Realistic and practical stochastic models for ocean waves – from Gaussian to non-Gaussian models

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Stochastic models for ocean waves

There is a need for stochastic wave models that can produce crest-trough and/or front-back asymmetric waves:

- The Gaussian models – introduced around 1950, inspired by Steve Rice, Random noise theory for electrical circuits
- 2nd and higher order Stokes waves
- New: Stochastic Lagrange waves
The Draupner wave

New Years Day 1995 an extreme waves way registered at the Norwegian North Sea platform Draupner. Extreme crest-trough asymmetry.

Height from crest to bottom was

\[ H = 26.6m = 8.6\sigma \]

The wave is well described by the hydrodynamic equations - but the relation between the energy spectral distribution is not clear – so no probabilistic statement about the probability of occurrence under different sea conditions can be made – at least not at present.
Wave steepness is important for shipping safety

Ship capsizing and ship damage depends not only of wave height but also on wave shape: “Wall of water”
Why using a stochastic Lagrange model?

- The (linear) Gaussian wave model allows exact computation of wave characteristic distributions
- **BUT**
- produces crest-trench and front-back stochastically symmetric waves.
- The modified stochastic Lagrange model can produce (2006) crest-trench and **NEW!!** front-back asymmetric waves
- **AND**
- still allows for exact computation of wave characteristic distributions!
The Gaussian wave model

In the Gaussian wave model, the surface elevation process $W(t, s)$ is a Gaussian random process with time parameter $t$ and space coordinate $s = (u, v)$:

- Elevation $W(t, s)$ has a Gaussian distribution
- Slope in space, $\partial W/\partial x$, $\partial W/\partial y$ and rising speed $\partial W/\partial t$ are also Gaussian
- Software exists for calculation of statistical wave characteristic distributions (crest, trough heights, period, steepness etc)
- Weak physics connection – models only vertical movement
- Statistically symmetric crest/trough and front/back properties
In the Gaussian model the height $W(t,s)$ is an integral of harmonics with random phases and amplitudes:

$$\kappa = (\kappa_x, \kappa_y) = \kappa(\cos \theta, \sin \theta)$$

$$\omega = \omega(\kappa) = \sqrt{g \kappa \tanh \kappa h}$$

$$W(t,s) = \int e^{i(\kappa s - \omega t)} \, d\zeta(\kappa, \omega)$$

$$= \int_{\omega=0}^{\infty} \int_{\theta=-\pi}^{\pi} e^{i(\kappa s - \omega t)} \, d\zeta(\omega, \theta)$$

with $S(\omega, \theta) = \text{the "orbital spectrum" and } \zeta(\omega, \theta) \text{ is a Gaussian complex "spectral process".}$
Physically based models for individual water particles. Example: 2D-waves. Take a particle with initial horizontal location $u$ called the reference coordinate. At time $t$ is has moved to location $x(t, u)$ and is at height $w(t, u)$.

With $\omega =$ frequency, $\kappa =$ wave number, $\omega = \sqrt{g\kappa}$,

$$w(t, u) = a \cos(\kappa u - \omega t)$$
$$x(t, u) = u - a \sin(\kappa u - \omega t)$$

Particles move in circles – become ellipses for finite depth
The stochastic Lagrange model –

Describes vertical and horizontal movements of individual surface water particles. Use the Gaussian model

\[ W(t, s) = \int e^{i(\kappa s - \omega t)} d\zeta(\kappa, \omega) \]

for the vertical movement of a particle with (initial) reference coordinate \( s = (u, v) \) and write

\[ \Sigma(t, s) = \begin{pmatrix} X(t, s) \\ Y(t, s) \end{pmatrix} = \text{horizontal location at time } t \]

Make \( \Sigma(t, s) \) stochastic, and correlated with \( W(t, s) \).
– with horizontal Gaussian movements

Use the same Gaussian spectral process as in \( W(t, s) \) to generate the horizontal variation


\[
\Sigma(t, s) = \begin{pmatrix} X(t, s) \\ Y(t, s) \end{pmatrix} = s + \int H(\theta, \kappa) e^{i(\kappa s - \omega t)} d\zeta(\kappa, \omega)
\]

where the filter function \( H \) depends on water depth \( h \):

\[
H(\theta, \kappa) = i \frac{\cosh \kappa h}{\sinh \kappa h} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}
\]
The stochastic Lagrange model

The 3D stochastic first order Lagrange wave model is the triple of Gaussian processes

\[(W(t, s), \Sigma(t, s)) = (W(t, s), X(t, s), Y(t, s))\]

All covariance functions and auto-spectral and cross-spectral density functions for \(\Sigma(t, s)\) follow from the orbital spectrum \(S(\omega, \theta)\) and the filter equation.
The free stochastic 2D Lagrange model

- In the 2D stochastic linear Lagrange model the \( W(t, s) \) process is the height of the sea surface at randomly changing location

\[
X_M(t, s) = s + \int H(\kappa) e^{i(\kappa s - \omega t)} \, d\zeta(\kappa, \omega)
\]

which is a linear filtration of the \( W \)-process (\( M = \) Miche-wave)

- \( H(\omega) = i \frac{\cosh \kappa h}{\sinh \kappa h} \)

- gives peaked, crest-trough asymmetric waves both in space and in time
Lagrange space waves, time fixed

A Lagrange space wave at time $t_0$ is the parametric surface

$$L(x) : s \mapsto (X_M(t_0,s), W(t_0,s))$$
A Gaussian field $W(t, s)$ is observed along a level curve of another Gaussian field $X(t, s)$. $W(t, X^{-1}(t, 25))$
The Lagrangian space wave distribution

The height of the 2D space wave at time $t_0$ at location $x$ is

$$W(t_0, X(t_0, x)^{-1})$$

**PROBLEM:** there may be many solutions to

$$X(t_0, u) = x$$

**Occupation density** $\mu(w)$:

$$E \left( \# \left\{ u \in \mathbb{R} \text{ such that } \begin{array}{l} X(t_0, u) = x \\ W(t_0, u) \leq w \end{array} \right\} \right) = \int_{-\infty}^{w} \mu(y) \, dy.$$
Occupation density

Aberg and Lindgren (2008) show that

\[
\mu(w) = \frac{1}{\sqrt{2\pi r_{ww}}} \exp \left( -\frac{w^2}{2r_{ww}} \right) \\
\times \left\{ m(w) \left( 2\Phi \left( \frac{m(w)}{\sigma_{x_u \cdot w}} \right) - 1 \right) + \frac{2\sigma_{x_u \cdot w}}{\sqrt{2\pi}} e^{-\frac{m(w)^2}{2\sigma_{x_u \cdot w}^2}} \right\}
\]

where \( m(w) \) and \( \sigma_{x_u \cdot w}^2 \) are the conditional expectation and variance of \( \partial X(t, u)/\partial u \) given \( W(t, u) = w \), and \( r_{ww} = V(W(t, u)) \).
Occupation density – Jonswap spectrum

Dash-dotted curve: Gaussian model.
Dotted curve: non-Gaussian factor.
Consequence for rogue waves?

- Gaussian model ⇒ Lagrangian model have the same $W(t, u)$ but locations are different
- Lagrangian waves have smaller standard deviation $\sigma$ than the Gaussian process $W(t_0, u)$
- More waves are characterized as rogue waves even if they can be explained within the Gaussian/Lagrangian framework!

<table>
<thead>
<tr>
<th>water depth</th>
<th>4m</th>
<th>8m</th>
<th>16m</th>
<th>32m</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{D(\text{Lagrangian})}{D(\text{Gaussian})}$</td>
<td>0.894</td>
<td>0.964</td>
<td>0.987</td>
<td>0.993</td>
<td>0.994</td>
</tr>
</tbody>
</table>

- Difference not big enough to influence “rogue waves”
- Average sea level (in space or time) is negative for the Lagrange model, it may however have consequences for abnormal crest heights?
Front-back asymmetry in space and time

The simplest Lagrange models gives waves that are statistically front-back symmetric – time and space coordinates can be reversed without changing the statistical properties. The front-back asymmetry depends on the correlation between the $W$- and $X$-processes and their time and space derivatives:

\[
\begin{align*}
  r_{ww} &= V(W(t, u)) \\
  r_{wx} &= \text{Cov}(W(t, u), X(t, u)) \\
  r_{0w} &= \text{Cov}(W(t, u), \frac{\partial W(t, u)}{\partial t}) \\
  r_{0x} &= \text{Cov}(W(t, u), \frac{\partial X(t, u)}{\partial u})
\end{align*}
\]

etc
The important covariance matrix

$$
\Sigma = \begin{pmatrix}
    r_{tt}^{ww} & r_{tu}^{ww} & r_{tt}^{wx} & r_{tu}^{wx} & 0 & r_{t0}^{wx} \\
    r_{ut}^{ww} & r_{uu}^{ww} & r_{ut}^{wx} & r_{uu}^{wx} & 0 & r_{u0}^{wx} \\
    r_{tt}^{xw} & r_{tu}^{xw} & r_{tt}^{xx} & r_{tu}^{xx} & r_{t0}^{xw} & 0 \\
    r_{ut}^{xw} & r_{uu}^{xw} & r_{ut}^{xx} & r_{uu}^{xx} & r_{u0}^{xw} & 0 \\
    0 & 0 & r_{0t}^{wx} & r_{0u}^{wx} & r_{00}^{ww} & r_{00}^{wx} \\
    r_{0t}^{xw} & r_{0u}^{xw} & 0 & 0 & r_{00}^{xw} & r_{00}^{xx}
\end{pmatrix}
$$

Front-back asymmetry depends on $r_{uu}^{wx}$ and some other covariances – zero or non-zero!
Front-back asymmetric Lagrange waves

To get realistic front-back asymmetry one needs a model with external input from wind, for example, by a parameter $\alpha$:

$$\frac{\partial^2 X(t, s)}{\partial t^2} = \frac{\partial^2 X_M(t, s)}{\partial t^2} + \alpha W(t, s)$$

The filter function from vertical $W(t, u)$ to horizontal $X(t, u)$ is then

$$H(\omega) = i \frac{\cosh \kappa h}{\sinh \kappa h} + \frac{\alpha}{(-i\omega)^2} = \rho(\omega) e^{i\theta(\omega)},$$

Implies an extra phase shift ($\theta = \pi/2$ in the free model)

$$X(t, u) = u + \int e^{i(\kappa u - \omega t + \theta(\omega))} \rho(\omega), d\zeta(\omega, \kappa)$$
Modified Lagrange space and times waves – $\alpha \neq 0$
Wave characteristics at level crossings

Some important variables that can be observed:

AS  Asyncronous slope in space
AT  Asyncronous slope in time
SS  Slope in space at level crossings in space
TT  Slope in time at level crossing in time
ST  Slope in space at level crossing in time = space slope of wave profile when wave hits the deck of a platform
VT  Profile velocity at level crossing in time = the velocity of the hitting wave
SS: Slope in space wave

The slope of the space wave surface at an arbitrary crossing of a level \( \nu \) at time \( t_0 \) is

\[
L = \frac{\partial W(t_0, u) / \partial u}{\partial X(t_0, u) / \partial u}
\]

conditioned on a crossing of level \( \nu \) by \( W(t_0, u) \). Slope at an upcrossing \( u_k \) in a Gaussian process \( W \) has a Rayleigh distribution, regress \( X_u(t_0, u_k) \) on \( W(t_0, u_k) = \nu \), \( W_u(t_0, u_k) = R\sqrt{r_{uu}^{ww}} \):

\[
W_u(t_0, u_k) \overset{\mathcal{L}}{=} R\sqrt{r_{uu}^{ww}}, \quad R \text{ standard Rayleigh}
\]

\[
X_u(t_0, u_k) \overset{\mathcal{L}}{=} 1 + \nu \frac{r_{0u}^{wx}}{r_{ww}} + R \frac{r_{uu}^{wx}}{\sqrt{r_{uu}^{ww}}}
\]

\[
+ U \sqrt{r_{uu}^{xx} - \frac{(r_{0u}^{wx})^2}{r_{ww}} - \frac{(r_{uu}^{wx})^2}{r_{uu}^{ww}}},
\]

This gives the following distribution of the Lagrange slope:
Slope of asymmetric Lagrange space waves

Theorem: The slope of a Lagrange space process observed at (up)crossings of a level \( v \) has the same distribution as

\[
\frac{aR}{1 + bv + cR + dU}
\]

where \( a, b, c, d \neq 0 \) depend on the correlations between the \( W \)- and the \( X \)-process, and \( R, U \) are independent standard Rayleigh and normal variables.

\( c = 0 \) if \( \alpha = 0 \); gives front-back symmetry. Lindgren and Aberg: 2008
Space waves with front-back asymmetry

Averaged up- (solid) and down- (dashed) crossings waves
CDFs of slopes as function of level (space)
$$TT, \ ST, \ VT: \ \text{Slopes of Lagrange time waves}$$

With $X(t, 0)^{-1} = \{u; X(t, u) = 0\}$, the time wave at location 0 is

$$L(t, 0) = \mathcal{W}(t, X(t, 0)^{-1})$$

The time wave has a crossing of level $\nu$ at location 0 at time $t$ if, simultaneously, $X(t, u) = 0$ and $\mathcal{W}(t, u) = \nu$.

- The time slope at the crossing is

$$\mathcal{W}_t(t, u) - \mathcal{W}_u(t, u) \cdot \frac{X_t(t, u)}{X_u(t, u)}$$

- The space slope at the crossing is

$$\mathcal{W}_u(t, u)/X_u(t, u))$$
Crossing event in time wave

\[ W(t,u) = 2, \text{ upcrossing, } X(t,u) = 0 \]
\[ W(t,u) > 2 \]
\[ W(t,u) = 2, \text{ downcrossings, } X(t,u) = 0 \]
\[ W(t,u) < 2 \]
Case TT: With

\[ N(y) = \#\{(t, u) \in [0, 1] \times \mathbb{R}; W(t, u) = \nu, X(t, u) = 0, \]

\[ \frac{W_t(t, u) - W_u(t, u)}{X_u(t, u)} \leq y \}, \]

\[ E(N(y)) = \int_{-\infty}^{\infty} g_y(u) f_{W(0, u), X(0, u)}(\nu, 0) \, du, \]

where \( f_{W(0, u), X(0, u)}(\nu, 0) \) is a Gaussian density function, and

\[ g_y(u) = E\left( |W_t(0, u)X_u(0, u) - W_u(0, u)X_t(0, u)| \right) \]

\[ \times \mathbf{1}(\frac{W_t(0, u) - W_u(0, u)}{X_u(0, u)} \frac{X_t(0, u)}{X_u(0, u)} \leq y) \]

\[ \left| W(0, u) = \nu, X(0, u) = 0 \right) \]
CDF for slopes in Lagrange time waves

The ratio $E(N(y))/E(N(\infty))$ is the CDF of the slope or other interesting wave quantity at level crossings for the time wave.

No explicit formula for $g_y(u)$ exists - easy to make Monte Carlo simulation with a six-dimensional Gaussian distribution.
TT: CDF for time slopes at time crossings
ST: CDF for space slopes at time crossings

- $h = \infty$, $\alpha = 0$
- $h = 64$, $\alpha = 0$
- $h = 8$, $\alpha = 0$
- $h = \infty$, $\alpha = 0.4$
- $h = 64$, $\alpha = 0.4$
- $h = 8$, $\alpha = 0.4$
- $h = \infty$, $\alpha = 0.8$
- $h = 64$, $\alpha = 0.8$
- $h = 8$, $\alpha = 0.8$
References


▶ http://www.maths.lth.se/matstat/wafo/