Stochastic 3D Lagrange waves, asymmetry properties and consequences for marine safety

Georg Lindgren and Finn Lindgren

Lund University and Norwegian University of Science and Technology

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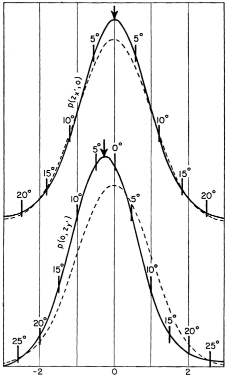
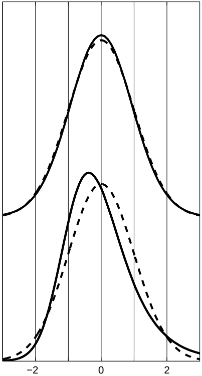
The Cox and Munk experiment 1954 on wave asymmetry –

compared to Lagrange asymmetry:





Why using a stochastic Lagrange model?

- The (linear) Gaussian wave model allows exact computation of wave characteristic distributions (the WAFO toolbox)
- produces crest-trough and front-back stochastically symmetric waves
- The modified stochastic Lagrange model can produce
  (2006) crest-trough asymmetric waves (2D)
  (2009) front-back asymmetric waves (2D)
  (2011) front-back asymmetric waves (3D)
- still allows for exact computation of wave characteristic distributions!
- Software exists for calculation of statistical wave characteristic distributions (crest, trough heights, period, steepness etc)

Gaussian generator and the orbital spectrum

In the Gaussian model the vertical height $W(t, \mathbf{s})$ of a particle at the free surface at time t and location $\mathbf{s} = (u, v)$ is an integral of harmonics with random phases and amplitudes:

$$\boldsymbol{\kappa} = (\kappa_x, \kappa_y) = \kappa(\cos\theta, \sin\theta)$$

$$\omega = \omega(\boldsymbol{\kappa}) = \sqrt{g\kappa \tanh \kappa h}$$

$$W(t, \mathbf{s}) = \int_{\omega=0}^{\infty} \int_{\theta=-\pi}^{\pi} e^{i(\boldsymbol{\kappa}\mathbf{s} - \omega t)}\, d\zeta(\omega, \theta)$$

with $S(\omega, \theta) =$ the "orbital spectrum" and $\zeta(\omega, \theta)$ is a Gaussian complex "spectral process".

The stochastic Lagrange model –

Describes vertical and horizontal movements of individual surface water particles. Use

$$W(t, \mathbf{s}) = \int e^{i(\boldsymbol{\kappa}\mathbf{s} - \omega t)} \, d\zeta(\boldsymbol{\kappa}, \omega)$$

for the vertical movement of a particle with (intitial) reference coordinate $\mathbf{s} = (u, v)$ and write

$$\boldsymbol{\Sigma}(t, \mathbf{s}) = \begin{pmatrix} X(t, \mathbf{s}) \\ Y(t, \mathbf{s}) \end{pmatrix} = \text{horizontal location at time } t$$

Make $\boldsymbol{\Sigma}(t, \mathbf{s})$ stochastic.

– with horizontal Gaussian movements

Use the same Gaussian spectral process as in $W(t, \mathbf{s})$ to generate the horisontal variation

Fouques,Krogstad,Myrhaug,Socquet-Juglard (2004), Gjøsund (2000,2003) Aberg, Lindgren, Lindgren (2006,2007,2008,2009,2011), Guerrin (2009)

$$\mathbf{\Sigma}(t, \mathbf{s}) = \begin{pmatrix} X(t, \mathbf{s}) \\ Y(t, \mathbf{s}) \end{pmatrix} = \mathbf{s} + \int \boldsymbol{H}(\theta, \kappa)\, e^{i(\boldsymbol{\kappa}\mathbf{s} - \omega t)}\, d\zeta(\boldsymbol{\kappa}, \omega)$$

where the filter function $\boldsymbol{H}$ depends on water depth h:

$$\boldsymbol{H}(\theta, \kappa) = i\, \frac{\cosh \kappa h}{\sinh \kappa h} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

The stochastic Lagrange model

The 3D stochastic first order Lagrange wave model is the triple of Gaussian processes

$$(W(t,\mathbf{s}),\mathbf{\Sigma}(t,\mathbf{s})) = (W(t,\mathbf{s}),X(t,\mathbf{s}),Y(t,\mathbf{s}))$$

All covariance functions and auto-spectral and cross-spectral density functions for $\mathbf{\Sigma}(t,\mathbf{s})$ follow from the orbital spectrum $S(\omega,\theta)$ and the filter equation.

Front-back asymmetric Lagrange waves

To get realistic front-back asymmetry one needs a model with external input from wind, for example, by a parameter α:

$$\frac{\partial^2 X(t,s)}{\partial t^2} = \frac{\partial^2 X_M(t,s)}{\partial t^2} + \alpha W(t,s)$$

The filter function from vertical $W(t,u)$ to horizontal $X(t,u)$ is then

$$H(\omega) = i\,\frac{\cosh \kappa h}{\sinh \kappa h} + \frac{\alpha}{(-i\omega)^2} = \rho(\omega)\, e^{i\theta(\omega)},$$

Implies an extra phase shift ($\theta = \pi/2$ in the free model)

$$X(t,u) = u + \int e^{i(\kappa u - \omega t + \theta(\omega))} \rho(\omega), d\zeta(\omega,\kappa)$$

For 3D waves the filter function needs to take directional spreading into account

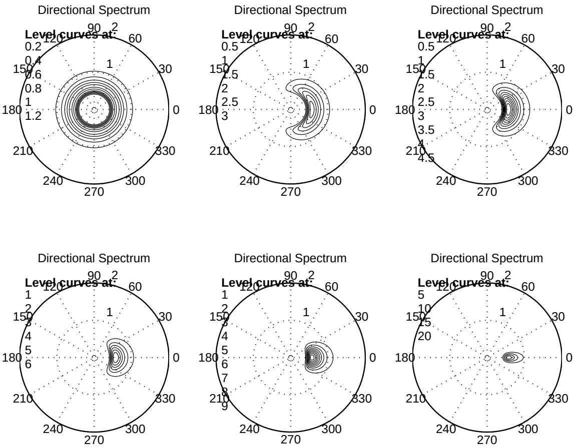
An example:

$$\boldsymbol{H}(\theta,\kappa) = \frac{\alpha}{(i\omega)^2}\cdot\begin{pmatrix}\cos^2(\theta)\,|\cos(\theta)|\\ \cos^2(\theta)\sin(\theta)\,\mathrm{sign}(\cos\theta)\end{pmatrix} + i\frac{\cosh\kappa h}{\sinh\kappa h}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}.$$

to take care of wind blowing in positive x-direction.

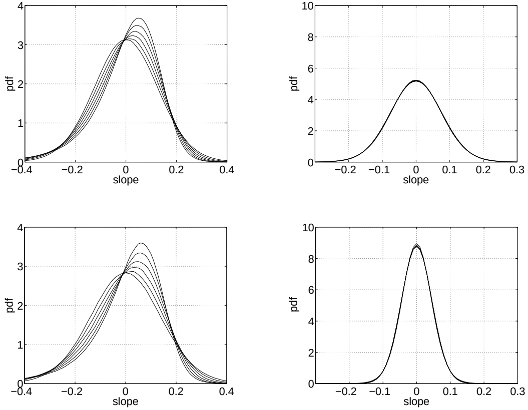
Front-back asymmetry depends on directional spreading in the orbital spectrum

Example specta - Pierson-Moskowitz with different spreading



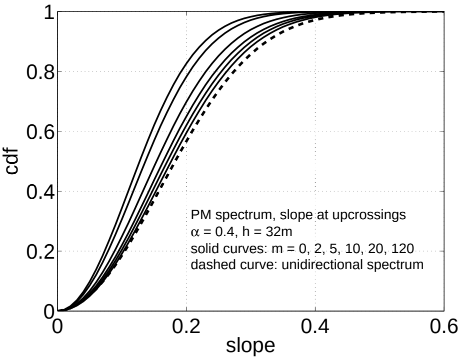
Slope PDF across wind and along wind, asynchrone

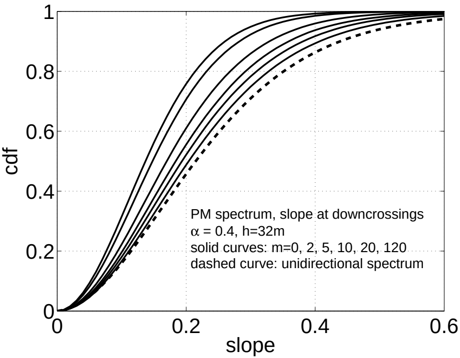
Top: Large spreading – Bottom: Little spreading
Left: Along wind – Right: Across wind; cf. Cox and Munk



Slope CDF for different directional spreading

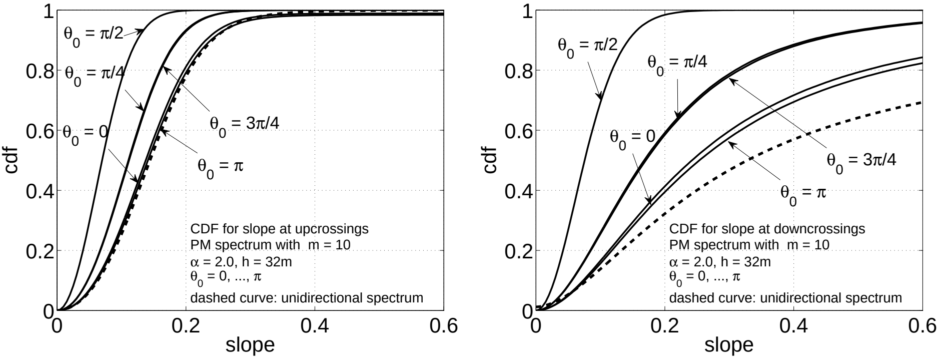
Left: slope at upcrossings – Right: Slope CDF at downcrossings
Moderate linkage parameter: $\alpha = 0.4$





Slope CDF for different heading; Strong linkage: $\alpha = 2$

Left: slope at upcrossings – Right: Slope CDF at downcrossings



References

- Aberg, S. (2007). Wave intensities and slopes in Lagrangian seas. *Adv. Appl. Prob.*, **39**
- Aberg, S. and Lindgren, G. (2008). Height distribution of stochastic Lagrange ocean waves. *Prob. Engng Mech*, **23**
- Lindgren, G. (2006). Slepian models for the stochastic shape of individual Lagrange sea waves. *Adv. Appl. Prob.* **38**
- Lindgren, G. (2009). Exact asymmetric slope distributions in stochastic Gauss-Lagrange ocean waves. *Appl. Ocean Res*, **31**
- Lindgren, G. (2010). Slope distribution in front-back asymmetric stochastic Lagrange time waves. *Adv. Appl. Prob.*, **42**
- Lindgren, G. and Aberg, S. (2008). First order stochastic Lagrange models for front-back asymmetric ocean waves. *J. OMAE*, **131**
- WAFO-group, (2011). *WAFO – a Matlab Toolbox for Analysis of Random Waves and Loads; Tutorial for WAFO Version 2.5*, URL http://www.maths.lth.se/matstat/wafo