution \( G_{A'}(x) \). It is clear from equation \((4.8)\) that \( G_{A'}(x) \) restricted to \( [0, A] \) also solves \((4.8)\), and therefore by uniqueness it must coincide with \( G_A(x) \). Thus, the solution \( G_A(x) \) can be extended to the whole segment \( [0, N] \), and \( G(x) = 0 \) for \( x > N \) from the definition of \( G(x) \).

\[
\square
\]

Finally, recalling that \( \Delta_1 = x \) is itself continuously distributed with the density \( f(x) \), we have
\[
\mathbb{P}(0 \text{ is not covered}) = 1 - \int_0^\infty \mathbb{P}_x(0 \text{ is covered}) f(x) \, dx \\
= 1 - \int_0^\infty f(x) G(x) \, dx.
\]

5 Generalized model on a line

Suppose now that the inter-arrival times of the renewal process are distributed on the whole real line, with the process starting at \(-\infty\). We will calculate the probability that \( 0 \) is not covered in this two-sided generalized model.

Let \( u := X_1 > 0 \) be the smallest positive and \(-v := X_0 < 0 \) be the largest negative node of the renewal process. Let \( X_1 < X_2 < \ldots < X_n < \ldots \) be the nodes of the process on the positive half-line and \( X_0 > X_{-1} > \ldots > X_{-n} > \ldots \) be the nodes on the negative half-line. Thus, \( X_i - X_{i-1}, i \in \mathbb{Z} \), are iid random variables with the density \( f \).

Consider the following positive random variables: \( U \), the distance from 0 to the last renewal, that is, \( 0 - X_0 \), and \( V \), the distance from 0 until the
next renewal, that is \( X_1 - 0 \). Then, since 0 is not covered at all if and only if it is not covered by a circle centered either at \( X_1 \) or at \( X_0 \), reducing this situation to two one-sided models, we have

\[
\mathbb{P}(\text{point } 0 \text{ is not covered}) = \int_{0}^{\infty} \int_{0}^{\infty} (1 - G(u))(1 - G(v)) h(u, v) \, du \, dv,
\]

where \( h(u, v) \) is the joint density function of \( U \) and \( V \), and \( G(x) = \mathbb{P}_x(0 \text{ is covered}) \) for the one-sided model introduced in Section 4. We have also used the fact that in the two-sided model the event “0 is not covered by a circle centered at \( X_1 \)” is independent of the event “0 is not covered by a circle centered at \( X_0 \)”.

Next we will use a well-known result about the joint distribution of \( U \) and \( V \) (see, e.g. [4]). Consider a renewal process started at time zero for which the inter-arrival times with a finite mean \( \mu \) have cumulative distribution function \( F \). Let \( t > 0 \) and \( U_t \) and \( V_t \) be, respectively, the time since the last by \( t \) renewal and the time to the next renewal after \( t \). Then, if \( F \) is non-arithmetic (see, for example [1] or [4])

\[
\lim_{t \to \infty} \mathbb{P}(U_t > u, V_t > v) = \frac{1}{\mu} \int_{u+v}^{\infty} (1 - F(x)) \, dx,
\]

(5.9)

where \( u, v \geq 0 \) and

\[
\mu = \int_{0}^{\infty} u F(du) < \infty.
\]

(5.10)

Since in the generalized two-sided model the renewal process starts at time \(-\infty\), we can use (5.9) to find the density \( h(u, v) \) thus obtaining that

\[
h(u, v) = \frac{\partial^2}{\partial u \partial v} \mathbb{P}(U > u, V > v) = \frac{\partial^2}{\partial u \partial v} \frac{1}{\mu} \int_{u+v}^{\infty} (1 - F(x)) \, dx
\]

\[
= \frac{1}{\mu} f(u + v).
\]

Consequently,

\[
\mathbb{P}(\text{point } 0 \text{ is not covered}) = \frac{1}{\mu} \int_{0}^{\infty} \int_{0}^{\infty} (1 - G(x))(1 - G(y)) f(x + y) \, dx \, dy.
\]

where \( \mu \) is given by (5.10) and \( G \) is the unique solution of (4.8).
6 Some examples, generalizations, and open questions

The theorems in the previous two sections imply that the equation (4.8) for continuously distributed distances between the centers always has a unique solution and that using this solution one can find the probability \( q = q(f) \) \((q_+ = q_+(f) \ \text{resp.})\) that a randomly chosen point of \( \mathbb{R} \) (or point 0 in \( \mathbb{R}_+ \) resp.) is not covered by a circle. However, solving the integral equation (4.8) explicitly seems to be too hard for majority of the “usual” distributions. It has, however, been solved for an exponential distribution by Daley et. al. (2000) yielding \( q_+ = e^{-\frac{b}{a}} \approx 0.607 \) and \( q = e^{-1} \approx 0.368 \).

Here we present the solution of (4.8) for a uniform \( U[a, b] \) distribution. It implicitly covers two cases: \( a = 0 \) and \( a > 0 \).

\[
G(x) = \begin{cases} 
1, & 0 \leq x < a/2; \\
\frac{a \cdot 2e^{\frac{a-x}{a-b}}}{2e^{\frac{a-b}{a-b}} - 1}, & a/2 \leq x < b/2; \\
\frac{a-2x}{b-a} + 2e^{\frac{b-x}{a-b}}, & b/2 \leq x < b-a/2 \\
0, & \text{otherwise.}
\end{cases}
\]

Therefore,

\[
q_+ = \begin{cases} 
\frac{1}{4} - \frac{6}{2(6-a)} + 4e^{-1/2} - 2e^{-a/(2(b-a))}, & b \geq 2a; \\
2e^{\frac{6-b}{6-a}} - \frac{e^{\frac{b}{a}}}{4(b-a)^2}, & 3a/2 \leq b < 2a; \\
1, & b \leq 3a/2
\end{cases}
\]

where for \( a = 0 \) we get the result as a limit: \( q_+ = 4e^{-1/2} - 7/4 \approx 0.676 \).

Incidentally, \( q_+ \) does not even depend on \( b \), because of the same reason why this number did not depend on the parameter of Poisson distribution in Daley et. al. (2000) case: one can always rescale the line such that the distribution between the centers of germs will be \( U[0, 1] \).

Similarly,

\[
q = \frac{48e^{-1} - 17}{3} \cdot \frac{b-a}{b+a}
\]

which monotonely increases in \( b/a \) from 0 to \( 16e^{-1} - 17/3 \approx 0.219 \) when \( a = 0 \).
A possible generalization of the lilypond model is outlined below. Suppose that each germ has a certain random lifetime, after which it cannot grow. In one-dimensional model this means that to each node $i$ we put in correspondence a nonnegative random variable $T_i$, for simplicity let $T_i$’s be iid with a common cdf $W(x) := \mathbb{P}(T \leq x)$. Each germs grows at the same constant speed until either it hits a neighbor or the time reaches $T$. If as before $G(x)$ denotes the probability that point 0 is covered from the right given the distance to the leftmost positive node is $x$, then $G(x)$ satisfies a slightly modified version of (4.8):

$$
G(x) = (1 - W(x)) \times \left[ \int_x^\infty f(u) \, du - \int_0^x G(u) f(u + x) \, du \right].
$$

Then the probabilities $q_+$ and $q$ that 0 is not covered in one- and two-sided model respectively can be computed in the same manner as above. For example, if the nodes of the germs form a Poisson point process with intensity $\lambda$, and the lifetime of germs has an exponential distribution with parameter $\mu$, then

$$
q_+ = e^{-\lambda/(\mu+2\lambda)}, \quad q = e^{-2\lambda/(\mu+2\lambda)}.
$$

Finally we note that we were able to solve (4.8) explicitly only for uniformly and exponentially distributed distances between the grains. However, it would be interesting to know if one can provide an explicit solution for any other distributions. Also, the analogue of the lilypond model in higher dimensions, even for Poisson-distributed centers of germs, still remains a challenging task.

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