EXTREME VALUES AND CROSSINGS FOR THE
$\chi^2$-PROCESS AND OTHER FUNCTIONS OF
MULTIDIMENSIONAL GAUSSIAN PROCESSES,
WITH RELIABILITY APPLICATIONS

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Abstract

Extreme values of non-linear functions of multivariate Gaussian processes
are of considerable interest in engineering sciences dealing with the safety of
structures. One then seeks the survival probability

$$P((X_1(t), \ldots, X_n(t)) \in S, \quad \text{all } t \in [0, T]),$$

where $X(t) = (X_1(t), \ldots, X_n(t))$ is a stationary, multivariate Gaussian load
process, and $S$ is a safe region. In general, the asymptotic survival probability
for large $T$-values is the most interesting quantity.

By considering the point process formed by the extreme points of the vector
process $X(t)$, and proving a general Poisson convergence theorem, we obtain
the asymptotic survival probability for a large class of safe regions, including
those defined by the level curves of any second- (or higher-) degree polyno-
mial in $(x_1, \ldots, x_n)$. This makes it possible to give an asymptotic theory for
the so-called Hasofer-Lind reliability index, $\beta = \inf_{x \in S} \|x\|$, i.e. the smallest
distance from the origin to an unsafe point.

CONVERGENCE OF POINT PROCESSES; EXTREMAL THEORY; RELIABILITY; CHI-
SQUARED PROCESSES; CROSSINGS; MAXIMA; SAFETY INDEX; SAFETY OF STRUC-
TURES

1. Probabilistic structural analysis

Probabilistic structural analysis tries to answer the question: what is the
probability that a certain mechanical (or other) structure will survive when it is
subject to a random load. The load is then usually defined by some $n$-variate
random variable $X = (X_1, \ldots, X_n)$, and one seeks the probability that $X$
falls in some more or less well-defined safe region, which is specific for the structure.
As an example, let $X_1$ and $X_2$ be loads acting in two perpendicular directions,
and suppose that the structure is equally sensitive for loads in all directions.
Then the safe region is the set $S = \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 \leq r^2\}$ for some constant

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$r$, and the object could then be to design the structure so that $P\{X \in S\} \geq 1 - \alpha$ for some small $\alpha$. Other examples of this type, which appear in the engineering sciences, are sets like

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 + \max (x_2, x_3) \leq r\}$$

$$S = \{(x_1, x_2) \in \mathbb{R}^2; \max (a_1 x_1 + a_2 x_2, b_1 x_1 + b_2 x_2) \leq r\},$$

and their generalizations in higher dimensions.

Suppose that the load variables $X_1, \cdots, X_n$ have been rotated to being uncorrelated and standardized to have mean zero and unit variance, and let $S$ be the safe region for these transformed variables. The smallest distance from the origin to the unsafe region,

$$\beta = \inf_{x \not\in S} \|x\|,$$

is then sometimes called the Hasofer–Lind reliability index, which has been used as a simple measure of the reliability of the structure; see Hasofer and Lind (1974) and Lind (1977). Of course, it is only in very restricted classes of safe regions that there is any simple relationship between $\beta$ and the probability $P\{X \in S\}$, but in such cases it is in principle possible to evaluate the probability of failure, $1 - P\{X \in S\}$, as a function of $\beta$, if, say, $X_1, \cdots, X_n$ are independent Gaussian variables.

This is the static formulation of probabilistic structural analysis. In a more dynamic setting one supposes that the structure is subject to loads which are not constant in time but random functions in which case the problem transforms into a boundary crossing problem for multidimensional continuous stochastic processes. One then has an $n$-variate stochastic process $X(t) = (X_1(t), \cdots, X_n(t))$ with more or less known characteristics, and asks for the distribution of the number of times $X(t)$ leaves the safe region $S$ through its boundary $\partial S$. The structure survives at least the time $T$ if $X(t)$ has no crossings of $\partial S$ for $0 \leq t \leq T$, (and of course $X(0) \in S$). In this case the connection between the reliability index $\beta$ and the survival time distribution is even more complex than in the simple static setting. However, some results have been obtained and used in recent years for extreme circular regions, $S_r = \{(x_1, \cdots, x_n) \in \mathbb{R}^n; \sum x_i^2 \leq r^2\}$. For the special case that $X_1(t), \cdots, X_n(t)$ are independent, standardized Gaussian processes, Sharpe (1978) derives the asymptotic form of the probability $P\{X(t) \in S\}$ for all $t \in [0, T]$ when $T \to \infty$ and $r \to \infty$ in a coordinated way.

An alternative way to formulate this is to consider the $\chi^2$-process

$$\chi^2(t) = \sum_{k=1}^n X_k^2(t) = \|X(t)\|^2.$$
and its upcrossings of a level $u^2$ as $u \to \infty$. As $u$ increases, the level-crossings become rarer, and one has to consider an ever-increasing time interval $[0, T]$ in order to get a non-trivial limit for the distribution of the number of upcrossings. If $T \to \infty$ and $u \to \infty$ together, so that the mean number of $u^2$-upcrossings between 0 and $T$ remains constant and equal to $\tau > 0$ say, then it was shown by Sharpe (under fairly restrictive conditions on the correlation properties of $(X_1(t), \ldots, X_n(t))$ that the number of $u^2$-upcrossings by $\chi^2(t)$ for $0 \leq t \leq T$ has an asymptotic Poisson distribution and that, consequently,

$$(1.1) \quad P\{X(t) \in S \text{ for all } t \in [0, T]\} = P\{\chi^2(t) \leq u^2 \text{ for all } t \in [0, T]\} \to e^{-\tau}.$$  

Since there exists a simple formula that relates $u$ to $\tau$, one can in this way obtain the correspondence between the reliability index $\beta (= \tau)$ and the survival probability $e^{-\tau}$.

In this paper we shall see that the convergence (1.1) is in fact only one simple consequence of a more general convergence result on extreme boundary crossings for a multivariate Gaussian process, and see how the consequences of this general theorem allow us to evaluate the asymptotic survival distribution for a more general class of safety boundaries $\partial S$. This paper generalizes results presented in Lindgren (1980b) for bivariate Gaussian processes. In the present paper, proofs will be omitted when they duplicate those previously given.

2. Notation and basic facts

Let $X_1(t), \ldots, X_n(t)$ be independent, stationary Gaussian processes with zero mean and unit variance, and with covariance functions $r_k(t) = \text{Cov}(X_k(s), X_k(s + t))$, admitting the expansions

$$(2.1) \quad r_k(t) = 1 - \lambda_{2k} t^2/2 + o(t^2) \quad \text{as} \quad t \to 0.$$  

Suppose furthermore that each $X_k(t)$ has continuously differentiable sample paths. Then $\text{Var}(X_k(t)) = \lambda_{2k} = -r_k'(0)$ and

$$\chi^2(t) = \sum_{k = 1}^{n} X_k^2(t) = \|X(t)\|^2$$

defines a stationary $\chi^2$-process with continuously differentiable sample paths. If all the spectral moments $\lambda_{2k}$ are equal it is natural to say that all $X_k(t)$ have the same time scale, and that $\chi^2(t)$ has homogeneous time scales. If not only the spectral moments, but the whole covariance functions are equal, we say that the $\chi^2$-process is homogeneous. In a homogeneous $\chi^2$-process all components have the same time scale, while not all $\chi^2$-processes with equal time scales need to be homogeneous.

The interpretation of the $\chi^2$-process as the squared distance from the origin to the point $X(t) = (X_1(t), \ldots, X_n(t)) \in \mathbb{R}^n$, will be very useful in the sequel.
when we look for the extremal properties of \( \chi^2(t) \), and we shall now exploit this interpretation to derive a formula for the expected number of upcrossings for a \( \chi^2 \)-process.

Let

\[
S_u^0 = \{ (x_1, \cdots, x_n); \| x \| < u \}
\]

be the open \( n \)-dimensional sphere with radius \( u \), and denote its boundary by

\[
S_u = \{ (x_1, \cdots, x_n); \| x \| = u \}.
\]

If \( \tilde{N}_u(T) \) is the number of \( u \)-upcrossings by \( \chi^2(t) \) in the time interval \( [0, T] \), then \( \tilde{N}_u(T) \) is equal to the number of exits through \( S_u \) from \( S_u^0 \) by the vector process \( X(t), 0 \leq t \leq T \). To generalize this notion, let \( S \subseteq S_u \) be any subset of the boundary \( S_u \) and \( I = (a, b) \) a real interval, and define

\[
\tilde{N}(I \times S) = \# \{ t \in I; X(t) \in S \text{ and, for some } \varepsilon > 0, X(t-\varepsilon) \in S_u^0, X(t+\varepsilon) \notin S_u \cup S_u^0, \text{ for all } \tau \text{ with } 0 < \tau < \varepsilon \}
\]

to be the number of exits through \( S \) during the time interval \( I \).

**Theorem 2.1.** If \( \chi^2(t) = \sum_{k=1}^{n} X_k^2(t) \) is a \( \chi^2 \)-process

\[
E(\tilde{N}_u(1)) = \mu_n(u) = C_n u^{n-1} e^{-u^2/2}.
\]

Here

\[
C_n = (2\pi)^{-(n+1)/2} \int_{S_1} \left( \sum_{k=1}^{n} \lambda_{2k} y_k^2 \right)^{1/2} ds(y),
\]

with \( \lambda_{2k} = \text{Var}(X_k^2(t)) \) while \( ds(y) \) is the Haar measure on \( S_1 \) with total mass \( 2\pi^{n/2}/\Gamma(n/2) \). In particular, if \( \chi^2(t) \) has homogeneous time scales with \( \lambda_{2k} = \lambda_2 \),

\[
C_n = \left( \frac{\lambda_2}{\pi} \right)^{1/2} \frac{1}{2^{(n-1)/2} \Gamma(n/2)}. \tag{2.4}
\]

**Proof.** A simple proof of (2.2), (2.4) for a homogeneous process can be found, e.g. in Hasofer (1976) or Sharpe (1978), who use the general formula for the average number of upcrossings

\[
E(\tilde{N}_u(1)) = f_{\chi^2(0)}(u^2) E\left\{ \left( \frac{d}{dt} \chi^2(t) \right)^+ \bigg| \chi^2(t) = u^2 \right\}, \tag{2.5}
\]

\((x^+ = \max(0, x)). We shall here prove (2.2), (2.3) by means of a representation of (2.5) in the form of a surface integral over \( S_u \), hereby following Belayev (1968). As was shown by Belayev (1968) the mean number of exits from \( S_u^0 \) through a regular surface \( S_u \) can be expressed as a surface integral

\[
\int_{S_u} E((\nu_x, X'(0))^+ \big| X(0) = x) f_{X(0)}(x) \, ds(x), \tag{2.6}
\]
where $\nu_x$ is the normal of unit length at the point $x$ on the surface $S_u$ directed from $S_u^0$ into $S_u^*$, and $(x, y) = \sum x_k y_k$; for a proof that (2.6) equals (2.5), see for example Lindgren (1980a).

In this case, when $S_u$ is the $u$-ball,

$$\nu_x = u^{-1} x$$

and $(\nu_x, X'(0)) = u^{-1} \sum_{k=1}^n x_k X'_k(0)$, so since $X_1(0), \ldots, X_n(0), X'_1(0), \ldots, X'_n(0)$ are independent, and the conditional distribution of $(\nu_x, X'(0)) | X(0) = x$ is normal with mean zero and variance $u^{-2} \sum x_k^2 \lambda_{2k}$, we have

$$E((\nu_x, X'(0))^+ | X(0) = x) = \frac{1}{u \sqrt{2\pi}} \left( \sum_{k=1}^n x_k^2 \lambda_{2k} \right)^{\frac{1}{2}}.$$

Since furthermore

$$f_{X(0)}(x) = (2\pi)^{-n/2} \exp \left( -\frac{1}{2} \sum_{k=1}^n x_k^2 \right) = (2\pi)^{-n/2} \exp \left( -\frac{1}{2} u^2 \right),$$

if $\|x\| = u$ we see that (2.6) is equal to

$$\int_{S_u} \frac{1}{u \sqrt{2\pi}} \left( \sum_{k=1}^n x_k^2 \lambda_{2k} \right)^{\frac{1}{2}} \frac{1}{(2\pi)^{n/2}} \exp \left( -\frac{1}{2} \sum_{k=1}^n x_k^2 / 2 \right) ds(x)$$

$$= \frac{u^{-1} e^{-u^2/2}}{(2\pi)^{(n+1)/2}} \int_{S_u} \left( \sum_{k=1}^n x_k^2 \lambda_{2k} \right)^{\frac{1}{2}} ds(x).$$

By changing variables, $x_k = u y_k$, $ds(x) = u^{-1} ds(y)$, we arrive at the desired formula for the mean number of exits,

$$\frac{u^{-n} e^{-u^2/2}}{(2\pi)^{(n+1)/2}} \int_{S_u} \left( \sum_{k=1}^n y_k^2 \lambda_{2k} \right)^{\frac{1}{2}} ds(y).$$

**Remark 2.2.** To evaluate $E(\tilde{N}_u(1))$ in particular cases with non-homogeneous time scales one can express (2.3) by means of spherical coordinates

$$y_1 = \sin \theta_1,$$

$$y_k = \sin \theta_k \cdot \cos \theta_{k-1} \cdot \cdots \cdot \cos \theta_1,$$

$$y_n = \cos \theta_{n-1} \cdot \cdots \cdot \cos \theta_1,$$

where $-\pi/2 \leq \theta_k \leq \pi/2$, $k = 1, \ldots, n - 2$, $-\pi < \theta_{n-1} \leq \pi$, and

$$ds(y) = \prod_{i=1}^{n-2} \cos^i \theta_{n-1-i} \ d\theta_1 \cdots \ d\theta_{n-1}.$$
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**Theorem 2.3.** The mean number of exits from $S^0_u$ into $S'_u$ which pass through the subset $uS$ on $S_u$ is given by

$$E(\hat{N}((0, 1] \times uS)) = \mu_n(u, S) = C_n(S)u^{n-1}e^{-u^2/2},$$

where

$$C_n(S) = (2\pi)^{(n+1)/2} \int_S \left( \sum_{k=1}^{n} \lambda_{2k}y_k^2 \right)^{1/2} ds(y).$$

If $\chi^2(t)$ has homogeneous time scales with $\lambda_{2k} = \lambda_2$ then

$$C_n(S) = \left( \frac{\lambda_2}{\pi} \right)^{1/2} \frac{1}{2^{(n-1)/2}\Gamma(n/2)} \frac{|S|}{|S_1|},$$

where $|S|$ and $|S_1|$ are the areas of $S$ and of the unit ball, respectively.

**Remark 2.4.** $C_n(S)/C_n$ defines a probability measure on $S_1$. If the process $X(t)$ is ergodic this measure is equal to the long-run distribution of the location of the exits by $X(t)$ across any sphere $S_u$ in the sense that, with probability one,

$$\frac{C_n(S)}{C_n} = \lim_{T \to \infty} \frac{\#\{\text{exits across } uS \text{ by } X(t), 0 \leq t \leq T\}}{\#\{\text{exits across } S_u \text{ by } X(t), 0 \leq t \leq T\}}.$$}

This aspect of $C_n(S)/C_n$ is further developed in Lindgren (1980a), Theorem 4.5, in connection with general alarm problems.

For future use, let

$$c_n(x) = \left( \sum_{k=1}^{n} \lambda_{2k}x_k^2 \right)^{1/2} \int_{S_1} \left( \sum_{k=1}^{n} \lambda_{2k}x_k^2 \right)^{1/2} ds(x)$$

be the density of this probability measure with respect to the Haar measure $ds(x)$.

**3. Asymptotic Poisson character of exits**

We now turn to the statistical properties of the number of exits for $X(t)$ across the ball $S_u$ and their locations in time and space. This will also give the distributional properties of the number of $u^2$-upcrossings for the (homogeneous or non-homogeneous) $\chi^2$-process.

As can be expected from extreme value theory for univariate Gaussian processes, a suitable condition for the asymptotic Poisson character of $N_u(t)$ as $u, T \to \infty$ is the rather weak condition

$$r_k(t) \log t \to 0 \quad \text{as} \quad t \to \infty, \ k = 1, \cdots, n,$$

together with the indispensable (2.1); see e.g. Berman (1971) and Leadbetter et al. (1979).
From now on, let

(3.2) \[ \mu_n(u) = E(\hat{N}_u(1)) = C_n u^{-1} e^{-u^{3/2}} \]
denote the mean number of \( u^2 \)-upcrossings per time unit for the \( \chi^2 \)-process, or, which is the same, the mean number of exits across

\[ S_u = \{(x_1, \cdots, x_n) \in \mathbb{R}^n; \|x\| = u\} \]

by the vector process \( X(t) = (X_1(t), \cdots, X_n(t)) \).

Consider the cylinder in the \( (n+1) \)-dimensional space, defined by

\[ (0, 1] \times S_1 = \{(t, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}; 0 < t \leq 1, \|x\| = 1\}, \]

and transform the exits by \( X(t), 0 < t \leq T \), across \( S_u \) and represent them as points on \( (0, 1] \times S_1 \), in such a way that an exit at time \( t \in (0, T] \) at the point \( x \in S_u \) is represented by the point \((T^{-1}t, u^{-1}x) \in (0, 1] \times S_1 \). This procedure will define a random set of points on \( (0, 1] \times S_1 \), i.e. a point process, which we simply denote by \( N_u \).

As an illustration, we take \( n = 2 \), in which case \((0, 1] \times S_1 \) is a circular cylinder with the unit circle as base and axis along the unit time axis. In Figure 1 we have illustrated the process \( u^{-1}X(tT) \) and its exits across the cylinder. Note that an exit at a point \((t, x_1, x_2)\) corresponds to an exit of the \( X \)-process at the time \( tT \) with \( X(tT) = (ux_1, ux_2) \).

![Figure 1](image)

We shall now let \( u \to \infty \) so that \( \mu_n(u) \to 0 \) and there are, with probability approaching one, no \( u \)-exits in any interval of fixed length in the \( X \)-process. To compensate this we let \( T \to \infty \) so that

(3.3) \[ T \mu_n(u) \to \tau > 0. \]

Taking logarithms and using (3.2) we can invert this and obtain

\[ u^2 - 2(n - 1) \log u = 2(\log T - \log \tau + \log C_n) + o(1) \]
or, since \( u^2/2 \log T \to 1 \) and therefore \( 2 \log u - \log \log T \to \log 2, \)

(3.4) \[ u^2 = 2 \log T + (n - 1) \log \log T + \log 2^{n-1} - 2 \log \tau + 2 \log C_n + o(1). \]
As in the univariate case, the point process of upcrossings of the increasing level will be Poisson in character as \( u, T \to \infty \), \( T \mu_n(u) \to \tau \). Therefore define the tentative limiting point process \( N \) to be a Poisson process on \( (0, 1] \times S_1 \) with the intensity measure equal to \( \tau dt \times (\sum \lambda_{2k} x_k^2)^{3/2} ds(x) \), where \( dt \) denotes Lebesgue measure and \( ds(x) \) denotes the Haar measure on the \( n \)-dimensional unit ball \( S_1 \), invariant under rotations and with total mass equal to \( |S_1| = 2\pi^{n/2}/\Gamma(n/2) \). In particular, the mean number of points in \( N \) on the entire cylinder \((0, 1] \times S_1 \) will be \( \tau \).

**Theorem 3.1.** If the components of the Gaussian process \( X(t) = (X_1(t), \ldots, X_n(t)) \) are independent and have covariance functions which satisfy (2.1) and (3.1), the time- and space-normalized point process \( N_u \) of exits tends in distribution to a Poisson process \( N \) on \((0, 1] \times S_1 \) with intensity \( \tau dt \times (\sum \lambda_{2k} x_k^2)^{3/2} ds(x) \), if

\[
T \mu_n(u) \to \tau > 0. 
\]

**Corollary 3.2.** The normalized values \((u^{-1}X_1(t), \ldots, u^{-1}X_n(t))\) of the components of a \( \chi^2 \)-process at the times of the \( u^2 \)-upcrossings for \( \chi^2(t) \), \( 0 \leq t \leq T \) will form a point process on \( S_1 \) which will be asymptotically Poisson with intensity measure \( \tau (\sum \lambda_{2k} x_k^2)^{3/2} ds(x) \) if \( T \mu_n(u) \to \tau \).

If \( \chi^2(t) = \sum_{k=1}^n X_k^2(t) \) is a \( \chi^2 \)-process, the time-normalized point process of upcrossings of the level \( u^2 \) will tend in distribution to a Poisson process with intensity \( \tau \).

The proof of the theorem rests on the general criteria for convergence in distribution of point processes derived by Kallenberg (1976). In this setting we define a rectangle on \( S_1 \) to be any connected set \( J \) on \( S_1 \) bounded by a finite number of great-circle pieces. The family of all rectangles on \( S_1 \) forms a \( \text{DC} \)-semiring, while the family of sets which are finite unions of rectangles is a \( \text{DC} \)-ring. If \( I = (a, b] \subseteq (0, 1] \) is a time-interval and \( J \) a rectangle on \( S_1 \), \( I \times J = \{(t, x); t \in I, x \in J\} \) is a rectangle on the \( (n+1) \)-dimensional cylinder \((0, 1] \times S_1 \) with base \( S_1 \) while \( uJ \) is simply a rectangle on \( S_u \).

By Theorem 4.7 in Kallenberg (1976) it is then sufficient to prove that

(a) \( E(N_u(I \times J)) \to E(N(I \times J)) \) for \( I = (a, b] \), \( J \) a rectangle on \( S_1 \),

(b) \( P\{N_u(U) = 0\} \to P\{N(U) = 0\} \) for \( U = \bigcup_{k=1}^m I_k \times J_k \) where \( I_k \) and \( J_k \) are sets of this type.

Here (a) is trivial by construction, while (b) is somewhat more complicated than the proof of Theorem 2.1 in Lindgren (1980b).

For each \( x \in S_1 \), define the Gaussian process generated by the length of the orthogonal projection of \( X(t) \) on the ray \( \{ux, u \in R\} \), i.e.

\[
\xi_x(t) = \langle X(t), x \rangle = \sum_{k=1}^n x_k X_k(t)
\]
and denote its maximum over \((t, x) \in I \times J\) by
\[
M(I \times J) = \sup \{ \xi_x(t); t \in I, x \in J \}.
\]
The following lemma corresponds to Lemma 2.2 in Lindgren (1980b).

**Lemma 3.3.** There exists \(h_0 > 0\) such that for \(I = (a, b], b - a = h < h_0,\)
\[
P\{M(I \times J) > u\} = h \mu_n(u, J)\{1 + \rho_u(J)\},
\]
where
\[
\lim_{u \to \infty} \sup_{J} |\rho_u(J)| = \rho(J) \to 0 \quad \text{as} \quad \text{diam} (J) \to 0
\]
uniformly for \(J \subseteq S_1\).

**Proof.** We can regard \(\xi_x(t)\) as an \((n + 1)\)-dimensional random field with parameter \((t, x), t \in \mathbb{R}, x \in S_1,\) and covariance function
\[
r_{xy}(t) = \text{Cov}(\xi_x(0), \xi_y(t)) = \sum_{k=1}^{n} x_k y_k r_k(t) = \sum_{k=1}^{n} x_k y_k (1 - \lambda_{2k} t^2/2 + o(t^2)).
\]
Even if \(\xi_x(t)\) is not stationary in \(x\) it can be approximated locally by a stationary field with asymptotically the same maximum distribution. To be more specific, there exist to each \(x \in S_1,\) covariance functions \(\tilde{r}_x(t) = 1 - \tilde{\lambda}_{2x} t^2/2 + o(t^2)\) and \(\overline{r}_x(t) = 1 - \overline{\lambda}_{2x} t^2/2 + o(t^2),\) as \(t \to 0,\) such that the inequalities
\[
(3.5) \quad r_x(t) \cdot \langle x, y \rangle \leq r_{xy}(t) \leq \overline{r}_x(t) \cdot \langle x, y \rangle
\]
hold for all \(t\) near 0, and \(y\) near \(x.\) One can e.g. choose \(\tilde{\lambda}_{2x}\) and \(\overline{\lambda}_{2x}\) both close to \(\lambda_{2x} = \sum_{k=1}^{n} \lambda_{2k} x_k^2\) but satisfying
\[
\tilde{\lambda}_{2x} < \inf_{||y-x|| < \delta} \sum_{k=1}^{n} \lambda_{2k} y_k^2 < \sup_{||y-x|| < \delta} \sum_{k=1}^{n} \lambda_{2k} y_k^2 < \overline{\lambda}_{2x}
\]
and then take
\[
\tilde{r}_x(t) = \cos t \sqrt{\tilde{\lambda}_{2x}}, \quad \overline{r}_x(t) = \cos t \sqrt{\overline{\lambda}_{2x}}.
\]
By taking \(\delta\) small we can also obtain that \(\tilde{\lambda}_{2x}/\overline{\lambda}_{2x}\) is uniformly close to 1 for \(x\) on \(S_1.\)

Now let \(x\) be fixed in \(J\) and define a process \(\tilde{X}(t) = (\tilde{X}_1(t), \cdots, \tilde{X}_n(t))\) of independent Gaussian components each with covariance function \(\tilde{r}_x(t),\) and similarly \(X(t)\). Define \(\tilde{M}(I \times J)\) and \(M(I \times J)\) by substituting \(\tilde{X}\) and \(X\) for \(X\) in the definition of \(M(I \times J).\) Then the well-known Slepian’s lemma for Gaussian processes, together with (3.5) implies
\[
P\{\tilde{M}(I \times J) > u\} \leq P\{M(I \times J) > u\} \leq P\{\tilde{M}(I \times J) > u\}
\]
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(see Leadbetter (1974), Theorem 5.3). Here $\bar{X}(t)$ is a stationary field for which

$$\frac{1}{\mu_n(u)} P\{\bar{M}(I \times J) > u\} \rightarrow h |J|/|S_1|$$

as $u \rightarrow \infty$ (see Bickel and Rosenblatt (1973), Lemma 5), and similarly for $\bar{M}(I \times J)$. Since $\mu_n(u) = (\tilde{\lambda}_{2x}/\lambda_{2x})^{3/2}$ is close to 1 if $\delta$ is small, and furthermore

$$\mu_n(u, J) = \mu_n(u) |J|/|S_1|$$

if $\text{diam}(J) < \delta$, we have that

$$\lim_{u \rightarrow \infty} \frac{1}{\mu_n(u, J)} P\{M(I \times J) > u\} = \frac{|S_1|}{|J|} \lim_{u \rightarrow \infty} \frac{1}{\mu_n(u)} P\{\bar{M}(I \times J) > u\}$$

$$= h \cdot (\tilde{\lambda}_{2x}/\lambda_{2x})^{3/2},$$

which is close to $h$ if $\text{diam}(J)$ is small, and this holds uniformly in $J$ on $S_1$. A similar argument for $\text{lim sup}$ concludes the proof.

For the rest of the proof of Part (b) we assume that

$$U = \bigcup_k I_k \times \left( \bigcup_j J_{kj} \right),$$

where $I_k = (a_k, b_k], 0 < a_k < b_k < \cdots < 1$ and $J_{kj}, j = 1, \cdots, j_k$ are disjoint for every $k$. Furthermore we can without restriction assume each $J_{kj}$ to have $\text{diam}(J_{kj})$ as small as necessary.

The following two lemmas are simple generalizations of the bivariate cases.

Lemma 3.4. If $I = (a, b], b - a < h_0$, and

$$\hat{N}_u(I \times J) = \#\{t \in I; X(t) \in uJ, \text{outcrossing}\}$$

then

$$\frac{1}{\mu_n(u, J)} |P\{M(I \times J) > u\} - P\{\hat{N}_u(I \times J) \geq 1\}| \rightarrow 0$$

as $u \rightarrow \infty$.

Lemma 3.5. If $T, u \rightarrow \infty$ so that $T\mu_n(u) \rightarrow \tau$, then

$$P\{N_u(U) = 0\} - P\left( \bigcap_k \bigcap_j \{M(T. I_k \times J_{kj}) \leq u\} \right) \rightarrow 0.$$

The next step in the proof of Theorem 3.1 is to replace the maxima $M(T. I_k \times J_{kj})$ of $\xi_x(t)$ by the maxima taken over a discrete set of $t$- and $x$-values, coarse enough to catch all extreme values but still making it possible to treat $\xi_x(t)$ and $\xi_x$-t as independent. We therefore define a $p$-net $\sum = \sum_p$ of
points on $S_1$ to be a finite set $\Sigma = \{x_1, \ldots, x_p\}$ such that

\[(3.6) \quad \sup_{x \in S_1} \inf_{x_i, x_j \in \Sigma} \|x - x_i\| \leq p\]

and

\[(3.7) \quad \inf_{x_i \neq x_j} \|x_i - x_j\| \geq c_np\]

for some constant $c_n$ (in fact, there is a $p$-net with $c_n < 2/\sqrt{n} - 1$ for all small $p$-values). Note that (3.6) implies that for some $i$,

\[(3.8) \quad \xi_i(t) = \|X(t)\|^2 - \|X(t) - \xi_i(t)\|^2 \geq \|X(t)\|^2 (1 - p^2),\]

from which we can derive the following lemma.

Lemma 3.6. Let $p, q \to 0$ as $u \to \infty$ so that $up \to a > 0$, $uq \to b > 0$, and write

\[\nu_{a,b}(J) = \lim_{u \to \infty} \inf_{q \mu_n(u, J)} \frac{1}{P_{\max_{0} < u < \max_{q} \mu_n(u, J)}} \left\{ \max_{x_i \in J} \xi_i(0) \leq u < \max_{x_i \in J} \xi_i(0) \right\}\]

and $\tilde{\nu}_{a,b}(J)$ for the corresponding lim sup. Then

\[\lim \nu_{a,b}(J) = \lim \tilde{\nu}_{a,b}(J) = 1\]

if $a, b \to 0$ so that $a^2/b \to 0$.

Proof. Concentrating on $\nu_{a,b}(J)$ we first observe that (3.8) implies that

\[P_{\max_{x_i \in J} \xi_i(0) \leq u < \max_{x_i \in J} \xi_i(0)}\]

\[(3.9) \quad \geq P_{\|X(0)\| \leq u < \|X(q)\| (1 - p^2)_{1/2}, \frac{X(0)}{\|X(0)\|} \in J}\]

\[= P_{u^2 + \alpha_u - 2q \|X(0)\| H_q < \|X(0)\|^2 \leq u^2, \frac{X(0)}{\|X(0)\|} \in J}\],

where we have introduced the normalized difference quotient

\[H_q = \frac{\|X(q)\|^2 - \|X(0)\|^2}{2q \|X(0)\|}\]

and

\[\alpha_u = -u^2 \left(1 - \frac{1}{1 - p^2}\right) \to b^2 \quad \text{as} \quad u \to \infty.\]
Conditioning on each component $X_k(0) = x_k$ we have the conditional distributions

$$(X_k(q) \mid X_k(0) = x_k) \leq x_k r_k(q) + Y_k \sqrt{1 - r_k^2(q)},$$

where $Y_1, \cdots, Y_n$ are independent $N(0, 1)$-variables. Introducing this into the definition of $H_q$, we obtain after some rearrangement, with $x = (x_1, \cdots, x_n)$, $r_k = r_k(q)$,

$$(H_q \mid X(0) = x) \overset{L}{=} \xi_u + \beta_u + \varepsilon_u,$$

where

$$\xi_u = \frac{\sum_{k=1}^{n} x_k Y_k r_k \sqrt{1 - r_k^2}}{q \|x\|},$$

$$\beta_u = -\frac{\sum_{k=1}^{n} x_k^2 (1 - r_k^2)}{2q \|x\|},$$

$$\varepsilon_u = \frac{\sum_{k=1}^{n} Y_k^2 (1 - r_k^2)}{2q \|x\|}.$$

Conditioning on $X(0) = x$ in (3.9) we can therefore write

$$(3.10) \quad P\left\{ u^2 + \alpha_u - 2q \|X(0)\| H_q < \|X(0)\|^2 \leq u^2, \frac{X(0)}{\|X(0)\|} \in J \right\} = E_{X(0)} \left\{ P\left\{ u^2 + \alpha_u - 2q \|X(0)\| (\xi_u + \beta_u + \varepsilon_u) \leq \|X(0)\|^2 \leq u^2, \frac{X(0)}{\|X(0)\|} \in J \mid X(0) \right\} \right\}.$$

Here, the conditional probability is zero if $\|X(0)\| > u$, while for $X(0) = x = \|x\| \cdot x_0$, $\|x\|^2 = u^2 - h < u^2$, it is equal to

$$P\left\{ H_q \geq \frac{h + \alpha_u}{2q \|x\|} \mid X(0) = x \right\} = P\left\{ \xi_u + \beta_u + \varepsilon_u \geq \frac{h + \alpha_u}{2q \|x\|} \right\},$$

and as is easily seen $q \|x\| \rightarrow b$, $\alpha_u \rightarrow a^2$, and

$$\beta_u \sim -\frac{uq \sum_{k=1}^{n} x_k^2 \lambda_{2k}}{2 \|x\|^2} \rightarrow -\frac{b}{2} \sum_{k=1}^{n} x_k^2 \lambda_{2k} = -\frac{b}{2} \lambda_{2x_0},$$

say, while the random variable $\varepsilon_u$ tends in probability to zero as $u \rightarrow \infty$. Finally
the random variable \( \xi_u \) is \( N(0, \sigma_{u_x}^2) \) where

\[
\sigma_{u_x}^2 = \frac{\sum_{k=1}^{n} x_k^2 r_k^2 (1 - r_k^2)}{q^2 \|x\|^2} \rightarrow \sum_{k=1}^{n} x_k^2 \lambda_{2k} = \lambda_{2x_0}
\]

so that

\[
P\left( \xi_u + \beta_u + \varepsilon_u \geq \frac{h + \alpha_u}{2q \|x\|} \right) \rightarrow 1 - \Phi\left( \frac{h}{2b \sqrt{\lambda_{2x_0}}} + \gamma_{a,b} \right),
\]

where

\[
\gamma_{a,b} = \frac{a^2}{2b \lambda_{2x_0}} + \frac{b}{2} \sqrt{\lambda_{2x_0}} \rightarrow 0 \quad \text{as} \quad a, b \rightarrow 0, \frac{a^2}{b} \rightarrow 0.
\]

Since further, \( \|X(0)\|^2 \) and \( X(0)/\|X(0)\| \) are independent, \( \chi^2(n) \) and uniformly distributed over \( S_1 \), respectively, it is now a simple task to evaluate the expectation (3.10) and obtain that asymptotically under the stated conditions on \( a, b \),

\[
P\left( \max_{x_i \in J} \xi_{x_i}(0) \leq u < \max_{x_i \in J} \xi_{x_i}(q) \right)
\]

\[
\equiv \int_{x_0 \in J} \int_{v=0}^{u^2} f_{\|X(0)\|^2}(v) P\left( \xi_u + \beta_u + \varepsilon_u \geq \frac{u^2 - v + \alpha_u}{2q \sqrt{v}} \right) dv \, dx_0
\]

which in turn is asymptotically bounded from below by

\[
qu^{n-1} e^{-u^2/2(2\pi)^{-(n+1)/2}} \int_{x_0 \in J} x_k^2 \lambda_{2k} \, ds(x_0) = q \mu_{n}(u, J)
\]

which proves that \( \lim \inf \nu_{a,b}(J) \geq 1 \). Treating \( \tilde{\nu}_{a,b}(J) \) in a similar way gives that \( \lim \sup \tilde{\nu}_{a,b}(J) \leq 1 \), from which the conclusion follows.

The lemma makes it possible to approximate the maximum \( M(I \times J) \) by the maximum of \( \xi_{x_i}(t_i) \), as the following lemma shows, the proof of which is exactly as that of Lemma 2.6 in Lindgren (1980b).

**Lemma 3.7.** If \( U = \bigcup_i I_i \times (\bigcup_i J_i) \), \( u_p \rightarrow a > 0 \), \( u_q \rightarrow b > 0 \) and \( T\mu_n(u) \rightarrow \tau > 0 \) as \( T, u \rightarrow \infty \), then

\[
\limsup_{u \rightarrow \infty} \left| P\left( \bigcap_i \bigcap_{x_i \in J_i} \left\{ \max_{q \in T, \xi_{x_i}(jq) \geq u} \right\} \right) - P\left( \bigcap_i \bigcap_{J_i} \{M(T, I_i \times J_i) \leq u\} \right) \right|
\]

\[
\leq \limsup_{u \rightarrow \infty} \sum_i T(b_i - a_i) \sum_i \mu_n(u, J_i)(1 - \nu_{a,b}(J_i))
\]

\[
= \frac{\tau}{C_n} \sum_i C_n \left( \bigcup J_i \right)(1 - \nu_{a,b}(J_i)).
\]

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Next, split each $J_{il}$ into subrectangles $B_r$ of diam $(B_r) \leq d$, and let $h > 0$ be as in Lemma 3.3.

**Lemma 3.8.** If $r_k(t) \to 0$ as $t \to \infty$, $k = 1, \cdots, n$ and for each $k$ and $\varepsilon > 0$,

$$\frac{T}{q_p^{2(n-1)}} \sum_{\varepsilon = jq = T} |r_k(jq)| e^{-u^2/(1 + |r_k(jq)|)} \to 0$$

as $T, u \to \infty$ so that $T\mu_n(u) \to \tau$, $u_p \to a > 0$, $u_q \to b > 0$, then

(a) \[ \lim_{u \to \infty} \sup_{\varepsilon = jq = T} \left| P\left( \max_{x \in J_{il}} \xi_{x_{il}}(j) \leq u \right) - \prod_{r=1}^m P\left( \max_{x \in B_r, jq \in J_{il}} \xi_{x_{il}}(j) \leq u \right) \right| = 0 \]

(b) \[ \lim_{u \to \infty} \left( P\left( \bigcap_{i=1}^n \left\{ \max_{x \in J_{il}} \xi_{x_{il}}(j) \leq u \right\} \right) - \prod_{r=1}^m P\left( \max_{x \in B_r, jq \in J_{il}} \xi_{x_{il}}(j) \leq u \right) \right) = 0. \]

The proof of the lemma is a straightforward generalization of the bivariate proof, and it will therefore be left out.

**Proof of Theorem 3.1.** First note that (3.11) is satisfied if $r_k(t) \log t \to 0$, $t \to \infty$; see Lemma 2.8 in Lindgren (1980b). Lemmas 3.6–3.8 then give that

$$\lim_{u \to \infty} \sup_{\varepsilon = jq = T} \left| P\left( \bigcap_{i=1}^n \left\{ M(T, I_i \times J_{il}) \leq u \right\} \right) - \prod_{r=1}^m P\left( M(T, I_i \times J_{il}) \leq u \right) \right| = 0. \tag{3.12}$$

Now take any one $J_{il}$ and split it into subrectangles $B_1, B_2, \cdots$ with diam $(B_k) \leq d$, so that exactly $m$ of these are needed to cover $J_{il}$, and write $n = [T, I_i/h]$, $[\ ]$ denoting integer part,

\[ \lim_{u \to \infty} \sup_{\varepsilon = jq = T} \left| P\left( \bigcap_{i=1}^n \left\{ M(T, I_i \times J_{il}) \leq u \right\} \right) - \prod_{r=1}^m P\left( M((0, h] \times B_r) \leq u \right) \right| = 0. \]

as $d \to 0$, while by Lemma 3.3,

$$\prod_{r=1}^m P\left( (0, h] \times B_r \leq u \right) \prod_{r=1}^m (1 - h\mu_n(u, B_r) - h\mu_n(u, B_r)\rho_u(B_r))^n \to \prod_{r=1}^m \exp \left\{ \frac{-1}{\tau} \frac{C_n(B_r)}{C_n} (1 + \rho(B_r)) \right\} \]

as $u \to \infty$, where $|\rho(B_r)| \leq \limsup_{u \to \infty} |\rho_u(B_r)|$. Since $\rho(B_r) \to 0$ as $d \to 0$ we obtain the limit

$$\exp \left\{ -\tau \frac{|I|}{C_n(J)/C_n} \right\} = P\{N(I \times J) = 0\},$$

and then (3.12) and Lemma 3.5 finish the proof.
4. Complete Poisson character of extremes

The Poisson character of exits across the sphere $S_u$ is coupled to the extremal distribution of the $\chi^2$-process, in the sense that

$$\lim_{T \to \infty} P\left\{ \max_{0 \leq t \leq T} \chi^2(t) \leq u^2_T \right\} = \lim_{T \to \infty} P\{N_{\mu}(T) = 0\} = e^{-\tau}$$

if $T \mu_n(u_T) \to \tau$. Of course, it would be possible to prove a Poisson convergence theorem for non-circular expanding sets defined by the level curves of some function $g(x)$, and as a consequence get the extremal distribution of the process $g(X(t))$, $0 \leq t \leq T$. However, we shall accomplish all this by giving one single theorem about the extremes of the vector process $X(t)$, which then will yield the extremal distributions for a wide class of non-Gaussian processes.

We start by defining a fundamental space transformation. Let, as before, $\mu_n(u) = C_n u^{-n} \exp(-u^2/2)$ be the average number of exits per time unit across the centered sphere $S_u$ with radius $u$ by the $n$-variate Gaussian process $X(t)$, the constant $C_n$ depending on the covariance structure of $X(t)$. Note that $\mu_n(u)$ is decreasing for $u \geq n - 1$. Now define, for each $T$, a transformation $h^*_T$, $R^n_0 \to R^n_0$, where $R^n_0 = \{x \in R^n; \|x\| > 0\}$, by requiring that

$$h^*_T(x) = \frac{x}{\|x\| T \mu_n(\|x\|)} \quad \text{if} \quad \|x\| \geq \sqrt{n - 1}$$

and $\|h^*_T(x)\| < 1/(T \mu_n(\sqrt{n - 1}))$ otherwise. Then consider the non-Gaussian process

$$X_T(t) = h^*_T(X(t)).$$

The reason for this is that if $\|x\|$ and $T$ are large and $\|h^*_T(x)\| = r$, then $\mu_n(\|x\|) = 1/rT$, which means that the process $X_T(t)$ has on the average $1/rT$ exits per time unit across the ball $S_r$ with radius $r$, and consequently $1/r$ exits in the entire interval $[0, T]$. After a time-normalization, Theorem 3.1 implies that the point process of exits across the cylinder $(0, 1] \times S_r$, by $(t, X_T(tT))$, $t \in (0, 1]$ is asymptotically Poisson with intensity measure

$$r^{-1} c_n(r^{-1}x) \, dt \times r^{-(n-1)} \, ds(x) = r^{-1} dt \times c_n(r^{-1}x) \, ds_1(r^{-1}x),$$

where $ds(x)$ is the Haar measure on $S_r$ with total mass $|S_r| = r^{n-1} |S_1|$, and $c_n$ is defined by (2.7).

We shall now identify the most extreme parts of the curve $(t, X_T(t))$, $t \in (0, T]$, and the point process they generate. Let $t_1$ be such that

$$\sup_{0 < t \leq T} \|X(t)\| = \|X(t_1)\|.$$
let $t_2$ be the time for the most extreme part of $X(t)$ in $I_i$, i.e.

$$\sup_{t \in I_i} \|X(t)\| = \|X(t_2)\|.$$ 

Proceeding recursively, with $I_n = I_{n-1} \cap (t_{n-1} - \varepsilon, t_{n-1} + \varepsilon)$, we define $t_n$ so that

$$\sup_{t \in I_n} \|X(t)\| = \|X(t_n)\|,$$

always taking the leftmost point if more than one $t$-value is possible. This will give a finite set of points, here called $\varepsilon$-extremes,

$$(T^{-1}t_k, X_T(t_k)), \quad k = 1, 2, \ldots$$

constituting a point process in $(0, 1] \times R^n$. We denote this point process by $N_T$. Further, let $N$ be a Poisson process in $(0, 1] \times R^n$ with an intensity measure which is absolutely continuous with respect to Lebesgue measure and which at the point $(t, rx)$, $r > 0$, $x \in S_1$, is equal to

$$dt \times c_n(x) \, ds_1(x) \times r^{-2} \, dr,$$

i.e. the product of Lebesgue measure and a measure $\Gamma$ on $R^n$. We shall later use $\gamma(x)$ to denote the density of the measure $\Gamma$ with respect to Lebesgue measure in $R^n$. 

Theorem 4.1. Under the conditions of Theorem 2.1, the time and space normalized point process $N_T$ of $\varepsilon$-extremes $\{(T^{-1}t_k, X_T(t_k)), \, k = 1, 2, \ldots\}$ converges in distribution as $T \to \infty$ to the Poisson process $N$ in $(0, 1] \times R^n$ with intensity measure

$$dt \times c_n(x) \, ds_1(x) \times r^{-2} \, dr, \quad r > 0, \|x\| = 1.$$

The theorem is proved in exactly the same way as the bivariate counterpart in Lindgren (1980b), and no extra complications arise in the non-homogeneous case.

As a simple corollary of the theorem we get the following result about the probability of no visit to a region $A$.

A set $A \subseteq R^n$ is called ray-shaped if $x \in A \Rightarrow rx \in A$ for all $r \geq 1$.

Theorem 4.2. Suppose $A \subseteq R^n$ is ray-shaped, $\Gamma(A) < \infty$, $\Gamma(\partial A) = 0$, and that the hypotheses of Theorem 4.1 are satisfied. Then

$$P\{X_T(t) \in A, \text{ some } t \in (0, T]\} \to 1 - P\{N(A) = 0\} = 1 - e^{-\Gamma(A)}$$

as $T \to \infty$.

Proof. Obviously

$$\liminf_{T \to \infty} P\{X_T(t) \in A, \text{ some } t \in (0, T]\} \geq \liminf_{T \to \infty} P\{N_T(A) \geq 1\} = 1 - P\{N(A) = 0\},$$

as $T \to \infty$.
while the reverse inequality can be obtained by considering the number of $\varepsilon$-separated points in $A$, as in the proof of Theorem 3.2 in Lindgren (1980b).

We shall now make a small extension of Theorem 4.1 by considering the extremes of $p$ of the components of $X(t)$ separately.

Write $X^{(1)}(t) = (X_1(t), \ldots, X_p(t))$, $X^{(2)}(t) = (X_{p+1}(t), \ldots, X_n(t))$, $q = n - p$, and consider now the $\varepsilon$-extremes $t_k$ of $X^{(1)}(t)$ only. Denote by $N^{(1,2)}_T$ the point process in $[0, 1] \times R_p \times R^3$ formed by the points

$$(T^{-1} t_k, X^{(1)}_T(t_k), X^{(2)}_T(t_k)),$$

where now $X^{(1)}_T(t)$ is obtained from $X^{(1)}(t)$ by the transformation $h_T^p(x)$, defined by (4.1).

Further, let $N^{(1,2)}$ be a Poisson process in $[0, 1] \times R_p \times R^q$ with intensity measure $dt \times \Gamma^{(1,2)} = dt \times \Gamma \times \phi(x^{(2)}) dx^{(2)}$, where $\phi(x^{(2)})$ is the $q$-variate normal density of $X^{(2)}(t)$.

If we wish, we can regard $X^{(2)}(t_k)$ as a set of extra information attached to the point $(T^{-1} t_k, X^{(1)}_T(t_k))$, and use the terminology of marked point processes.

**Theorem 4.3.** If the hypotheses of Theorem 2.1 are satisfied, the point process $N^{(1,2)}_T$ of marked $\varepsilon$-extremes converges in distribution as $T \to \infty$ to a Poisson process $N^{(1,2)}$ in $[0, 1] \times R_p \times R^q$ with intensity measure

$$dt \times c_p(x^{(1)}) ds^p(x^{(1)}) r^{-2} dr \times \phi(x^{(2)}) dx^{(2)},$$

where $ds^p(x^{(1)})$ is the Haar measure on the $p$-dimensional unit sphere.

**Proof.** One has to check (a) and (b) as in the proof of Theorem 2.1, and use that the Poisson character of $T^{-1} t_k$ implies asymptotic independence of the $X^{(2)}(t_k)$.

The following theorem states that for certain sets $A \subseteq R_p \times R^q$ we can calculate the probability that the process $(X^{(1)}_T(t), X^{(2)}_T(t))$ pays no visits to $A$ for $t \in [0, T]$, asymptotically as the probability that $N^{(1,2)}(A) = 0$. It will be used in the following section to compute the asymptotic extreme value distribution for non-linear functions of multivariate Gaussian processes.

The set $A \subseteq R_p \times R^q$ is called ray-shaped in $x^{(1)}$ if $(x^{(1)}, x^{(2)}) \in A$ implies that $(rx^{(1)}, x^{(2)}) \in A$ for all $r \geq 1$. For each such set $A$, let

$$A_\varepsilon = \{ (x^{(1)}, x^{(2)}); \exists x^{(2)}, \|x^{(2)} - \bar{x}^{(2)}\| < \varepsilon, (x^{(1)}, \bar{x}^{(2)}) \in A \}$$

be an expansion of $A$ in the $x^{(2)}$-directions by the amount $\varepsilon > 0$.

**Theorem 4.4.** Suppose the hypotheses of Theorem 4.3 are satisfied, and let $A \subseteq R_p \times R^q$ be ray-shaped in $x^{(1)}$, with $\Gamma^{(1,2)}(A) \ll \infty$, $\Gamma^{(1,2)}(\partial A) = 0$, and
assume
\[\Gamma^{(1,2)}(A_{\varepsilon}) \to \Gamma^{(1,2)}(A) \quad \text{as} \quad \varepsilon \to 0\]
(4.2)

\[\limsup_{T \to \infty} TP\{(X^{(1)}_T(t), X^{(2)}_T(t)) \in A, \text{some} \ t \in (0, 1]\} < \infty.\]
(4.3)

Then
\[P\{(X^{(1)}_T(t), X^{(2)}_T(t)) \in A, \text{some} \ t \in (0, T]\} \to 1 - P\{N^{(1,2)}(A) = 0\}
= 1 - \exp (-\Gamma^{(1,2)}(A))\]
as \(T \to \infty\).

Note that if \(\Gamma^{(1,2)}(A) < \infty\), then (4.2) implies that for each \(d > 0\),
\[r_d = \inf \{\|x^{(1)}\|; (x^{(1)}, x^{(2)}) \in A, \text{some} \ |x^{(2)}| \leq d\} > 0,\]
(4.4)
since otherwise \(\Gamma^{(1,2)}(A_{\varepsilon}) = \infty\). Furthermore, (4.2) and (4.3) are automatically satisfied if \(r_\infty > 0\).

**Proof.** We have to prove that there is, with high probability, no \(\varepsilon\)-extremes outside \(A_{\varepsilon}\) which extend into \(A\). To see this, define for \(\varepsilon > 0\) the event
\[B_\varepsilon = \{\exists s, t \in (0, T]; \|X^{(1)}_T(s)\| \geq \|X^{(1)}_T(t)\|, (X^{(1)}_T(t), X^{(2)}_T(t)) \in A, \|X^{(2)}(s) - X^{(2)}(t)\| \geq \varepsilon\},\]
and let \(B'_\varepsilon\) be defined similarly, but with \(s, t\) restricted to \((0, 1]\) instead of \((0, T]\).
Then
\[
\limsup_{T \to \infty} P\{(X^{(1)}_T(t), X^{(2)}_T(t)) \in A, \text{some} \ t \in (0, T]\}
\leq \limsup_{T \to \infty} P\{N^{(1,2)}(A_{\varepsilon}) \geq 1\} + \limsup_{T \to \infty} P(B_\varepsilon)
= 1 - P\{N^{(1,2)}(A_{\varepsilon}) = 0\} + \limsup_{T \to \infty} P(B_\varepsilon)
= 1 - \exp (-\Gamma^{(1,2)}(A_{\varepsilon})) + \limsup_{T \to \infty} P(B_\varepsilon).
\]
Since obviously
\[
\liminf_{T \to \infty} P\{(X^{(1)}_T(t), X^{(2)}_T(t)) \in A, \text{some} \ t \in (0, T]\}
\geq \liminf_{T \to \infty} P\{N^{(1,2)}(A_{\varepsilon}) \geq 1\} = 1 - P\{N^{(1,2)}(A) = 0\}
= 1 - \exp (-\Gamma^{(1,2)}(A))
\]
all we need to show is that $P(B_\varepsilon) \to 0$ as $T \to \infty$ (since we have assumed $\Gamma^{(1,2)}(A_\varepsilon) \to \Gamma^{(1,2)}(A)$ as $\varepsilon \to 0$). But

$$P(B_\varepsilon) \leq 2TP(B'_\varepsilon),$$

and if we write

$$\omega(h) = \sup \{\|X^{(2)}(s) - X^{(2)}(t)\|; s, t \in (0, 1], |s - t| \leq h\}$$

then $H_\varepsilon = \inf \{h; \omega(h) \geq \varepsilon\}$ is a strictly positive r.v. with a distribution function $F_{H_\varepsilon}$ so we have

$$P(B'_\varepsilon) = \int_0^\infty P(B'_\varepsilon \mid H_\varepsilon = h) \, dF_{H_\varepsilon}(h).$$

Here

$$P(B'_\varepsilon \mid H_\varepsilon = h) \leq P(\exists s, t \in (0, 1]; |s - t| \geq h, \|X^{(1)}_T(s)\| > \|X^{(1)}_T(t)\|, (X^{(1)}_T(t), X^{(2)}(t)) \in A)$$

and this can be seen to be $o(T^{-1})$ using (4.4) and properties of correlated normal variables. Since furthermore $P(B'_\varepsilon \mid H_\varepsilon = h)$ is bounded by

$$P((X^{(1)}_T(t), X^{(2)}(t)) \in A, \text{some } t \in (0, 1]),$$

(4.3) implies that $TP(B'_\varepsilon \mid H_\varepsilon = h)$ is bounded as $T \to \infty$, and we conclude that $TP(B'_\varepsilon) \to 0$ as $T \to \infty$, thus completing the proof.

5. Extreme values of non-linear functions

The results of Section 4 will now be used to find the asymptotic form and the correct location and scale parameters for the distribution of

$$\max_{0 \leq t \leq T} g(X(t))$$

as $T \to \infty$, for wide class of functions $g$. The reliability index $\beta$, defined in Section 1, will turn out to be an essential measure of the safety of a structure subject to a stochastically varying load $X(i)$, but it has to be supplemented by the local structure of the boundary of the safe region near the point where it is closest to the origin. Some examples make this clear.

Example 5.1. For the (homogeneous or non-homogeneous) $\chi^2$-process

$$\chi^2(t) = \sum_{k=1}^n X_k^2(t)$$

one immediately obtains from Theorem 2.1 that

$$P\left\{ \max_{0 \leq t \leq T} \chi^2(t) \leq u_\varepsilon^2 \right\} \to e^{-\varepsilon}$$
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if \( T\mu_n(u_T) \to \tau > 0 \), i.e.

(5.1) \( u_T^2 = 2 \log T + (n-1) \log \log T + \log 2^{n-1} - 2 \log \tau + 2 \log C_n + o(1) \)

where \( C_n \) is defined by (2.3).

Example 5.2. The class of concomitants to the \( \chi^2 \)-process was introduced in Lindgren (1980b) as the class of processes \( Z(t) = g(X(t)) \), such that there is a function \( g^* \) for which

(5.2) \( g(x) - \|x\|^2 + 2g^*(x/\|x\|) \to 0 \)

as \( \|x\| \to \infty \), and

\[
g^*(x) \begin{cases} > -\infty & \text{for all } x \text{ with } \|x\| = 1 \\ < +\infty & \text{for } x \in S \subseteq S_1 \end{cases}
\]

for some set \( S \) with non-zero measure. As was shown in Theorem 4.2 in Lindgren (1980b), for such a process

\[
P \left\{ \max_{0 \leq t \leq T} Z(t) \leq u_T^2 \right\} \to \exp \left\{ -\tau \int_{\|x\|=1} c_n(x)e^{-g^*(x)} \, ds(x) \right\}
\]

if \( T\mu_n(u_T) \to \tau \). To see this here, consider the level surface \( g(x) = u^2 \) and the transformation (4.1),

\[
h^n_T(x) = \frac{x}{\|x\| T\mu_n(\|x\|)}.
\]

If \( u^2 \) and \( \|x\| \) are large, and \( g(x) = u^2 \), (5.2) implies that

\[
T\mu_n(\|x\|) = TC_n \|x\|^{n-1} e^{-\|x\|^2/2} \sim TC_n u^{-1} e^{-u^2/2 - g^*(x/\|x\|)} \to \tau e^{-g^*(x/\|x\|)}
\]

so that

\[
h^n_T(x) \sim \frac{x}{\|x\| \tau^{-1} e^{g^*(x/\|x\|)}}.
\]

If the set \( A \subseteq R^n_0 \) is defined by

\[
A = \{ x; \|x\| > \tau^{-1} e^{g^*(x/\|x\|)} \},
\]

we see that

\[
P \left\{ \max_{0 \leq t \leq T} Z(t) > u_T^2 \right\}
\]

is asymptotically equal to

\[
P \{ X_T(t) \in A, \text{ some } t \in [0, T] \},
\]
which, by Theorem 4.2, tends to \(1 - \exp(-\Gamma(A))\), where

\[
\Gamma(A) = \int_{\|x\|=1}^{\infty} \int_{r=r^{-1}\exp^{-x}}^\infty r^{-2} d\rho_c(x) ds(x)
\]

\[
= \tau \int_{\|x\|=1} c_n(x) e^{-\sigma^2(x)} ds(x).
\]

The class of concomitants to the \(\chi^2\)-process is rather restricted since the level curves \(g(x) = u^2\) differ from the sphere, \(\|x\| = u\), by a quantity of the order \(1/u\). For example, polynomial functions of \(X_1(t), \cdots, X_n(t)\) are in general not concomitants to the \(\chi^2\)-process.

We first present three examples to outline the main ideas, and then verify some conditions in a theorem.

**Example 5.3.** For \(n = 2\), the second-order functions

\[g(x) = x_1^2 + ax_2^2\]

where \(0 < a < 1\) or \(a < 0\), have level curves which are ellipses or hyperbolas. The extreme values of the process

\[g(X(t)) = X_1^2(t) + aX_2^2(t)\]

will be located at points where \(X_1^2(t)\) is extremely large, but the height of the maxima of course will depend on the values of \(X_2(t)\). We can here use Theorem 4.4 (identifying \(X^{(1)}(t) = X_1(t), X^{(2)}(t) = X_2(t)\)) and transform only the \(x_1\)-coordinate,

\[R_0 \times R \ni (x_1, x_2) \rightarrow \left(\frac{x_1}{|x_1| T\mu_1(|x_1|)}, x_2\right) \in R_0 \times R,\]

where

\[
\mu_1(u) = \frac{1}{2\pi} \sqrt{\lambda_{21}} e^{-u^2/2}.
\]

At the level curves, \(x_1^2 + ax_2^2 = u^2\), one has

\[
T\mu_1(|x_1|) = \frac{T}{2\pi} \sqrt{\lambda_{21}} e^{-x_1^2/2} = \frac{T}{2\pi} \sqrt{\lambda_{21}} e^{-u^2/2} \cdot e^{ax_2^2/2}
\]

\[= T\mu_1(u) \cdot e^{ax_2^2/2}\]

which means that if \(T\mu_1(u) = \tau\), the level curves are transformed into the sets \(\{\pm \tau^{-1} \exp(-ax_2^2/2), x_2\}, x_2 \in R\}. To use Theorem 4.4, we take

\[A = \{(x_1, x_2) \in R_0 \times R; |x_1| > \tau^{-1} e^{-ax_2^2/2}\}\]
and conclude that
\[
\lim_{T \to \infty} P\left\{ \max_{0 \leq t \leq T} X_1^2(t) + aX_2^2(t) > u^2 \right\}
\]
\[
= \lim_{T \to \infty} P\{(X_1^{(1)}(t), X^{(2)}(t)) \in A, \text{ some } t \in [0, T]\}
\]
\[
= P\{N^{(1,2)}(A) \geq 1\} = 1 - \exp(-\Gamma^{(1,2)}(A)).
\]

Since the measure $\Gamma^{(1,2)}$ has density $x_1^{-2}\phi(x_2)$ with respect to Lebesgue measure,
\[
\Gamma^{(1,2)}(A) = 2 \int_{x_2=-\infty}^{\infty} \int_{x_1=\tau e^{-\alpha x_2^2/2}}^\infty x_1^{-2}\phi(x_2) \, dx_1 \, dx_2
\]
\[
= 2 \int_{x_2=-\infty}^{\infty} \tau e^{\alpha x_2^2/2} \phi(x_2) \, dx_2
\]
\[
= 2\pi\sqrt{1-a},
\]
and we conclude that
\[
P\left\{ \max_{0 \leq t \leq T} X_1^2(t) + aX_2^2(t) \leq u_T^2 \right\} \rightarrow e^{-2\pi\sqrt{1-a}}
\]
if
\[
T\mu_1(u_T) = \frac{T}{2\pi} \sqrt{\lambda_{21}} \exp(-u_T^2/2) \rightarrow \tau,
\]
i.e.
\[
(5.3) \quad u_T^2 = 2 \log T + 2 \log (\sqrt{\lambda_{21}/2\pi}) - 2 \log \tau + o(1).
\]

The example reveals quite an interesting feature of non-linear functions. If $a < 1$, $\max_{0 \leq t \leq T} X_1^2(t)$ and $\max_{0 \leq t \leq T} X_1^2(t) + aX_2^2(t)$ are of the same order of magnitude, $u_T^2 = 2 \log T + O(1)$, while for $a = 1$ we get the $\chi^2$-process, whose maximum is of the order $\lambda_2 T = 2 \log T + \log \log T + O(1)$. Of course, the practical consequences of this may be minor due to the extremely slow increase of $\log \log T$.

Example 5.4. For the parabolic function $x_1^2 + ax_2$ the order of the maximum is determined by $x_1^2$. Making the same transformation as in Example 5.3,
\[
R_0 \times R \ni (x_1, x_2) \rightarrow \left(\frac{x_1}{|x_1| T\mu_1(|x_1|)}, x_2\right) \in R_0 \times R,
\]
the level curves $x_1^2 + ax_2 = u^2$ are transformed into the sets
\[
\{ (\pm \tau^{-1} e^{-ax_2}, x_2) \}.
\]
With

\[ A = \{(x_1, x_2); |x_1| > \tau^{-1} e^{-ax_2}\}, \]

\[ \Gamma^{(1,2)}(A) = 2 \int_{x_2 = -\infty}^{\infty} \int_{x_1 = \tau^{-1} e^{-ax_2}}^{\infty} x_1^{-2} \phi(x_2) \, dx_1 \, dx_2 \]

\[ = 2\tau \int_{x_2 = -\infty}^{\infty} e^{ax_2} \phi(x_2) \, dx_2 = 2\tau e^{a/2}, \]

so that Theorem 4.4 implies

\[ P\left\{ \max_{0 \leq t \leq T} X_1^2(t) + aX_2(t) \leq u_T^2 \right\} \rightarrow \exp(-2\tau e^{a/2}) \]

if \( T\mu_1(u_T) \rightarrow \tau \).

**Example 5.5.** In higher dimensions the extremal properties of \( g(X(t)) \) may depend mainly on a subset of the \( X_k(t) \)-variables. If \( 0 < a < 1 \), the level surfaces

\[ x_1^2 + x_2^2 + ax_3^2 = u^2 \]

are ellipsoids and the extremes are determined by \( X_1^2(t) + X_2^2(t) \). Write \( x^{(1)} = (x_1, x_2) \), \( x^{(2)} = x_3 \), and define the transformation

\[ R^2 \times R \ni (x^{(1)}, x^{(2)}) \rightarrow \left( \frac{x^{(1)}}{\|x^{(1)}\|} T\mu_2(\|x^{(1)}\|), x^{(2)} \right), \]

which takes the level surfaces \( x_1^2 + x_2^2 + ax_3^2 = u^2 \) into the set

\[ \{(\tau^{-1} x^{(1)} e^{-a(x^{(2)})^2/2}, x^{(2)}); \|x^{(1)}\| = 1, |x^{(2)}| \leq u/\sqrt{a}\} \]

if \( T\mu_2(u) = \tau \). With

\[ A = \{(x^{(1)}, x^{(2)}) \in R^2 \times R; \|x^{(1)}\| > \tau^{-1} e^{-a(x^{(2)})^2/2}\} \]

one has

\[ \Gamma^{(1,2)}(A) = \int_{x_2 = -\infty}^{\infty} \int_{x_1 = \tau^{-1} e^{-ax_2}}^{\infty} c_2(x) r^{-2} \phi(x) \, ds(x) \, dr \, dy \]

\[ = \int_{\|x\| = 1} c_2(x) \, ds(x) \cdot \tau/\sqrt{1 - a} = \tau/\sqrt{1 - a}, \]

and consequently, by Theorem 4.4,

\[ P\left\{ \max_{0 \leq t \leq T} X_1^2(t) + X_2^2(t) + aX_3^2(t) \leq u_T^2 \right\} \rightarrow e^{-\tau/\sqrt{1 - a}} \]

where the same normalization, \( T\mu_2(u_T) \rightarrow \tau \), applies as for \( X_1^2(t) + X_2^2(t) \).
The following theorem summarizes the extremal properties of homogeneous second-degree polynomials in $X_1(t), \cdots, X_n(t)$,

$$g(X(t)) = X(t)GX(t)^T,$$

where $G$ is symmetric with eigenvalues $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, (and without loss of generality $\gamma_1 = 1$). Suppose $G$ is diagonalized by the orthogonal matrix $V$, $G = VT^TV$, $\Gamma = \text{diag} (\gamma_i)$, and write $\Lambda = \text{diag} (\lambda_{2i})$, $\lambda_{2i} = \text{Var} (X_{i}(t))$, $L = V^T \Lambda V$. Then the components of $Y(t) = X(t)V$ are Gaussian and independent with derivative $Y'(t)$ having covariance matrix $L$, and

$$g(X(t)) = \sum_{i=1}^{n} \gamma_i Y_i^2(t).$$

We now assume there are exactly $p$ eigenvalues equal to one, $\gamma_1 = \cdots = \gamma_p = 1 > \gamma_{p+1}$, and shall see that the extreme value distribution of $g(X(t))$ is of the same order of magnitude as that of a $\chi^2$-process with $p$ degrees of freedom.

Write

$$C_p = \int \left( \sum_{i=1}^{p} l_i y_i^2 \right)^{\frac{1}{2}} ds^p(y),$$

$$(y = (y_1, \cdots, y_p), ds^p(y)$ the Haar measure on the $p$-dimensional unit sphere), and observe that $C_p$ is independent of the orthogonal transformation $V$ chosen to diagonalize $G$.

**Theorem 5.6.** If the hypotheses of Theorem 2.1 are satisfied and

$$g(X(t)) = X(t)GX(t)^T = \sum_{i=1}^{n} \gamma_i Y_i^2(t)$$

$$(\gamma_1 = \cdots = \gamma_p = 1 > \gamma_{p+1} \geq \cdots \geq \gamma_n, Y(t) = X(t)V, G = VT^TV),$$

then

$$P\left( \max_{0 \leq t \leq T} g(X(t)) \leq u_T^2 \right) \rightarrow \exp \left( -\tau - \prod_{i=p+1}^{n} (1 - \gamma_i)^{\frac{1}{2}} \right)$$

when $T \rightarrow \infty$ so that $T\mu_p(u_T) = TC_p u_T^{-1} \exp (-u_T^2/2) = \tau$, i.e.

$$u_T^2 = 2 \log T + (p - 1) \log \log T + \log 2^{p-1} - 2 \log \tau + \log C_p + o(1).$$

**Proof.** Write $q = n - p$, and use Theorem 4.4 identifying $X^{(1)}(t)$ and $X^{(2)}(t)$ with $(Y_{1}(t), \cdots, Y_{q}(t))$ and $(Y_{p+1}(t), \cdots, Y_{n}(t))$, respectively, and note that the transformation

$$R_0^p \times R^q \ni (x^{(1)}, x^{(2)}) \rightarrow \begin{pmatrix} x^{(1)} \\ \| x^{(1)} \| T\mu_p(\| x^{(1)} \|) \\ x^{(2)} \end{pmatrix} \in R_0^p \times R^q$$
then transforms the level curves \( g((x^{(1)}, x^{(2)})) = u^2 \) into the set
\[
\left\{ x^{(1)} \tau^{-1} \exp \left[ -\frac{1}{2} \sum_{i=1}^{q} \gamma_{p+i}(x^{(2)}_i)^2 \right], \|x^{(1)}\| = 1 \right\}.
\]

If we write
\[
A = \left\{ (x^{(1)}, x^{(2)}); \|x^{(1)}\| \geq \tau^{-1} \exp \left[ -\frac{1}{2} \sum_{i=1}^{q} \gamma_{p+i}(x^{(2)}_i)^2 \right] \right\}.
\]

Theorem 4.4 therefore implies that
\[
P\left( \max_{0 \leq t \leq T} \sum_{i=1}^{n} \gamma_i Y_i^2(t) \geq u^2 \right) \to 1 - e^{-\Gamma(1,2)(A)},
\]
where, writing
\[
r(x^{(2)}) = \tau^{-1} \exp \left( -\frac{1}{2} \sum_{i=1}^{q} \gamma_{p+i}(x^{(2)}_i)^2 \right),
\]
\[
\Gamma^{(1,2)}(A) = \int_{(x^{(1)}, x^{(2)}) \in A} \gamma(x^{(1)}) \phi(x^{(2)}) \, dx^{(1)} \, dx^{(2)}
\]
\[
= \int_{x^{(2)}} \int_{\|x^{(1)}\| = 1} \int_{r = r(x^{(2)})}^{\infty} r^{-2} c_p(x^{(1)}) \phi(x^{(2)}) \, dr \, ds^p(x^{(1)}) \, dx^{(2)}
\]
\[
= \tau \int_{x^{(2)}} \exp \left[ \frac{1}{2} \sum_{i=1}^{q} \gamma_{p+i}(x^{(2)}_i)^2 \right] \phi(x^{(2)}) \, dx^{(2)}
\]
\[
= \tau \prod_{i=1}^{q} (1 - \gamma_{p+i}).
\]

When checking Conditions (4.2) and (4.3) one can obviously assume \( \gamma_{p+1} = \cdots = \gamma_n = \gamma < 1 \). If \( \gamma \leq 0 \) then \( r_n = \inf \|x^{(1)}\| \); \((x^{(1)}, x^{(2)}) \in A\), some \( x^{(2)} > 0 \), in which case (4.2) and (4.3) are automatically satisfied so that it suffices to consider the case \( 0 < \gamma < 1 \).

Then
\[
A = \{(x^{(1)}, x^{(2)}); \|x^{(1)}\| > \tau^{-1} \exp \left[ -\frac{1}{2} \gamma \|x^{(2)}\|^2 \right]\}
\]
and
\[
\Gamma^{(1,2)}(A) = \tau \int_{x^{(2)}} \exp \left[ \frac{1}{2} \gamma (\|x^{(2)}\| + \varepsilon)^2 \right] \phi(x^{(2)}) \, dx^{(2)} < \infty
\]
which implies (4.2) by dominated convergence.

To see that (4.3) holds, we shall estimate the probability of at least one visit to \( A \) by means of the expected number of exits through certain simple
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boundaries. For \( k = 0, 1, \cdots \), write \( \nu_k = k, \, r_k = \tau^{-1}e^{-\nu_k/2} \), and define

\[
C_k^{(1)} = \{(x^{(1)}, x^{(2)}); \|x^{(1)}\| = r_k, \, \nu_k - 1 \leq \|x^{(2)}\| < \nu_k\}, \\
C_k^{(2)} = \{(x^{(1)}, x^{(2)}); \, r_{k+1} < \|x^{(1)}\| \leq r_k, \, \|x^{(2)}\| = \nu_k\}.
\]

Then \( \bigcup_{k=1}^{\infty} C_k^{(1)} \cup C_k^{(2)} \) is the boundary of a set \( C \) such that \( A \subseteq C \subseteq A_1 \). If we let \( N_k^{(i)}, \, i = 1, 2 \), denote the number of exits into \( C \) through \( C_k^{(i)} \) by \( (X_T^{(1)}(t), X_T^{(2)}(t)), \, t \in (0, 1] \) we therefore have

\[
P\{(X_T^{(1)}(t), X_T^{(2)}(t)) \in A, \, \text{some} \, t \in (0, 1]\} \leq P\{(X_T^{(1)}(0), X_T^{(2)}(0)) \in A\} + \sum_{k=1}^{\infty} E(N_k^{(1)} + N_k^{(2)}).
\]

Here, by stationarity,

\[
E(N_k^{(1)} + N_k^{(2)}) = \frac{1}{T \mu_k} P\{\nu_{k-1} \leq \|X^{(2)}(0)\| < \nu_k\} + \mu_q(\nu_k) P\{r_{k+1} < \|X^{(1)}(0)\| \leq r_k\}.
\]

Writing \( \mu^0 = \sup_\nu \mu_\nu(\nu) \) we then define \( u_k = u_k(T) \equiv \sqrt{p-1} \) such that for \( r_k \equiv 1/T \mu^0 \),

\[
T \mu_p(u_k) = 1/r_k,
\]

so that

\[
P\{r_{k+1} < \|X^{(1)}(0)\| \leq r_k\} = P\{u_{k+1} < \|X^{(1)}(0)\| \leq u_k\}
\]

for \( k \) such that \( r_{k+1} \equiv 1/T \mu^0 \). Since \( \|X^{(1)}(0)\|^2 \) and \( \|X^{(2)}(0)\|^2 \) have \( \chi^2 \)-distributions with \( p \) and \( q \) degrees of freedom

\[
P\{u_{k+1} < \|X^{(1)}(0)\| \leq u_k\} \leq c(u_k - u_{k+1}) \mu_p(u_{k+1}) = \frac{c(u_k - u_{k+1})}{Tr_{k+1}}
\]

and

\[
P\{\nu_{k-1} \leq \|X^{(2)}(0)\| < \nu_k\} \leq c(\nu_k - \nu_{k-1}) \mu_q(\nu_{k-1})
\]

for some constant \( c > 0 \). Since \( \nu_k - \nu_{k-1} = 1 \) we therefore have

\[
\sum_{k=1}^{\infty} E(N_k^{(1)} + N_k^{(2)}) \leq \sum_{k=1}^{\infty} \frac{c}{Tr_k} \mu_q(\nu_{k-1})
\]

\[
+ \sum_{k : r_{k+1} \equiv 1/T \mu^0} \frac{c(u_k - u_{k+1})}{Tr_{k+1}} \mu_q(\nu_k) + \sum_{k : r_{k+1} < 1/T \mu^0} \mu_q(\nu_k)
\]

and this is easily seen to be \( T^{-1} \cdot O(1) \). Since \( TP\{(X_T^{(1)}(0), X_T^{(2)}(0)) \in A\} \to 0 \) as this shows (4.3), completing the proof of Theorem 5.6. \( T \to \infty \).

For non-homogeneous second-degree polynomials corresponding results hold. Since the general formulation is much less attractive than that in
Theorem 5.6, we contain ourselves with the example already given, Example 5.4.

The theory also applies for polynomial functions of higher order.

*Example 5.7.* The level curves

\[ x_1^4 + x_2^4 = u^4 \]

behave, from an extremal viewpoint, almost like straight lines at \( x_1 = \pm u \) and \( x_2 = \pm u \). We have to make two separate transformations,

\[
(x_1, x_2) \rightarrow \left( \frac{x_1}{|x_1| T \mu_1'(|x_1|)}, x_2 \right) \sim (\pm \tau_1^{-1}, x_2)
\]

and

\[
(x_1, x_2) \rightarrow \left( \frac{x_2}{|x_2| T \mu_1'(|x_2|)} \right) \sim (x_1, \pm \tau_2^{-1})
\]

if

\[
T \mu_1'(|x_1|) = \frac{T}{2\pi} \sqrt{\lambda_{21}} e^{-x_1^2/2} \rightarrow \tau_1 = \tau \sqrt{\lambda_{21}}
\]

and

\[
T \mu_1'(|x_2|) = \frac{T}{2\pi} \sqrt{\lambda_{22}} e^{-x_2^2/2} \rightarrow \tau_2 = \tau \sqrt{\lambda_{22}}.
\]

The exceedances by the process \( X_4(t) + X_2(t) \) of the level \( u^4 \) then form a point process which is composed of two asymptotically independent Poisson processes \( N_1 \) and \( N_2 \) with intensities \( 2\tau_1 = 2\tau \sqrt{\lambda_{21}} \) and \( 2\tau_2 = 2\tau \sqrt{\lambda_{22}} \), respectively, if

\[
\frac{T}{2\pi} e^{-u^2/2} \rightarrow \tau
\]

so that

\[
P\left\{ \max_{0 \leq t \leq T} X_4(t) + X_2(t) \leq u_T^4 \right\} \rightarrow e^{-2\tau(\sqrt{\lambda_{21}} + \sqrt{\lambda_{22}})}
\]

where

\[
u_T^2 = 2 \log T - 2 \log 2\pi - 2 \log \tau + o(1).
\]

(The fact that the limiting Poisson processes are independent can be seen in the same way as the asymptotic independence of exits through disjoint rectangles on the cylinder in Theorem 2.1.)

All the presented examples have in common that the asymptotic extremal distribution of \( g(X(t)) \) depends on the distance from the origin to the nearest
point on the level surface \( g(x) = u \), as well as on the geometrical properties of the level surface near that point. In Example 5.2 the level surfaces are uniformly close to a sphere, while in Examples 5.3, 5.4, and 5.7 the level curves deviate significantly from the inscribed circles, giving rise to a change in the order of magnitude in the maximum. In Example 5.5 finally, there is a subset of coordinates for which the level curves are spherical, and the maximum of \( g(X_1(t), X_2(t), X_3(t)) \) behaves like a function of \( X_1(t), X_2(t) \). For second-order homogeneous polynomials, the multiplicity of the largest eigenvalue determines the order of magnitude of the maximum, the exact location parameter being given by Theorem 5.6.

As a general rule, the safety assessment discussed in Section 1 could be dealt with as follows, as far as its asymptotic properties are concerned. For a given safe region, with boundary \( \partial S \), identify the point (or points) \( x^0 \) closest to the origin. By a simple rotation of coordinates, one can always ensure that \( x^0 \) falls on some of the coordinate axes.

If the boundary \( \partial S \) is almost spherical, with constant curvature, close to \( x^0 \), perform the transformation (4.1) and use Theorem 4.2 as in Example 5.2.

If the boundary \( \partial S \) is not spherical near \( x^0 \), make a partial transformation of some of the coordinates as in Examples 5.3–5.5, so that the unsafe region becomes ray-shaped in the transformed coordinates, and use Theorem 4.4.

The examples also indicate how one can handle the case when there is more than one point which is closest to the origin. The principle then is to treat what happens in all the different regions close to these nearest points as stochastically independent.

References


