Tutorial 4

1. Suppose we observe $Y_t, t \in I = [0,1]$, a stochastic process satisfying

$$dY_t = f(t)dt + \epsilon dW_t, \quad t \in I, \tag{1}$$

where $0 < \epsilon < 1$, (W_t) is a Wiener process and $f : I \to R$ is an unknown function. We suppose that $f \in L^2[0,1]$.

   (a) Let $\epsilon = n^{-\frac{1}{2}}$ and show that the model (1) is, for large n, equivalent to observing

$$y(i/n) \approx f(i/n) + Z_i, i = 1..., n \tag{2}$$

   where $Z_i's$ are iid $N(0,1)$ rvs. (Hint: set $y_t := (Y(t + 1/n) - Y(t))/(1/n)$)

   (b) Show that the model (1) is equivalent to observing

$$y_k = \theta_k + \epsilon\xi_k, k = 1, 2, ...$$

   where (ξ_k) is a sequence of iid $N(0,1)$ rvs. (Hint: let e_k be an orthonormal basis of $L^2[0,1]$...)

   (c) Suppose we observe

$$Y_i = f(i/n) + Z_i, i = 1, ..., n,$$

   where Z_i is a sequence of iid $N(0,1)$ rvs. Show that this is equivalent to observing

$$y_k \approx \theta_k + \epsilon\xi_k, k = 1, ..., n$$

   where (ξ_k) is a sequence of iid $N(0,1)$ rvs.(Hint: let e_k be an orthonormal basis of $L^2[0,1]$...)

2. Suppose that we observe a stochastic process satisfying (1). We wish to estimate f using a wavelet estimator. Let (Φ, Ψ) be an o.n.b. of $L^2[0,1]$, $\Psi_{j,k}(x) = 2^{\frac{j}{2}}\Psi(2^j x - k)$, $k = 0, ..., 2^j - 1, j \geq 1$, and define

$$\hat{\beta}_\kappa = \hat{\beta}_{j,k} := \int \Psi_{j,k}(t)dY_t.$$

Let $c_n = (\frac{\log n}{n})^{\frac{1}{2}}$, show that

   (a) $E(\hat{\beta}_\kappa - \beta_\kappa)^4 \leq C_1 c_n^4$.

   (b) there exists $\eta > 0$ such that $P(|\hat{\beta}_\kappa - \beta_\kappa| > \eta\frac{c_n}{2}) \leq c_2 c_n^4$.

   (c) Using the previous results and as well as results from the lecture notes, construct a non-linear wavelet estimator of f and describe its asymptotic properties w.r.t. MISE, when $f \in B^s_{r,r}$.