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Horseshoe-like patterns in first-order 3D random Gauss-Lagrange waves with directional spreading

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The Gauss–Lagrange model for ocean waves describes the vertical and horizontal movements of water particles as three correlated Gaussian fields. The model can produce irregular waves, asymmetric in both vertical and horizontal direction, and by judicious choice of a single skewness parameter the front–back asymmetry can be regulated to realistic values. In this paper, it is shown that this additive model for shallow waters can also produce horseshoe-like patterns around moderate to high wave crests. Such phenomena are usually analyzed and described as nonlinear interaction effects between different frequencies. The tool in the paper is a Slepian model for the three-dimensional movements conditioned on a wave crest.

1. Introduction

The Lagrange ocean wave model for irregular waves, described by Pierson [1], offers an attractive combination of simplicity and realism. The fundamental characteristic of the model is that it describes the vertical and horizontal movements of individual water particles, in contrast to the Euler model that describes the particle movements at fixed points. In later years, notable arguments for the Lagrange model have been brought forward by Gjøsund [2], and in further work in the same vein, reported in [3,4], while more mathematical studies have been presented in [5,6].

Even a first-order Lagrange model, without any interaction between elementary waves with different wavelengths, can generate waves with realistic crest–trough and front–back statistical asymmetry. With second-order interaction also more complicated nonlinearities, such as wave breaking and formation of horseshoe patterns can be reproduced.

One advantage of the Lagrangian model is that it permits detailed studies of the statistical properties under the assumption that the processes that describe the vertical and horizontal particle movements are Gaussian with a certain well-defined correlation structure. This assumption makes it possible to exploit the full machinery for conditioning in Gaussian processes, based on Rice formula and generalizations thereof,[7] to obtain explicit and exact statistical distributions of the characteristics of individual waves. A complete name for the model with Gaussian components would be the stochastic Gauss–Lagrange model.

Theoretical studies of the stochastic properties of the Lagrange models have recently been made by Lindgren [8] and coworkers [9–13]. All these works deal with unidirectional waves observed in time at a fixed location or along a line with a fixed direction in space,

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equivalent to 2D waves with one space parameter together with the time parameter. The full 3D-model was dealt with in [14], while an application to wave energy is described in [15].

In [16] Nouguiere and coworkers studied some statistical properties of the Lagrange model with similar methods and used the model in [17] to describe scattering in remote sensing. In [18] the same authors made a detailed hydrodynamic analysis of the second-order 3D Lagrange model and showed how the model can produce horseshoe wave patterns. Fouques et al. [3] showed that Monte Carlo simulated stochastic Lagrange models can produce realistic crest–trough asymmetry as well as front–back asymmetry, the latter for higher-order Lagrange models.

The random Lagrange model is easy to simulate in Monte-Carlo studies, and by judicious choice of a single skewness parameter even the first-order model can generate waves with a wide range of front–back skewness; this modification of the simplest Lagrange model was suggested in [11,12]. For an example of how to use it in a remote sensing setting, see the paper by Chen et al. [19], and also [20,21].

In the present paper, we investigate some further geometric properties of the 3D Lagrange wave field with directional spreading; cf. [14]. In particular, we present a Slepian model for the Lagrange components around maxima points in the 3D fields, which in turn allows a description of crescent- or horseshoe-like patterns in directional waves with front–back asymmetry. These results represent extensions of the Slepian representation in [22] of a Gaussian field near local maxima: in the 3D Gauss model level curves around local maxima are, on the average, ellipses, while they have a curved horseshoe-like shape in the linked Lagrange model.

One should note, however, that the horseshoe patterns in nonlinear and second-order wave models have a different geometry than the horseshoe-like patterns of the first-order Lagrange waves. These are simply an intrinsic “non-linear” property of that, basically linear, model.

The first-order 3D stochastic Lagrange model is presented in Section 2. Slepian models for the Lagrange components are derived in Section 3 and a simulation technique is described in Section 4. Simulation studies in Section 5 illustrated how the Lagrange model can generate horseshoe-like patterns.

A basic set of MATLAB® routines for Lagrange waves is collected in the WAFO module WafoL [23,24].

2. The 3D stochastic Gauss–Lagrange wave model

The 3D stochastic Gauss–Lagrange model is defined by three time-varying Gaussian fields, denoted \( W \) and \((X, Y)\), with time parameter \( t \) and space parameter \( s = (u, v)^T \). The field \( W(t, s) \) describes the vertical movements with time of an individual water particle with original location \( s \), also called its reference point. The fields \( X(t, s) \) and \( Y(t, s) \) describe the displacement in \( x \)- and \( y \)-direction of the same particle. Thus, the location at time \( t \) of a particle with original position \( s \) is \((u + X(t, s), v + Y(t, s), W(t, s))\) in \( \mathbb{R}^3 \).

With the notation

\[
\Delta(t, s) = s + \begin{pmatrix} X(t, s) \\ Y(t, s) \end{pmatrix},
\]
the 3D Lagrange model is defined as the tri-variate Gaussian process (cf. [1, Equation 19]),

\[(\Delta(t, s), W(t, s)), t \in \mathbb{R}, s \in \mathbb{R}^2.\]  

(1)

The height of the water surface at location \(\Delta(t, s) = (x, y)\) is equal to \(W(t, s)\). Due to the randomness in horizontal displacement, it may happen that there are many \(s\)-values that satisfy \(\Delta(t, s) = (x, y)^T\); then the surface height is not uniquely defined. The probability of this folding is negligible, for all but very shallow waters.

2.1. The vertical process \(W(t, s)\)

In the Gauss–Lagrange model, the process \(W(t, s)\) is a stochastic integral over wavenumber \(\kappa = (\kappa_x, \kappa_y) \in \mathbb{R}^2\), or, alternatively, over wave angular frequency \(\omega > 0\) and wave direction \(\theta \in (-\pi, \pi]\). Wave number and frequency/direction are related via the dispersion relation, which also includes water depth \(h\); see [25, Chapter 4]. With \(\kappa = ||\kappa|| = \sqrt{\kappa_x^2 + \kappa_y^2}\), the dispersion relation is \(\omega = \omega(\kappa) = \sqrt{g} \kappa \tanh(\kappa h) (\approx \sqrt{g} \kappa\) for large water depth), where \(g\) denotes the acceleration of gravity.

We denote by \(\tau\) and \(\sigma = (\sigma_x, \sigma_y)^T\) a time difference and a space difference, respectively. The covariance function of the field in space-time is then, with \(\kappa \sigma = \kappa_x \sigma_x + \kappa_y \sigma_y\),

\[r_{ww}(\tau, \sigma) = \text{Cov}(W(t, s), W(t + \tau, s + \sigma)) = \int_{\omega = 0}^{\infty} \int_{\theta = -\pi}^{\pi} \cos(\kappa \sigma - \omega \tau) S(\omega, \theta) \, d\omega \, d\theta,\]  

(2)

where \(S(\omega, \theta), \) for \(\omega > 0, -\pi < \theta \leq \pi\), is the directional spectrum of the field. In applications, \(S\) is called the orbital spectrum, and it is what is usually estimated from observations.

The vertical process is defined by

\[W(t, s) = \int_{\omega = -\infty}^{\infty} \int_{\theta = -\pi}^{\pi} e^{i(\kappa s - \omega t)} \, d\xi(\omega, \theta),\]  

(3)

where \(\xi(\kappa, \omega)\) is a Gaussian complex spectral process giving uniform phases and random amplitudes to the harmonic components; for details on how to interpret negative frequencies, see [14, Section 2.1.2].

2.2. The linked Lagrange model

In the first-order stochastic Gauss–Lagrange model the horizontal displacement is modeled as a bivariate Gaussian process, correlated with \(W(t, s)\) and expressed by a linear filter operation on the \(W\)-process.

To obtain realistic front–back asymmetries in the Lagrange model it is necessary to introduce a general transfer function

\[H(\theta, \kappa) = \begin{pmatrix} h_x(\theta, \kappa) \\ h_y(\theta, \kappa) \end{pmatrix} = \begin{pmatrix} \rho_x(\theta, \kappa) e^{i\psi_x(\theta, \kappa)} \\ \rho_y(\theta, \kappa) e^{i\psi_y(\theta, \kappa)} \end{pmatrix}.\]
The expression for the horizontal location is then
\[ \Delta(t, s) = s + \int_{\omega} \int_{\theta} \left( e^{i(\kappa_x u + \kappa_y v - \omega t + \psi_x(\theta, \kappa))} \rho_x(\theta, \kappa) e^{i(\kappa_x u + \kappa_y v - \omega t + \psi_y(\theta, \kappa))} \rho_y(\theta, \kappa) \right) d\xi(\omega, \theta). \]

In the simplest Lagrange model the transfer function is purely imaginary,
\[ H(\theta, \kappa) = H_M(\theta, \kappa) = i \frac{\cosh \kappa h}{\sinh \kappa h} \cdot \left( \cos \theta \sin \theta \right), \]
where the subscript \( M \) stands for Miche waves. We call this the free Lagrange model. It is seen that the free model introduces a constant phase shift of \( \psi_x = \psi_y = \pi/2 = 90^\circ \) between the vertical and horizontal processes, while the linked model induces a more general phase shift.

In the examples we will use a complex transfer function that, besides the imaginary component \( H_M(\theta, \kappa) \), contains a real term, whose form remains to be determined. Working with the 2D model, Lindgren and Åberg [13] argued for a wind driven dependence between the vertical and horizontal movements of the form
\[ \frac{\partial^2}{\partial t^2} X(t, s) = \frac{\partial^2}{\partial t^2} X_M(t, s) - \alpha W(t, s), \]
where the \textit{linkage parameter} \( \alpha \) determines the degree of front–back asymmetry. The corresponding transfer function is \( H_M(\kappa) + \alpha/\omega^2 \).

For the 3D case, we choose to keep the term \( \alpha/\omega^2 \), but have the freedom to modify it according to wave direction \( \theta \). The following choice is a simple alternative to that used in [14] and will be used in the examples in later sections,
\[ H(\theta, \kappa) = \frac{\alpha}{\omega^2} \cdot \left( \frac{\cos^2(\theta/2)}{\sin^2(\theta/2) \text{sign}(\sin(\theta/2))} \right) + i \frac{\cosh \kappa h}{\sinh \kappa h} \cdot \left( \cos \theta \sin \theta \right). \]  (4)

As will be shown in Section 5.3 the \( \alpha \)-parameter is also responsible for the horseshoe patterns which are the main motivation for the present paper.

2.3. \textit{The Lagrange space wave}

The first-order 3D Lagrange model for ocean waves was defined by (1) as the tri-variate Gaussian process \((\Delta(t, s), W(t, s)), t \in \mathbb{R}, s \in \mathbb{R}^2\). The time dependent Lagrange wave field can then be implicitly expressed as
\[ L(t, \Delta(t, s)) = W(t, s), \]  (5)
or, formulated explicitly, \( L(t, (x, y)) \) is the set of \( w \in \mathbb{R} \) such that, for some \( s \in \mathbb{R}^2 \),
\[ w = W(t, s), \quad x = X(t, s), \quad y = Y(t, s). \]  (6)

\textit{Remark 1} A complication in the model is that folding may occur, leading to multiple values of \( L \) in some areas, i.e. it can happen that \( \Delta(t_0, s_1) = \Delta(t_0, s_2) \) with \( s_1 \neq s_2 \), and \( W(t_0, s_1) \neq W(t_0, s_2) \). However, for realistic parameter values and water depth, the probability of folding is negligible.
The space wave field $L(t_0, (x, y))$, $(x, y) \in \mathbb{R}^2$, at time $t_0$ is obtained from (6) by setting $t = t_0$.

The time wave $L(x_0, y_0)(t)$, $t \in \mathbb{R}$, at location $(x_0, y_0)$, is defined from (6), by setting $(x, y) = (x_0, y_0)$.

If the inverse $\Delta^{-1}(t, (x, y)) = \{s; \Delta(t, s) = (x, y)^T\}$ is uniquely defined,

$$
L(t_0, (x, y)) = W(t_0, \Delta^{-1}(t_0, (x, y))),
$$

$$
L(x_0, y_0)(t) = W(t, \Delta^{-1}(t, (x_0, y_0))).
$$

Otherwise, the Lagrange wave takes multiple values.

3. A Slepian model near local wave crests

3.1. Slepian models and crossing events

Many geometrical characteristics of ocean waves are defined in terms of level crossings by any of the components. Wave periods and wave lengths are defined by means of mean level crossings by the height process $W(t, s)$, either in time or in any particular direction, or by zero crossings of any directional or temporal derivative. Wave steepness is usually defined as some combination of time and distance between crossings. The cited works [11–14] give many examples of wave characteristic distributions calculated by means of crossing theory in Gauss–Lagrange models.

The Gaussian structure of the Gauss–Lagrange model also makes it possible to construct an explicit representation of the fields in the neighborhood of any crossing event. Such models were introduced by Slepian [26] for a Gaussian process near its zero crossings, and further generalized in [27] to processes conditioned on a local maximum, and in [22] to Gaussian fields conditioned on a local maximum in $\mathbb{R}^n$. A Slepian model describes a Gaussian process near points in time and space when sampled by some type of crossing event. It typically consists of three terms, one deterministic depending on the type of crossing, one stochastic depending on the random local structure at the crossing, and one stochastic residual that takes care of the remaining variability.

The so-called quasi-deterministic representations of large waves, described in detail in [28], are Slepian models without the two random terms, which become negligible for very large waves; see Remarks 4 and 6.

Slepian models for the 2D Gauss–Lagrange model were derived in [8] and we will now extend those models to the 3D case and present an explicit representation of the Gaussian components conditioned on a wave crest in the Lagrange wave. The analysis will build on the Slepian model in [22] and it represents an extension of the analysis in [14].

3.2. A conditional distribution

The Lagrange wave field is obtained by a differentiable horizontal deformation of the Gaussian vertical field, which conserves the property of local maximum. Therefore, we first analyze the field $W(t_0, s)$ at a fixed time point, say, $t_0 = 0$. In the following we suppress the $t$-argument and simply write $W(s)$, etc. We have assumed the first- and second-order
derivatives to exists, with $W''(s) = W''_v(s)$, and define,

$$W'(s) = (W'_u(s), W'_v(s)), \quad W''(s) = (W''_u(s), W''_v(s), W''_{uv}(s)) = z(s),$$

$$Z(s) = \begin{pmatrix} W''_u(s) & W''_{uv}(s) \\ W''_v(s) & W''_{uv}(s) \end{pmatrix},$$

(We use the convention that $Z$ is the symmetric matrix formed by the elements of the vector $z$.) Then, $W$ has a local maximum at $s$ if $W'(s) = 0$ and $Z(s) < 0$, i.e. $Z(s)$ is negative definite.

Now, let $\sigma = (s^1, \ldots, s^m)$ be $m$ different points in $\mathbb{R}^2$, and define the following densities and conditional density functions, $x = (x_1, \ldots, x_m)$,

$$p_\sigma(u, v, z, x) \text{ for } W(0), W'(0), W''(0), W(s^1), \ldots, W(s^m),$$

$$p_\sigma(x \mid u, v, z) \text{ for } W(s^1), \ldots, W(s^m)$$

given $W(0) = u, W'(0) = v, W''(0) = z,$

$$p(u, v, z) \text{ for } W(0), W'(0), W''(0),$$

$$p(z \mid u, v) \text{ for } W''(0) \mid W(0) = u, W'(0) = v.$$  

We cite the following theorem [22, Theorem 1.1].

**Theorem 1** Given that $W(s)$ has a local maximum with height $u$ at $0$, the conditional distribution of $W(s^1), \ldots, W(s^m)$ has the density

$$\frac{\int_{Z<0} \left| \det Z \right| p_\sigma(u, 0, z, x) \, dz}{\int_{Z<0} \left| \det Z \right| p_\sigma(u, 0, z) \, dz}, \quad x \in \mathbb{R}^m. \tag{7}$$

With

$$q_u(z) = \frac{\left| \det Z \right| p(-z \mid u, 0)}{\int_{Z>0} \left| \det Z \right| p(-z \mid u, 0) \, dz} \text{ for } Z > 0, \tag{8}$$

(and 0 otherwise), the density (7) can be expressed as

$$\int q_u(z) \, p_\sigma(x \mid u, 0, -z) \, dz. \tag{9}$$

The density (9) has the typical structure for a Slepian models for a crossing-conditioned process. The factor $q_u(z)$ defines the distribution of characteristic initial values at the crossing point, in this case the second derivatives $Z$ at the maximum, where $W = u$ and $W' = 0$. The factor $p_\sigma(x \mid u, 0, -z)$ gives the conditional distribution of the surrounding field given the initial values. For a Gaussian field this conditional distribution is equal to the distribution of a regression function, depending on $Z$, plus a nonstationary Gaussian residual field, independent of $Z$; for details, see [22] and the examples in [8,27,29], and also [30, Section 8.4] for an overview.
3.3. Some notation and moment relations

To formulate the Slepian model in an explicit form we need notation for the second moments in the joint normal distribution of

\[ G = (W(0), W'(0), W''(0)), \]

\[ W(\sigma) = (W(s^1), \ldots, W(s^m)). \]

They have all mean zero and, obviously, we need only consider \( m = 2; \sigma = (s^1, s^2) \). With the spatial covariance function (2),

\[ r(\sigma) = \text{Cov}(W(0), W(\sigma)) = r^{W_0}(0, \sigma) \]

\[ = \int_0^\infty \int_{-\pi}^{\pi} \cos(\kappa \sigma) S(\omega, \theta) \, d\omega \, d\theta, \]

we have for \( i, j, k, l = u, v, \)

\[ \text{Cov}(W_i'(0), W_j'(0)) = -\text{Cov}(W_i(0), W_{ij}''(0)) = -r_{ij}''(0), \]

\[ \text{Cov}(W_{ij}''(0), W_k'(0)) = r_{ijkl}''(0), \]

\[ \text{Cov}(W_i'(0), W(s)) = -r_i'(s), \]

\[ \text{Cov}(W_{ij}''(0), W(s)) = r_{ij}''(s). \]

All other second moments are zero. The covariance matrix is partitioned as

\[ \Sigma(\sigma) = \begin{pmatrix}
\Sigma_{GG} & \Sigma_{GW}(\sigma) \\
\Sigma_{WG}(\sigma) & \Sigma_{WW}(\sigma)
\end{pmatrix} \]

\[ = \begin{pmatrix}
r(0) & 0 & \Sigma_{02} & \Sigma_{0W}(\sigma) \\
0 & \Sigma_{11} & 0 & \Sigma_{1W}(\sigma) \\
\Sigma_{20} & 0 & \Sigma_{22} & \Sigma_{2W}(\sigma) \\
r_{W0}(\sigma) & \Sigma_{W1}(\sigma) & \Sigma_{W2}(\sigma) & \Sigma_{WW}(\sigma)
\end{pmatrix}. \]

Here \( \Sigma_{GW}(\sigma) \) denotes the \( 6 \times 2 \) covariance matrix between the three \( G \)-components and the two \( W \)-variables \( W(s^1) \) and \( W(s^2) \). We use the notation as a generic notation for any size \( \sigma \), and similarly with \( \Sigma_{WW}(\sigma) \).

By the standard formulas for conditioning in normal distributions we can express the conditional mean and covariance matrix of \( W(\sigma) \) given \( G = y = (u, 0, z) \) as

\[ m_{W|G=y}(\sigma) = \mathbb{E}(W(\sigma)|G=y) = \Sigma_{WG}(\sigma) \Sigma_{GG}^{-1} (u, 0, z)^T \]

\[ = uA(\sigma) + zb(\sigma), \text{ say}, \]

\[ \Sigma_{W|G}(\sigma) = \text{Cov}(W(\sigma)|G) \]

\[ = \Sigma_{WW}(\sigma) - \Sigma_{WG}(\sigma) \Sigma_{GG}^{-1} \Sigma_{GW}(\sigma) \]

\[ = \Sigma_{WW}(\sigma) - (\Sigma_{W0}(\sigma) \Sigma_{W2}(\sigma)) \begin{pmatrix} r(0) & \Sigma_{02} \\ \Sigma_{20} & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{0W}(\sigma) \\ \Sigma_{2W}(\sigma) \end{pmatrix} \]

\[ - \Sigma_{W1}(\sigma) \Sigma_{11}^{-1} \Sigma_{1W}(\sigma). \]

(13)
The covariance matrix (13) defines a covariance function in \( \mathbb{R}^2 \) by
\[
C(s^1, s^2) = r(s^1 - s^2)
\]
\[
- (r(s^1) \Sigma_{W^2}(s^1)) \left( \begin{array}{c} r(0) \\ \Sigma_{20} \\ \Sigma_{22} \end{array} \right)^{-1} \left( \begin{array}{c} r(s^2) \\ \Sigma_{20} \\ \Sigma_{22} \end{array} \right) \\
- \Sigma_{W^1}(s^1) \Sigma_{11}^{-1} \Sigma_{1W}(s^2).
\]

(14)

Thus, \( p_\sigma(x \mid u, 0, -z) \) is the density function for a nonhomogeneous Gaussian field with mean value function \( uA(s) - zb(s) \) and covariance function \( C(s^1, s^2) \).

The density \( q_u(z) \) can be expressed in a similar way: \( p(z \mid u, 0) \) is a tri-variate Gaussian density with mean
\[
(\Sigma_{20} 0) \left( \begin{array}{c} r(0) \\ 0 \\ \Sigma_{11} \end{array} \right)^{-1} \left( \begin{array}{c} u \\ 0 \\ 0 \end{array} \right) = u \Sigma_{20}/r(0),
\]
and covariance matrix
\[
\Sigma_{22} - (\Sigma_{20} 0) \left( \begin{array}{c} r(0) \\ 0 \\ \Sigma_{11} \end{array} \right)^{-1} \left( \begin{array}{c} 0 \\ \Sigma_{02} \\ 0 \end{array} \right) = \Sigma_{20}, \text{ say}.
\]

Thus, for \( Z > 0 \) we have
\[
q_u(z) = k_u^{-1} \det Z \exp \left\{ -\frac{1}{2} (z + u \Sigma_{20})^T \Sigma_{20}^{-1} (z + u \Sigma_{20}) \right\},
\]
where \( k_u \) is a normalizing constant.

### 3.4. An explicit Slepian model for \( W \) near a maximum

We are now ready to formulate the Slepian model \( W_u(s) \) for \( W(s) \) near a local maximum with height \( u \). Like all Slepian models it contains three components, one deterministic regression part, one stochastic regression function depending on the random initial values at the maximum, and one residual nonstationary Gaussian field, independent of the initial values.

**Theorem 2** Let \( \xi \) be a random vector with probability density \( q_u(z) \), (8), over \( \mathbb{R}^3 \) and set, by (12),
\[
m'_u(\sigma; \xi) = uA(\sigma) - \xi b(\sigma).
\]

Further, let \( \delta_w(\sigma) \) be a nonhomogeneous residual Gaussian field over \( \sigma \in \mathbb{R}^2 \), independent of \( \xi \), with mean zero and covariance function \( C(s^1, s^2) \) given by (14).

Then, the long run empirical distribution of \( W(s_k + \sigma) \) over all local maxima \( s_k \in \mathbb{R}^2 \) with height \( u \) is equal to the distribution of the Slepian model \( W_u(s) \), \( s \in \mathbb{R}^2 \), defined as
\[
W_u(\sigma) = m'_u(\sigma; \xi) + \delta_w(\sigma) = uA(\sigma) - \xi b(\sigma) + \delta_w(\sigma), \sigma \in \mathbb{R}^2.
\]

**Remark 2** The phrase “all local maxima with height \( u \)” shall not be taken literally, since in a Gaussian field with continuous spectrum there are, with probability one, no local maxima with height exactly equal to \( u \). It rather means that \( W_u(\sigma) \) gives the correct description when weighted with the correct height distribution.
As described in the introduction one motivation for the present study is the formation of crescent waves with curved wavefronts near local wave crests. The regression terms $m^u_W(\sigma; \xi), m^u_X(\sigma; \xi), m^u_Y(\sigma; \xi),$ represent a type of average structure of the Lagrange field.
near a local maximum, although dependent of the maximum height and the random initial values $\zeta$. Thus, we can generate a first order Slepian representation without the residual processes $\delta_W, \delta_X, \delta_Y$.

**Definition 1** Let $\zeta$ be a random vector as in Theorem 2. The first-order Slepian representation of the Lagrange wave field near a local maximum is a random field $L^u(\sigma)$ over $\mathbb{R}^2$ that takes the value $m^u_W(\sigma; \zeta)$ at the point

$$(m^u_W(\sigma; \zeta) - m^u_W(0), m^u_Y(\sigma; \zeta) - m^u_Y(0)), \quad (17)$$

for $\sigma \in \mathbb{R}^2$. The field has a local maximum of height $u$ at the origin.

All regression terms in (17) have the same structure,

$$m^u_k(\sigma; \zeta) = uA_k(\sigma) - \zeta b_k(\sigma), \quad (18)$$

with $k = W, X, Y$, respectively. The deterministic functions $A_k, b_k$ are different linear combinations of the covariance functions in $\Sigma_k G(\sigma)$.

**Remark 4** (The Slepian models and quasi-determinism) The quasi-deterministic (QD) models, introduced by Boccotti [31] and presented in detail in [28], describe wave shapes in the neighborhood of selected extreme points in time and/or space. Compared to the structure of the Slepian model, (15) a QD-model consists of the first, deterministic, term only, the stochastic second and third terms becoming negligible for large $u$; cf. [22]. The exact form of the deterministic term depends on how the extreme events are selected – we will see an example in Remark 6.

### 4.2. Simulation of the field

The first-order Slepian model for a 3D Lagrange wave near a maximum can easily be simulated as a function of the random initial values $\zeta$. In the examples in the next section we compute, by numerical integration, the covariances $\Sigma_{GG}, \Sigma_{WG}(\sigma), \Sigma_{XG}(\sigma), \Sigma_{YG}(\sigma)$, in (16) for a discrete grid of $\sigma^d$-points. The initial values $\zeta$ are generated from the density (8) by the rejection method and then a realization of the first-order Slepian model with values $m^u_W(\sigma^d; \zeta)$ is generated at the randomly shifted and centered $\sigma^d$-points given by (17). The result is a realization over an irregular grid, and it can be smoothed by any 3D smoothing technique for such grids. The average first-order Lagrange field near a local maximum is obtained as the average of a large number of such smoothed fields.

**Remark 5** (Lagrange waves as nonlinear functions of Gauss waves) Each of the regression terms in (18) is a linear function of the random variables in $\zeta$. Thus, the average of the individual vertical and horizontal displacements are simply

$$\mathbb{E}(m^u_k(\sigma; \zeta)) = uA_k(\sigma) - \mathbb{E}(\zeta)b_k(\sigma).$$

However, since the Lagrange wave is a nonlinear function of the displacements, the Lagrange wave generated by the average displacements is not equal to the average of the Lagrange waves generated by the random displacements. Hence, the more complicated simulation procedure.
5. Horseshoe-like patterns

5.1. Horseshoe waves

Horseshoe, or crescent, waves have a characteristically curved crest ridge and concave wave back; for an illustration, see [32, Figure 4.13]. These are commonly explained in fluid dynamics as the result of high order, nonlinear, interactions between spectral components, and numerical simulations based on these principles generate realistic examples [33,34].

Figure 7 in [18] shows second-order Lagrange waves with horseshoe patterns. The purpose of the present paper is to illustrate that under certain conditions even the simplest Gauss–Lagrange model can generate horseshoe-like patterns with curved crest ridges and concave wave backs. In particular, these patterns are natural consequence of the directional deformation of the vertical Gaussian field in the linked Lagrange model, without any interaction between spectral components,

The examples will illustrate how the crest ridge curvature depends on four different parameters: the water depth $h$, the degree of directional spreading in the spectrum expressed by a parameter $m$, the crest height $u$, and the linkage parameter $\alpha$. The curvedness will decrease with increasing depth and increase with increasing spreading, crest height, and linkage. The most important of these parameters is the linkage parameter, $\alpha$, which also controls the front–back wave asymmetry. Only Lagrange models with front–back asymmetry exhibit horseshoe-like crest ridges.

All computations were performed in MATLAB® together with the wave packages WAFO and Wafol [23,24].

5.2. Examples

We will illustrate the theory on a model with Pierson–Moskowitz (PM) orbital frequency spectrum, with spectral density

$$ S(\omega) = \frac{5H_s^2}{\omega_p(\omega/\omega_p)^5} e^{-\frac{5}{4}(\omega/\omega_p)^{-4}}, \quad 0 < \omega < \omega_c, $$

where $H_s = 4\sqrt{V(W(t,u))}$ is the significant wave height in the linear model, and $\omega_p$ is the peak frequency, at which the spectral density has its maximum. We use a fixed value $H_s = 7$ m for the significant wave height, and assume a finite cut off frequency $\omega_c = 2.5$ rad/s. (The finite cut-off leads to 0.25% reduction of the significant wave height.) The peak period $T_p = 2\pi/\omega_p$ is set to $T_p = 13$ s.

The directional spreading around the main direction $\theta_0$ is taken to be frequency independent and defined by the cos $2\theta$-model, as

$$ S(\omega, \theta) = c(m) S(\omega) \cos^{2m} \left( \frac{\theta - \theta_0}{2} \right), $$

(19)

with three different values for the spreading parameter $m$. We use $m = 2, 5, 15$, in this example. In the literature, values have been used between $m = 10$ for wind waves and up to $m = 75$. The spectra are shown in Figure 1 for $\theta_0 = 0$. 

Figure 1. Directional PM spectra (19) for the $W(t, s)$-field with $H_s = 7$ m and $T_p = 13$ s; spreading parameter $m = 2, 5, 15$.

5.3. Results

Examples of average curved crest ridge are shown in Figures 2–4 for three different water depths. One is unrealistically low, $h = 16$ m, one more realistic, but still low, 32 m, and one shows the effect on deep water, $h = 500$ m. The figures also illustrate the degree of crest front/crest back asymmetry.

In Figure 2 the linkage is kept constant $\alpha = 2$ and the water is very shallow, $h = 16$ m, and there is a strong front–back asymmetry. Each plot shows contour curves for the average of 1000 replicates of the first-order Lagrange field. The figure illustrates the effect of conditioning crest height $u$ and the directional spreading parameter $m$. High spreading, $m = 2$, leads to strongly curved crest ridges for high waves, $u = 4$, with decreasing curvedness with decreasing spreading. For lower crest height, $u = 2$, is curvedness is reduced. One can also see the tendency to concave wave backs for spectrum with little spreading, $m = 15$, i.e almost uni-directional waves. The deformed shapes near the wave crests generalize the elliptic shapes for Gaussian waves derived in [22].

In Figure 3 the water depth is moderate, but still shallow, $h = 32$ m, and the wave curvedness is generally smaller. We illustrate the effect of varying crest height, $u = 2, 4$, (top/bottom), and linkage/front–back asymmetry, $\alpha = 1, 2, 3$, (left/right), keeping the spectral spreading at a high level, $m = 2$.

Figure 4 shows that horseshoe-like wave crests is an intrinsic property of the linked Lagrange model, coupled to the front–back asymmetry, regardless of water depth. The depth is $h = 500$ m and as seen, there is only a quantitative difference in the curvedness, compared to the shallow depth, $h = 32$ m.

Remark 6 (Comparison with a QD-model) The curved wave crest ridges in Figures 2–4 have their convex side pointing in the wave movement direction; this is an intrinsic nonlinear effect of the Lagrange construction. As an example of how a QD-model produce curved crest ridges we take the wave-group example in [35, Figure 2]. (We take the wave spectrum to be a directional JONSWAP spectrum with spreading parameter $m = 2$.) It is assumed that, at a measuring post at the origin, an exceptionally large wave crest is observed at time $\tau = 0$ followed by an (equally) deep trough -- this gives what Boccotti [28] calls the “second deterministic wave function.” With $T^*$ equal to the time for the first minimum of the time covariance function, the normalized QD-field is, in our notation (2),
Waves in Random and Complex Media

Figure 2. Effect of degree of directional spreading: Average of first-order Slepian model near local maximum with height $u = 2\ m$ and $u = 4\ m$. Directional PM spectrum with $H_s = 7\ m$; $T_p = 13\ s$; water depth $= 16\ m$; spreading parameter $m = 2, 5, 15$; linkage parameter $\alpha = 2$. The solid black curves connect points of local maxima along straight lines in the $x$-direction.

$$W_{QD}(\tau, \sigma) = \frac{r_{ww}(\tau, \sigma) - r_{ww}(\tau - T^*, \sigma)}{r_{ww}(0, 0) - r_{ww}(T^*, 0)}.$$ 

This function, observed at the origin, has a crest at $\tau = 0$ and a trough at $\tau = T^*$. Figure 5 shows how the field changes as it moves with time, in particular how the convexity of the crest and trough ridges change direction. At time $\tau = 0$ the crest ridge is curved with the convex side pointing left.

The fact that the crest ridge at the crest time is curved is an effect of the asymmetric selection criterion of extreme waves, based on a time series of observations. This model is valid for crest-to-trough waves, with the crest preceding the trough in time. For extreme trough-to-crest waves the result will be the opposite. This is most easily seen from the plot at $\tau = T^*$ in Figure 5. Changing the sign of the field, that plot would represent the crest in an extreme trough-to-crest wave, observed at the crest time. The preceding deep trough would be the (sign-changed) plot at $\tau = 0$. The crest ridge at crest time will then appear to be curved in the opposite direction. Hence, one can conclude that curved crest ridges in the QD-model are the result of the asymmetric sampling of extreme waves in the linear Gaussian model.

For the Lagrange model studied in this paper one selects local maxima in space over a large region, observed at a fixed time. Horseshoe-like patterns can then only appear in a linked model, which also produces front–back asymmetric waves. This sampling, by Boccotti [28] called the “first deterministic model,” will not produce statistically curved crest ridges in a Gaussian field; see [22].
Figure 3. Effect of linkage and front–back asymmetry, shallow water: Average of first-order Slepian model near local maximum with height \( u = 2 \text{ m} \) and \( 4 \text{ m} \). Directional PM spectrum with \( H_s = 7 \text{ m} \); \( T_p = 13 \text{ s} \); water depth \( h = 32 \text{ m} \); spreading parameter \( m = 2 \); linkage parameter \( \alpha = 1, 2, 3 \).

Figure 4. Effect of linkage and front–back asymmetry, deep water: Average of first-order Slepian model near local maximum with height \( u = 2 \text{ m} \) and \( 4 \text{ m} \). Directional PM spectrum with \( H_s = 7 \text{ m} \); \( T_p = 13 \text{ s} \); water depth \( h = 500 \text{ m} \); spreading parameter \( m = 2 \); linkage parameter \( \alpha = 1, 2, 3 \).
6. Final comments

The Gauss–Lagrange model with linked components can produce irregular waves with realistic asymmetry, and also, as shown in this paper, with curved wave crest profiles regardless of water depth. It allows detailed statistical analysis of wave characteristic distributions. The analysis shows that curved crest ridges are present already in the simple model, without interaction between frequencies, but, as remarked in the introduction, the model is a complement and not a substitute for nonlinear horseshoe pattern generation.

One can identify two important topics for future research. First, how do particle orbits predicted by the model agree with measured particle orbits in connection with asymmetric wave profiles? Second, what physically realistic linkage structure, cf. (4), can produce the best agreement between theory and observations. It seems as if published empirical data do not contain enough details to answer any of these questions.

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No potential conflict of interest was reported by the authors.

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