Recursive Estimation of Mixture Models with Applications in Video Analysis

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Background

- Part of a project to assess the safety of intersections by monitoring driving patterns.
- The project uses vehicle tracking based on a foreground/background segmentation.
- The tracking has experienced problems due to bad segmentation. Thus we want to improve the segmentation.

Gaussian Mixture Model

Introduce a model with local, per pixel, background and a global foreground. Using GMMs the distributions of the pixel values, $x_{ti}$, becomes,

$$p(x_{ti}) = \pi^F \sum_k \pi_k p(x_{ti} | \mu_k^F, \Sigma_k^F) + (1 - \pi^F) \sum_l \pi_l^B p(x_{ti} | \mu_l^B, \Sigma_l^B) .$$

Our goal is to estimate the parameters and classify each $x_{ti}$. But parameters might not be constant over time and the data arrives sequentially, i.e. we need to recursively update the parameters.
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\]

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Gaussian Mixture Model – Simplified

\[
p(x_{ti}) = \pi_F \sum_k \pi_k p(x_{ti} \mid \mu_k, \Sigma_k) + (1 - \pi_F) \sum_l \pi_l p(x_{ti} \mid \mu_l, \Sigma_l).
\]

To illustrate a simpler case is used,

\[
p(x_{ti}) = \sum_k \pi_k p(x_{ti} \mid \mu_k, \Sigma_k).
\]

Probability that pixel \( x_{ti} \) belongs to the \( k^{th} \)-mixture,

\[
P_{tik} = \frac{\pi_k p(x_{ti} \mid \mu_k, \Sigma_k)}{\sum_k \pi_k p(x_{ti} \mid \mu_k, \Sigma_k)}.
\]

Bayesian Formulation

Let \( \theta_T = \{ \pi_T, \mu_T, \Sigma_T \} \). Ideally we want a recursive Bayesian formulation,

\[
p(x_{ti} \mid \theta_T) = p(x_{ti} \mid \theta_T)p(\theta_T \mid x_{<i} \theta_T).
\]

An approximation is,

\[
p(x_{ti} \mid \theta_T) = p(x_{ti} \mid \theta_T)p(\theta_T \mid \Psi_T),
\]

where

\[
\Psi_T = g(\theta_{T-1}, \cdots, \theta_1) = \tilde{g}(\theta_{T-1}).
\]

But how do we select \( \Psi_T = \tilde{g}(\theta_{T-1}) \)?
Bayesian Formulation

Let \( \theta_T = \{ \pi_T, \mu_T, \Sigma_T \} \). Ideally we want a recursive Bayesian formulation,

\[
p(x_{\pi T}) = p(x_{\pi T} | \theta_T) p(\theta T | x_{<T}).
\]

An approximation is,

\[
p(x_{\pi T}) = p(x_{\pi T} | \theta_T) p(\theta T | \Psi T),
\]

where

\[
\Psi T = g(\theta T-1, \cdots, \theta_1) = \bar{g}(\theta T-1).
\]

But how do we select \( \Psi T = \bar{g}(\theta T-1) \)?

Bayesian Formulation – Updating Equations

\[
\ln L = \sum_{i=1}^{N} \ln p(x_{\pi i} | \theta_T) + \ln p(\theta_T | \Psi T).
\]

Suitable priors (Gelman et al., 2004; Ormoneit & Tresp, 1998):

\[
\pi_{1 \cdots K} \sim D(\beta_1, \cdots, \beta_K),
\]

\[
\mu_k | \Sigma_k \sim N(m_k, \eta_k^{-1} \Sigma_k),
\]

\[
\Sigma_k \sim IW(\xi_k, V_k).
\]

Using these priors the updating equation for \( \mu \) becomes,

\[
\mu_T = \frac{m_k \eta_k + \sum_i x_{T,i} P_{Tik}}{\eta_k + \sum_i P_{Tik}}.
\]

But how do we select \( \Psi T = \bar{g}(\theta T-1) \)?

Offline Estimates

Introduce a forgetting factor in the log-likelihood to handle varying parameters:

\[
\ln L = \sum_{t=1}^{T} \sum_{i=1}^{N} \alpha^{T-t} \ln p(x_{ti} | \theta).
\]

The EM-algorithm gives offline parameter estimates,

\[
\mu_k = \frac{\sum_{t,i} x_{T} P_{Tik} \alpha^{T-t}}{\sum_{t,i} P_{Tik} \alpha^{T-t}}.
\]
Offline Estimates cont.

Offline parameter estimate,

\[ \mu_k = \frac{\sum_{t,i} x_{t,i} P_{tik} \alpha^{T-t}}{\sum_{t,i} P_{tik} \alpha^{T-t}} . \]

Introduce the cumulative sums,

\[ S_{Tk} = \sum_{i=1}^{N} \sum_{t=1}^{T} P_{tik} \alpha^{T-t} = \alpha S_{T-1,k} + \sum_{i=1}^{N} P_{Tik} , \]

the offline estimates can be rewritten as online updating equations,

\[ \mu_{Tk} = \frac{\mu_{T-1,k} \alpha S_{T-1,k} + \sum_{i} x_{T,i} P_{Tik}}{\alpha S_{T-1,k} + \sum_{i} P_{Tik}} . \]

Selecting the Priors

Comparing the updating equations,

\[ \mu_{Tk} = \frac{m_k \eta_k + \sum_{i} x_{T,i} P_{Tik}}{\eta_k + \sum_{i} P_{Tik}} , \]

and

\[ \mu_{Tk} = \frac{\mu_{T-1,k} \alpha S_{T-1,k} + \sum_{i} x_{T,i} P_{Tik}}{\alpha S_{T-1,k} + \sum_{i} P_{Tik}} . \]

Gives a way of selecting the priors, e.g.

\[ \pi_{T,1:\cdots,K} \sim D(\alpha S_{T-1,1} + 1, \cdots, \alpha S_{T-1,K} + 1) , \]

\[ \mu_{Tk} \mid \Sigma_{Tk} \sim N(\mu_{T-1,k}, (\alpha S_{T-1,k})^{-1} \Sigma_{Tk}) , \]

\[ \Sigma_{Tk} \sim IW(\alpha S_{T-1,k} - d - 2, \alpha S_{T-1,k} \Sigma_{T-1,k}) , \]
Algorithm

1. Calculate posterior probabilities ($P_{Ti,k}$).
2. Recursively update the parameters.
3. Introduce new foreground components.
4. Transfer components representing stationary pixels from foreground to background.
5. Remove old, seldomly observed components.
Conclusions and Future Work

- Recursive parameter estimates in a Gaussian Mixture model.
- Bayesian interpretation of the recursive estimates.

On the applied side:
- Possible ways to speed up the algorithm.
- Object tracking using output from the algorithm.

On the statistical side:
- Utilise the spatial dependency.
- Fast methods for selecting the number of components in mixture models.