Abstract

The Gaussian wave model has been successfully used in ocean engineering for more than half a century. It is well understood, and there exists both exact theory and efficient numerical algorithms for calculation of the distribution of safety related wave characteristics, such as crest height and steepness. Its drawback is its lack of realism: it produces waves which are stochastically symmetric, both in the vertical and in the horizontal direction. From that point of view, the Lagrangian wave model is more realistic, but its stochastic properties have not been studied until quite recently. We present an explicit expression for the occupation density (approximately the univariate probability density) of the Lagrangian wave model. We also draw some conclusions about the definition of freak or rogue waves.

Keywords: Freak waves; Gaussian process; Occupation density; Rogue waves; Significant wave height; Wave height

1. Introduction

Stochastic wave models are derived as approximations from general ocean surface wave theory. The simplest of these approximations leads to a Gaussian process model for the free surface elevation, which has been used in ocean engineering with great success since the nineteen-fifties; e.g. [15,11]. Being reasonably realistic, its success stems from the fact that all its properties can be derived from the energy spectrum, which in turn can be estimated from recorded surface elevation data. It also fits well into linear ship dynamics models, and synthetic records can easily be generated for non-linear dynamics studies. Moreover, statistical distributions of important wave characteristics, such as wave steepness, crest amplitude and wave period, can be derived and computed in detail in the Gaussian model; see e.g. [1,4,13].

The Gaussian model is stochastically symmetric, both in the horizontal and in the vertical direction: time can be reversed and the elevation process turned upside down without any change in the statistical distributions. High crests surrounded by shallow troughs are just as likely as deep troughs surrounded by low crests. Real water waves usually have characteristic statistical asymmetry properties: crest-trough asymmetry, meaning that crests more often are peaked and troughs wide and shallow, than the opposite, and front-back asymmetry, which refers to the fact that wave fronts usually are steeper than the rear side of the waves. The Gaussian model does not reproduce these asymmetries, and it is therefore not ideal when one wants to describe statistical quantities like wave steepness or height distributions.

Another class of physically based wave models are the Lagrangian models, whose origin date back to the work of Lagrange [8]. These models consider not only the vertical movement but also the horizontal movement of the individual fluid particles and, in the stochastic formulation, these movements can be modeled by Gaussian processes. Stochastic Lagrange models have received little attention in the literature, but recently, studies by Gjosund, Socquet-Juglard et al. and Fouques et al. [7,14,6], have shown that a stochastic Lagrange model can produce crest-trough asymmetry as well as front-back asymmetry, the latter for higher order Lagrange models. Thus the stochastic Lagrange model can replicate waves with the desired asymmetry, at the same time as they retain enough of the Gaussian structure so that the extensive theory of Gaussian processes can be used in theoretical and numerical studies.
There are very few studies of the stochastic properties of Lagrange waves. This paper is based on recent results by Lindgren and Aberg [10,2] who derived the correlation structure of the Lagrange components, and used it to give the exact distributions of the wave slopes at level crossings; see also [3]. In this paper we present further studies of the stochastic properties of the Lagrange model. In particular we derive the occupation density of the Lagrangian space wave, i.e. the distribution of the height of the water surface, sampled in space at fixed time. To keep the presentation clear we present the results only for 2D-waves, i.e. waves with a time coordinate $t$ and a one-dimensional space coordinate $x$ measured along a straight line. For the full model, see the cited works.

2. The stochastic Lagrange model

2.1. The vertical Gaussian process

In the stochastic 2D Gaussian model the sea surface elevation $W(t, x)$ above the still water level at location $x$ at time $t$ is a Gaussian process with mean zero. To keep notation simple we consider unidirectional waves, and start with elementary waves $A \cos(\kappa x - \omega t + \phi)$ moving from left to right, with frequency $\omega > 0$ and wave number $\kappa > 0$. Denote the one-sided wave frequency spectrum by $S_+(\omega), \omega > 0$. The relation between the frequency $\omega$ and the wave-number $\kappa$ is given by the depth dependent dispersion relation

$$\omega^2 = g |\kappa| \tanh |\kappa|h,$$

with $h$ denoting water depth and $g$ the gravitational constant. The corresponding covariance function is

$$r(t, u) = \text{Cov}(W(s, x), W(s + t, x + u))$$

$$= \int_0^\infty \cos(\kappa u - \omega t) S_+(\omega) \, d\omega$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{i(\kappa x - \omega t)} S_+(\omega) \, d\omega.$$  \hfill (2)

In the symmetric form (2) we have used the symmetric spectrum $S_+(\omega) = S_-(-\omega)$ and defined the wave number in (1) to have the same sign as $\omega$.

The spectral representation of the process is

$$W(t, x) = \int_{-\infty}^\infty e^{i(\kappa x - \omega t)} \, d\xi(\omega),$$  \hfill (3)

where the complex valued Gaussian process has orthogonal increments such that

$$E \left( d\xi(\omega) \cdot d\xi(\omega') \right) = \begin{cases} 0, & \text{if } \omega \neq \omega', \\ 1/2 S_+(\omega), & \text{if } \omega = \omega'. \end{cases}$$

In this unidirectional formulation all waves travel from left to right. Waves in the opposite direction are obtained by letting $\kappa$ and $\omega$ in (3) solve the dispersion relation (1) and have different signs; the spectrum can be denoted by $S_-(-\omega)$. The general 2D process is the sum of the two independent processes. The total spectrum in this Gaussian wave model, $S(\omega) = S_+(\omega)$ for $\omega > 0$ and $= S_-(\omega)$ for $\omega < 0$, need not be symmetric.

2.2. The Lagrangian wave process

The Gaussian stochastic wave model is obtained as a superposition of independent harmonic waves $A \cos(\kappa x - \omega t + \phi)$ that describe the vertical movement of the water surface. In the Lagrangian wave model each particle on the surface is subject to a horizontal displacement, correlated with the vertical movement, $AH(\kappa) \cos(\kappa x - \omega t + \phi + \pi/2)$. The amplitude modulation $H(\kappa)$ is depth dependent. In the first order stochastic Lagrangian wave model the horizontal particle displacement is obtained as superpositions of the actions for each frequency.

Let $X(t, u)$ be the location at time $t$ of a water particle with original location $u$, sometimes referred to as its reference coordinate. More precisely, the stochastic Lagrangian wave model defines the horizontal displacement as the result of a linear filtering of the vertical movements.

$$W(t, u) = \int_{-\infty}^\infty e^{i(\kappa u - \omega t)} \, d\xi(\omega),$$

$$X(t, u) = u + \int_{-\infty}^\infty \frac{\cosh \kappa h}{\sinh \kappa h} e^{i(\kappa u - \omega t)} \, d\xi(\omega).$$  \hfill (4)

The spectral density $S(\omega)$ for the vertical process $W(t, u)$ is called the orbital spectrum.

The stochastic Lagrangian wave model, or simply the Lagrangian model, is the bivariate Gaussian process $(W(t, u), X(t, u))$: a water particle with reference coordinate $u$ is, at time $t$, located at $(X(t, u), u)$, and the height of the water surface above the still water level at location $X(t, u)$ is given by $W(t, u)$.

3. Lagrangian space waves

The stochastic Lagrange wave model consists of the vertical process $W(t, u)$ and the horizontal location process $X(t, u)$. When time is fixed to $t_0$ the particle with reference coordinate $u$ is located at position $X(t_0, u)$ and at height $W(t_0, u)$. A space wave is the parametric curve

$$u \rightarrow (X(t_0, u), W(t_0, u)).$$

Its distribution is independent of time, and in the sequel we usually assume $t_0 = 0$.

Fig. 1 shows simulated examples of space waves with Jonswap orbital spectrum; for details, see Section 4. The water depth varies from $h = 4$ m to $h = \infty$ m. As seen the individual space waves exhibit the typical crest-trough asymmetry, exaggerated here by the choice of unrealistically shallow water depths.

The figure also illustrates a peculiar behavior of the space waves, namely that they can fold and form loops. This occurs when $\partial X(t_0, u)/\partial u < 0$. Then more than one water particle occupies the same location, but at different heights. This unphysical behavior occurs with non-negligible intensity only at unrealistically low water depths. It can be disregarded in practice for moderate water depths, but the ambiguity implies that extra care is needed when one defines the statistical distribution of the level height.
expected number of level crossings by any non-stationary Gaussian process, and from its generalization to marked crossings; see [5] and [9]. We formulate the results in a theorem.

Define \( X_u(t, u) = \frac{\partial X(t, u)}{\partial u} \) and \( X_{uu}(t, u) = \frac{\partial^2 X(t, u)}{\partial u^2} \) and introduce the following notations for important variances and covariances, see [2], (for completeness we give formulas valid also for asymmetric spectra, with waves traveling in both positive and negative direction):

\[
\begin{align*}
r_{uu}^{wx} &= \text{Cov}(W(0, 0), X_u(0, 0)) = 0, \\
r_{0u}^{wx} &= \text{Cov}(W(0, 0), X_u(0, 0)) \\
&= \int_{-\infty}^{\infty} \left( \frac{\cosh |h|}{\sinh |h|} \right) S(\omega) \, d\omega, \\
r_{a0}^{xx} &= \text{Cov}(X_0(0), X_u(0, 0)) = 0, \\
r_{a0}^{xx} &= \text{Cov}(X_0(0), X_u(0, 0)) \\
&= \int_{-\infty}^{\infty} |h|^2 \left( \frac{\cosh |h|}{\sinh |h|} \right)^2 S(\omega) \, d\omega.
\end{align*}
\]

Further, write \( \phi(x) \) and \( \Phi(x) \) for the probability density and cumulative distribution function for the standardized normal distribution, and let \( \chi_A \) be the indicator function for the event \( A \), equal to 1 if \( A \) occurs.

**Theorem 1.** (a) For the stochastic Lagrangian space wave model, the expected number of particles occupying location \( x \), and with height less than \( w_0 \) are, respectively,

\[
E(N_x) = v_0 \\
= \int_{-\infty}^{\infty} f_{X_0}(0) \cdot E(\chi_{X_u(0, u) = 0}) \, du \tag{13}
\]

\[
= E(\chi_{X_0(0, u)}) \tag{14}
\]

\[
= 2 \Phi \left( \frac{1}{\sqrt{r_{0u}^{xx}}} \right) - 1 + 2 \sqrt{r_{0u}^{xx} \Phi \left( \frac{1}{\sqrt{r_{0u}^{xx}}} \right)}, \tag{15}
\]

\[
E(N_x(w_0)) = v_0(w_0) \\
= \int_{-\infty}^{w_0} f_{X_0}(0) \cdot \chi_{X_u(0, u) \leq w_0} \, du \tag{16}
\]

\[
= E(\chi_{X_0(0, u) \leq w_0}) \tag{17}
\]

\[
= \int_{-\infty}^{w_0} \frac{p_{W(0, u)}(w) \cdot E(\chi_{X_u(0, u) = 0})}{p_{WW}(0, u)} \, dw, \tag{18}
\]

where \( p_{W(0, u)}(w) \) is the normal probability density function of \( W(0, u) \).

(b) The conditional distribution of \( X_u(0, u) \) given \( W(0, u) = w \) is Gaussian with mean and variance given by

\[
m(w) = 1 + w \frac{r_{0u}^{xx}}{r_{uu}^{wx}}, \tag{19}
\]
The occupation density for the Lagrangian space wave is
\[
\mu(w) = \frac{1}{\sqrt{2\pi r_{ww}}} \exp \left( -\frac{w^2}{2r_{ww}} \right) \mu(w),
\]
\[
\times \left\{ m(w) \left( 2\Phi \left( \frac{m(w)}{\sigma_{xx}|w|} \right) - 1 \right) + \frac{2\sigma_{xx}|w|}{\sqrt{2\pi}} e^{-\frac{m(w)^2}{2\sigma_{xx}|w|}} \right\}. \tag{21}
\]

**Proof.** The steps between (13) and (14) and between (16) and (17) follow from the covariance relations (6)–(11): the covariance matrix between \( X(0, u), W(0, u), \) and \( X_u(0, u) \) is
\[
\begin{pmatrix}
 r_{xx} & 0 & 0 \\
 0 & r_{ww} & r_{wx} \\
 0 & r_{wx} & r_{uu}
\end{pmatrix};
\]
thus, \( X(0, u) \) is independent of \( (W(0, u), X_u(0, u)) \).

The explicit formulas (15) and (21) follow from direct integration. Finally, the conditional mean and variance in part (b) follow from the general properties of the bivariate Gaussian distribution. □

The ratio \( \mu_0(w)/\nu_0 \) has the formal form of a probability density function and integrates to one, but due to the presence of multiple points it cannot be directly interpreted as such.

**4. Examples and practical conclusions**

The Lagrangian space wave exhibits a crest-trough asymmetry, depending on water depth. We illustrate this effect and also draw some conclusions about the value of the significant wave height when it comes to evaluation of extreme wave height.

### 4.1. Orbital spectrum

For the orbital spectrum we use the standard Pierson-Moskowitz, PM, spectrum, modified to the peak enhanced Jonswap spectrum to illustrate the Lagrangian space distribution.

The PM spectrum is defined by two wave characteristic parameters, the *significant wave height*, \( H_s \), for the Gaussian wave model defined as four times the observed standard deviation of the surface elevation, and the *peak frequency* \( \omega_p \).

The Jonswap orbital spectrum is a modification of the PM spectrum with an extra *peak enhancement factor*, \( \gamma > 1 \). It has the spectral density, see e.g. \[12\],
\[
S_1(\omega) = \frac{5H^2_s}{\omega_p^2(\omega/\omega_p)^\gamma} e^{-\frac{1}{2}(\omega/\omega_p)^2/\sigma_\omega^2},
\]
with \( \gamma > 1 \) and \( \sigma_\omega \) taken as 0.07 for \( \omega < \omega_p \) and equal to 0.09 for \( \omega > \omega_p \). Instead of the peak frequency one often uses the peak period, \( T_p = 2\pi/\omega_p \).

### 4.2. Distributions depending on water depth

The first example illustrates the depth dependence of the Lagrangian space wave distribution. In order to illustrate the folding effect we have chosen unrealistically shallow waters with depth 2 m and 8 m. Fig. 2 shows the occupation density \( \mu(w) \) for the two water depths, in log scale and in linear scale. For comparison the figure also shows the density functions for the Gaussian space wave. As seen in the figure, the tendency of the Lagrangian wave to fold results in an increased level intensity at high levels.

Fig. 3 shows a histogram of observed occupations in a simulated Lagrangian space wave with standard Jonswap
The severity of a sea state is usually characterized by the significant wave height. In Gaussian wave theory crest-to-trough wave heights rarely exceed twice the significant wave height, and waves above that height are therefore called freak or rogue waves, not belonging to the normal wave process. (The height limit for being regarded as a rough wave is of course chosen somewhat arbitrarily.) In the Lagrangian space model, the crest and trough heights are defined by the Gaussian vertical component \( W(t_0, u) \) and the height of the local extremes are the same in the Gaussian and in the Lagrangian model. The standard deviation however, will be lower after the \( X \)-deformation, leading to more waves being classified as extreme. In simulations we have found the following approximate ratios between the standard deviations in the Gaussian model and in the Lagrangian model, depending on water depth; Table 1. The reduction is small but clear, and leads to the conclusion that some waves classified as rogue waves are in fact compatible with the model.

Due to the folding effect, the occupation density is not well suited for calculation of the tail of the crest height distribution. However, folding occurs any time \( X_u(t_0, u) < 0 \), downcrossing, and therefore the expected number of foldings per unit length can be calculated exactly by Rice’s formula, as

\[
n_{\text{fold}} = \frac{1}{2\pi} \sqrt{\frac{V(X_{uu}(0, 0))}{V(X_u(0, 0))}} e^{-1/(2V(X_u(0, 0)))}.
\]

For the Jonsvap spectrum with depth 32 m, the folding rate is about \( 10^{-3} \) per meter, i.e. virtually zero.

### References