Further development of the stochastic Lagrange model for asymmetric ocean waves – some new results since the first Seamocs meeting in Toulouse 2006

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Stochastic models for ocean waves

Stochastic wave models that can produce crest-trough and/or front-back symmetric or asymmetric waves:

- The Gaussian models – introduced around 1950, inspired by Steve Rice, Random noise theory for electrical circuits
- 2nd and higher order Stokes waves
- Stochastic Lagrange waves
- New: Laplace models – Aberg and Podgorski (2008), Wegener and Podgorski (2009), Wengang Mao (2009)
- New: Markov random field models related to stochastic partial differential equations – David Bolin, Finn Lindgren, Johan Lindström (2009)

Why using a stochastic Lagrange model?

- The (linear) Gaussian wave model allows exact computation of wave characteristic distributions
- BUT
- produces crest-trough and front-back stochastically symmetric waves.
- The modified stochastic Lagrange model can produce
  (2006) crest-trough and
  NEW!! front-back asymmetric waves
- AND
- still allows for exact computation of wave characteristic distributions!

The Gaussian wave model

In the Gaussian wave model, the surface elevation process $W(t, \boldsymbol{s})$ is a Gaussian random process with time parameter t and space coordinate $\boldsymbol{s} = (u, v)$:

- Elevation $W(t, \boldsymbol{s})$ has a Gaussian distribution
- Slope in space, $\partial W / \partial x$, $\partial W / \partial y$ and rising speed $\partial W / \partial t$ are also Gaussian
- Software exists for calculation of statistical wave characteristic distributions (crest, trough heights, period, steepness etc)
- Weak physics connection – models only vertical movement
- Symmetric crest/trough and front/back

Gaussian generator and the wave spectrum

In the Gaussian model the height $W(t,\mathbf{s})$ is an integral of harmonics with random phases and amplitudes:

$$\boldsymbol{\kappa} = (\kappa_x, \kappa_y) = \kappa(\cos\theta, \sin\theta)$$

$$\omega = \omega(\boldsymbol{\kappa}) = \sqrt{g\kappa \tanh \kappa h}$$

$$W(t,\mathbf{s}) = \int e^{i(\boldsymbol{\kappa}\mathbf{s}-\omega t)}\, d\zeta(\boldsymbol{\kappa},\omega)$$

$$= \int_{\omega=0}^{\infty} \int_{\theta=-\pi}^{\pi} e^{i(\boldsymbol{\kappa}\mathbf{s}-\omega t)}\, d\zeta(\omega,\theta)$$

with $S(\omega,\theta)$ = the "orbital spectrum" and $\zeta(\omega,\theta)$ is a Gaussian complex "spectral process".

Gerstner - Miche - Lagrange waves

Physically based models for individual water particles.
Example: 2D-waves. Take a particle with initial horizontal location u called the reference coordinate. At time t is has moved to location $x(t,u)$ and is at height $w(t,u)$.

With ω = frequency, κ = wave numer, $\omega = \sqrt{g\kappa}$,

$$w(t,u) = a\cos(\kappa u - \omega t)$$
$$x(t,u) = u - a\sin(\kappa u - \omega t)$$

Particles move in circles – become ellipses for finite depth

The stochastic Lagrange model –

Describes vertical and horizontal movements of individual surface water particles. Use

$$W(t, \mathbf{s}) = \int e^{i(\boldsymbol{\kappa}\mathbf{s} - \omega t)}\, d\zeta(\boldsymbol{\kappa}, \omega)$$

for the vertical movement of a particle with (intitial) reference coordinate $\mathbf{s} = (u, v)$ and write

$$\boldsymbol{\Sigma}(t, \mathbf{s}) = \begin{pmatrix} X(t, \mathbf{s}) \\ Y(t, \mathbf{s}) \end{pmatrix} = \text{horizontal location at time } t$$

Make $\boldsymbol{\Sigma}(t, \mathbf{s})$ stochastic.

– with horizontal Gaussian movements

Use the same Gaussian spectral process as in $W(t, \mathbf{s})$ to generate the horisontal variation

Fouques, Krogstad, Myrhaug, Socquet-Juglard (2004)
Gjøsund, (2000) and (2003)
Aberg, Lindgren (2006, 2007, 2008, 2009) Guerrin (2009)

$$\boldsymbol{\Sigma}(t, \mathbf{s}) = \begin{pmatrix} X(t, \mathbf{s}) \\ Y(t, \mathbf{s}) \end{pmatrix} = \mathbf{s} + \int \boldsymbol{H}(\theta, \kappa)\, e^{i(\boldsymbol{\kappa}\mathbf{s} - \omega t)}\, d\zeta(\boldsymbol{\kappa}, \omega)$$

where the filter function $\boldsymbol{H}$ depends on water depth h:

$$\boldsymbol{H}(\theta, \kappa) = i \, \frac{\cosh \kappa h}{\sinh \kappa h} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

The stochastic Lagrange model

The 3D stochastic first order Lagrange wave model is the triple of Gaussian processes

$$(W(t,\mathbf{s}),\mathbf{\Sigma}(t,\mathbf{s})) = (W(t,\mathbf{s}),X(t,\mathbf{s}),Y(t,\mathbf{s}))$$

All covariance functions and auto-spectral and cross-spectral density functions for $\mathbf{\Sigma}(t,\mathbf{s})$ follow from the orbital spectrum $S(\omega,\theta)$ and the filter equation.

The free stochastic 2D Lagrange model

- In the 2D stochastic linear Lagrange model the $W(t,s)$ process is the height of the sea surface at randomly changing location

$$X_M(t,s) = s + \int H(\kappa) e^{i(\kappa s - \omega t)}\, d\zeta(\kappa,\omega)$$

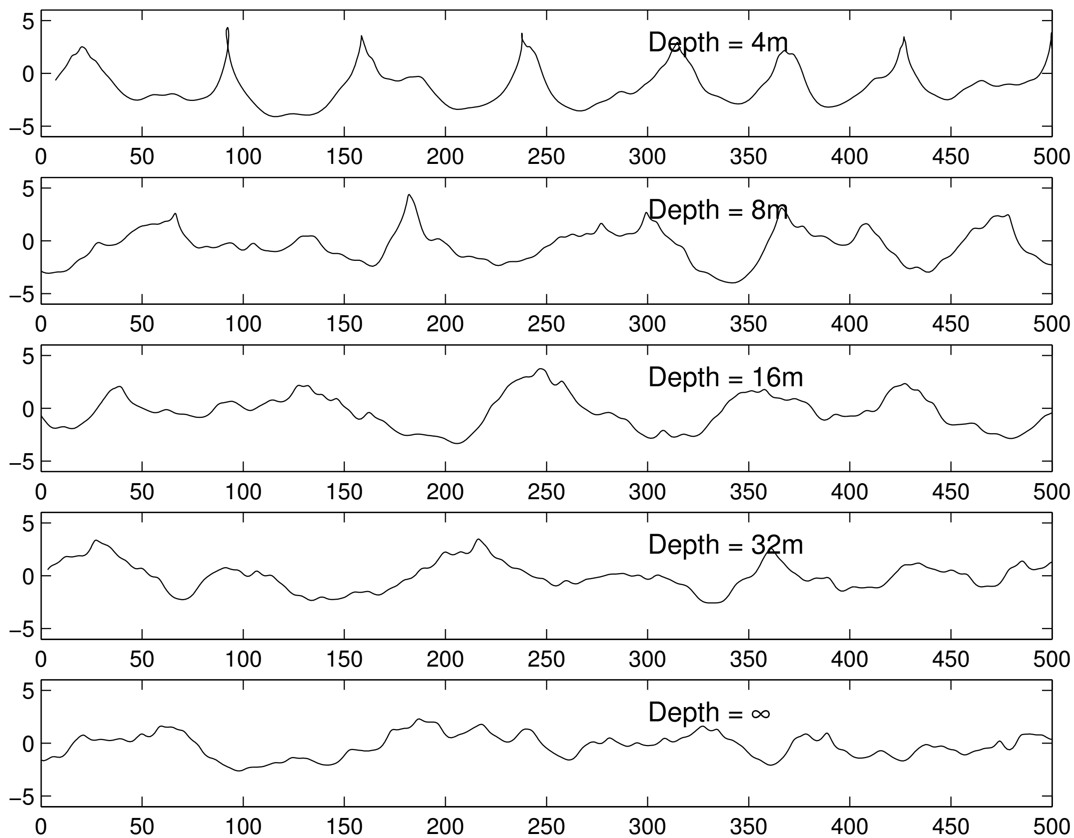
which is a linear filtration of the W-process (M = Miche-wave)

- $H(\omega) = i\,\frac{\cosh \kappa h}{\sinh \kappa h}$
- gives peaked, crest-trough asymmetric waves both in space and in time

Lagrange space waves, time fixed

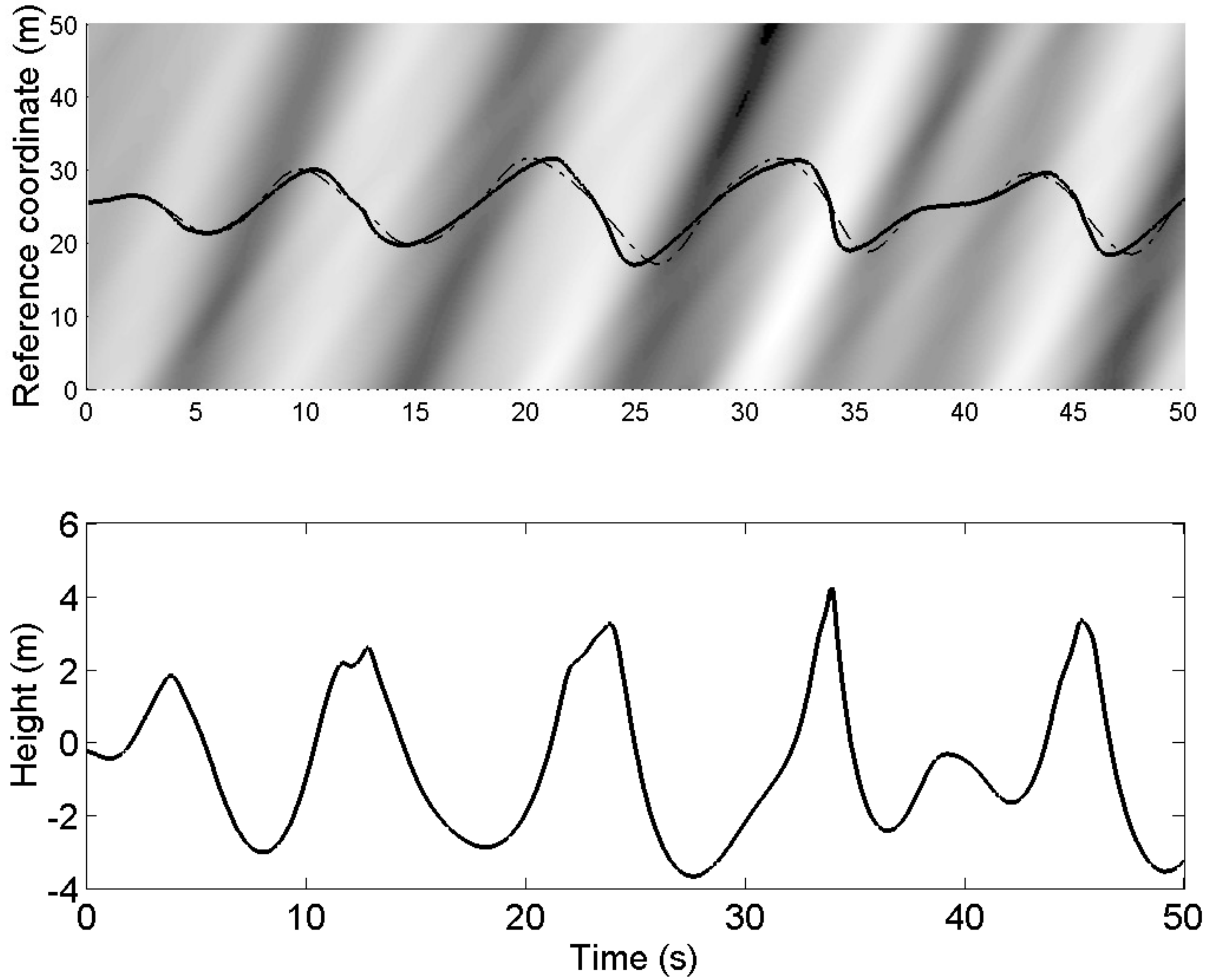
A Lagrange space wave at time t_0 is the parametric surface

$$L(x) : s \Rightarrow (X_M(t_0, s), W(t_0, s))$$



How time waves are generated

A Gaussian field $W(t,s)$ is observed along a level curve of another Gaussian field $X(t,s)$: $W(t, X^{-1}(t, 25))$



The Lagrangian space wave distribution

The height of the 2D space wave at time t_0 at location x is

$$W(t_0, X(t_0, x)^{-1})$$

PROBLEM: there may be many solutions to

$$X(t_0, u) = x$$

Occupation density $\mu(w)$:

$$E\left(\#\left\{u \in \mathbb{R} \text{ such that } \begin{array}{l} X(t_0, u) = x \\ W(t_0, u) \leq w \end{array}\right\}\right) = \int_{-\infty}^{w} \mu(y)\, dy.$$

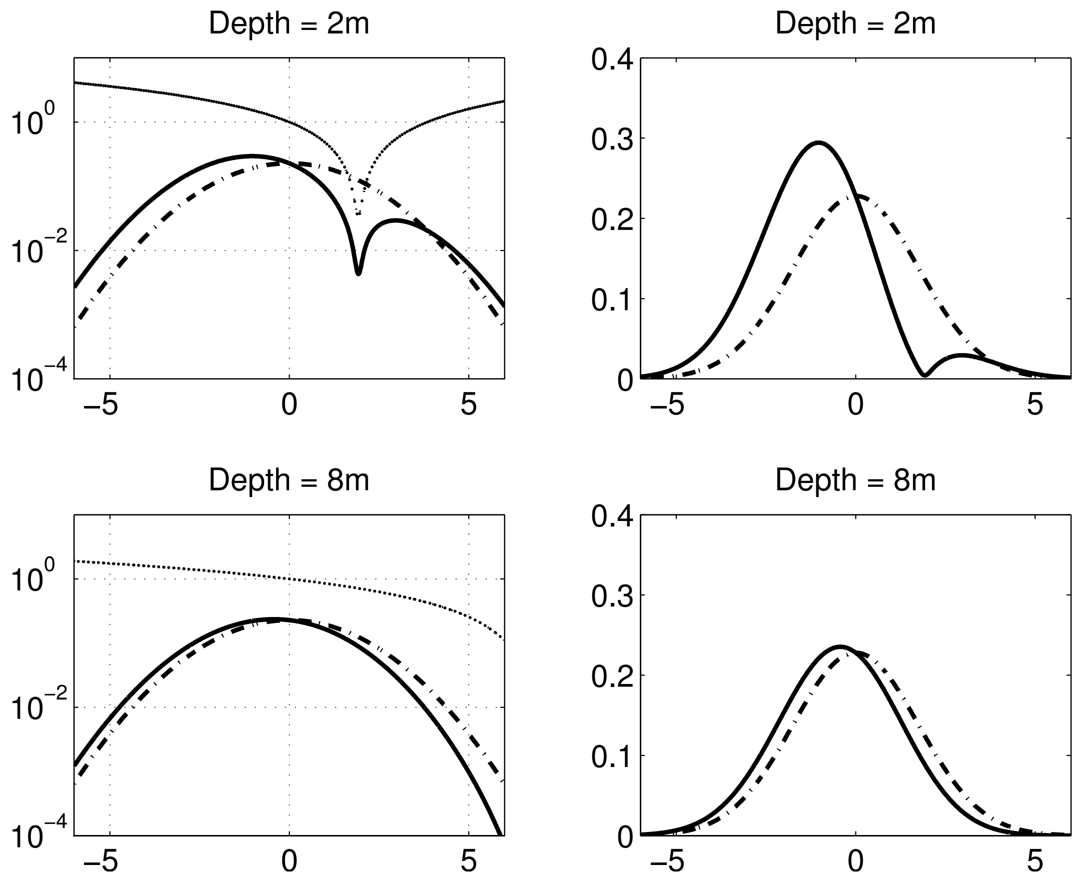
Occupation density

Aberg and Lindgren (2008) show that

$$\mu(w) = \frac{1}{\sqrt{2\pi r^{ww}}} \exp\left(-\frac{w^2}{2r^{ww}}\right)$$
$$\times \left\{ m(w) \left(2\Phi\left(\frac{m(w)}{\sigma_{x_u \cdot w}}\right) - 1\right) + \frac{2\sigma_{x_u \cdot w}}{\sqrt{2\pi}} e^{-\frac{m(w)^2}{2\sigma^2_{x_u \cdot w}}} \right\}$$

where $m(w)$ and $\sigma^2_{x_u \cdot w}$ are the conditional expectation and variance of $\partial X(t,u)/\partial u$ given $W(t,u) = w$, and $r^{ww} = V(W(t,u))$.

Occupation density – Jonswap spectrum



Dash-dotted curve: Gaussian model.

Dotted curve: non-Gaussian factor.

Consequence for rogue waves?

- Gaussian model $\Rightarrow$ Lagrangian model have the same $W(t,u)$ but locations are different
- Lagrangian waves have smaller standard deviation σ than the Gaussian process $W(t_0,u)$
- More waves are characterized as rogue waves even if they can be explained within the Gaussian/Lagrangian framework!
-

water depth	4m	8m	16m	32m	∞
$\frac{D(\text{Lagrangian})}{D(\text{Gaussian})}$	0.894	0.964	0.987	0.993	0.994

- Difference not big enough to influence “rogue waves”
- Average sea level (in space or time) is negative for the Lagrange model, it may however have consequences for abormal crest heights?

Symmetry or asymmetry

The front-back asymmetry depends on the correlation between the W- and X-processes and their time and space derivatives:

$$r^{ww} = V(W(t,u))$$
$$r^{wx} = Cov(W(t,u), X(t,u))$$
$$r^{ww}_{0t} = Cov(W(t,u), \frac{\partial W(t,u)}{\partial t})$$
$$r^{wx}_{0u} = Cov(W(t,u), \frac{\partial X(t,u)}{\partial u})$$

etc

The important covariance matrix

$$\mathbf{\Sigma} = \left(\begin{array}{cccc|cc} r^{ww}_{tt} & r^{ww}_{tu} & r^{wx}_{tt} & r^{wx}_{tu} & 0 & r^{wx}_{t0} \\ r^{ww}_{ut} & r^{ww}_{uu} & r^{wx}_{ut} & r^{wx}_{uu} & 0 & r^{wx}_{u0} \\ r^{xw}_{tt} & r^{xw}_{tu} & r^{xx}_{tt} & r^{xx}_{tu} & r^{xw}_{t0} & 0 \\ r^{xw}_{ut} & r^{xw}_{uu} & r^{xx}_{ut} & r^{xx}_{uu} & r^{xw}_{u0} & 0 \\ \hline 0 & 0 & r^{wx}_{0t} & r^{wx}_{0u} & r^{ww}_{00} & r^{wx}_{00} \\ r^{xw}_{0t} & r^{xw}_{0u} & 0 & 0 & r^{xw}_{00} & r^{xx}_{00} \end{array}\right)$$

Front-back asymmetry depends on r^{wx}_{uu} and some other covariances – zero or non-zero!

Front-back asymmetric Lagrange waves

To get realistic front-back asymmetry one needs a model with external input from wind, for example, by a parameter α:

$$\frac{\partial^2 X(t,s)}{\partial t^2} = \frac{\partial^2 X_M(t,s)}{\partial t^2} + \alpha W(t,s)$$

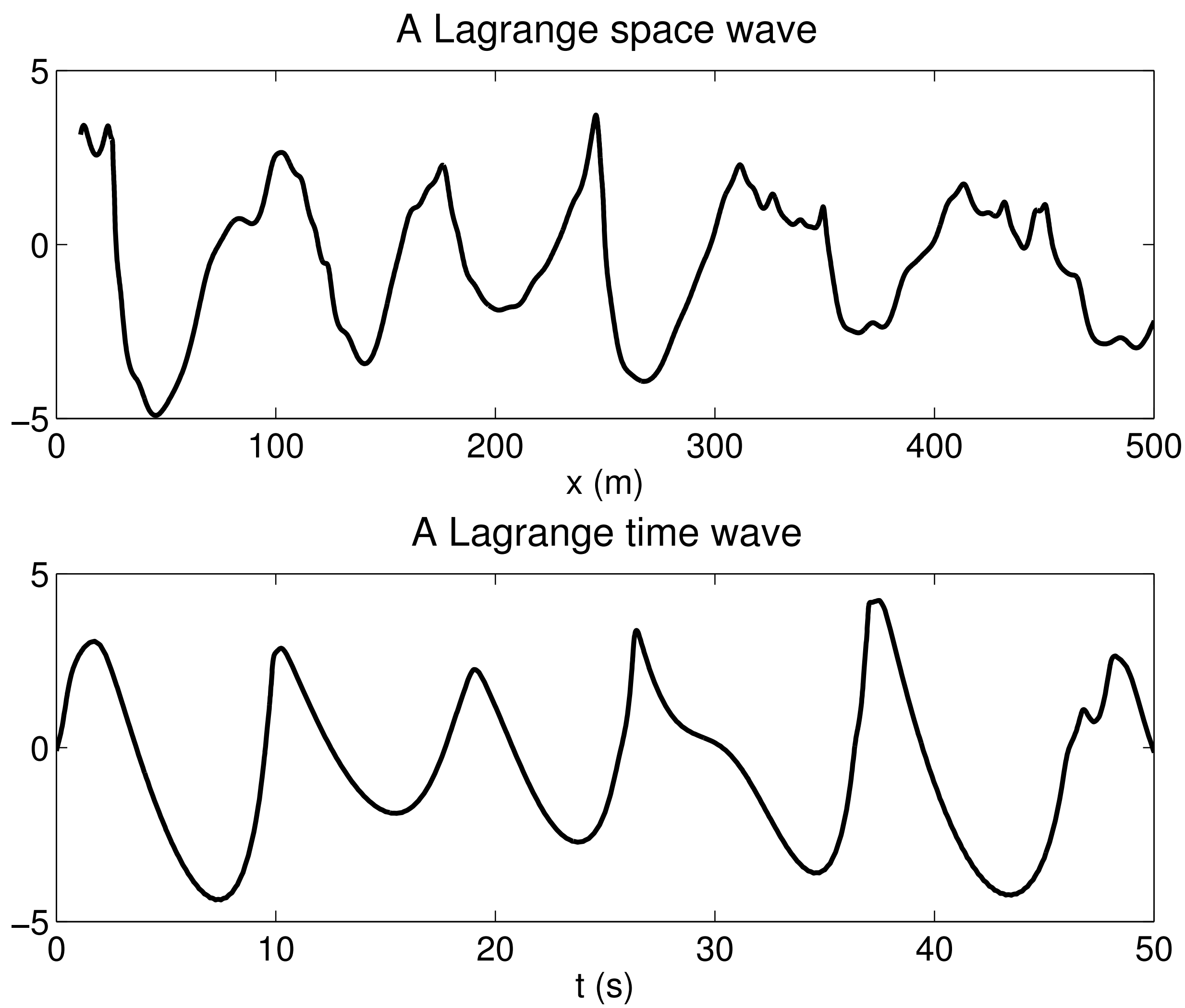
The filter function from vertical $W(t,u)$ to horizontal $X(t,u)$ is then

$$H(\omega) = i\,\frac{\cosh \kappa h}{\sinh \kappa h} + \frac{\alpha}{(-i\omega)^2} = \rho(\omega)\, e^{i\theta(\omega)},$$

Implies an extra phase shift ($\theta = \pi/2$ in the free model)

$$X(t,u) = u + \int e^{i(\kappa u - \omega t + \theta(\omega))} \rho(\omega), d\zeta(\omega,\kappa)$$

Modified Lagrange space and times waves – $\alpha \neq 0$



Wave characteristics at level crossings

Some important variables that can be observed:

- AS Asyncronous slope in space
- AT Asyncronous slope in time
- SS Slope in space at level crossings in space
- TT Slope in time at level crossing in time
- ST Slope in space at level crossing in time = space slope of wave profile when wave hits the deck of a platform
- VT Profile velocity at level crossing in time = the velocity of the hitting wave

SS: Slope in space wave

The slope of the space wave surface at an arbitrary crossing of a level v at time t_0 is

$$L = \frac{\partial W(t_0, u)/\partial u}{\partial X(t_0, u)/\partial u}$$

conditioned on a crossing of level v by $W(t_0, u)$. Slope at an upcrossing u_k in a Gaussian process W has a Rayleigh distribution, regress $X_u(t_0, u_k)$ on $W(t_0, u_k) = v$, $W_u(t_0, u_k) = R$:

$$\begin{aligned} W_u(t_0, u_k) &\overset{\mathcal{L}}{=} R\sqrt{r_{uu}^{ww}}, \\ X_u(t_0, u_k) &\overset{\mathcal{L}}{=} 1 + v\,\frac{r_{0u}^{wx}}{r^{ww}} + R\,\frac{r_{uu}^{wx}}{\sqrt{r_{uu}^{ww}}} \\ &\qquad + U\sqrt{r_{uu}^{xx} - \frac{(r_{0u}^{wx})^2}{r^{ww}} - \frac{(r_{uu}^{wx})^2}{r_{uu}^{ww}}}, \end{aligned}$$

This gives the following distribution of the Lagrange slope:

Slope of asymmetric Lagrange space waves

Theorem: The slope of a Lagrange space process observed at (up)crossings of a level v has the same distribution as

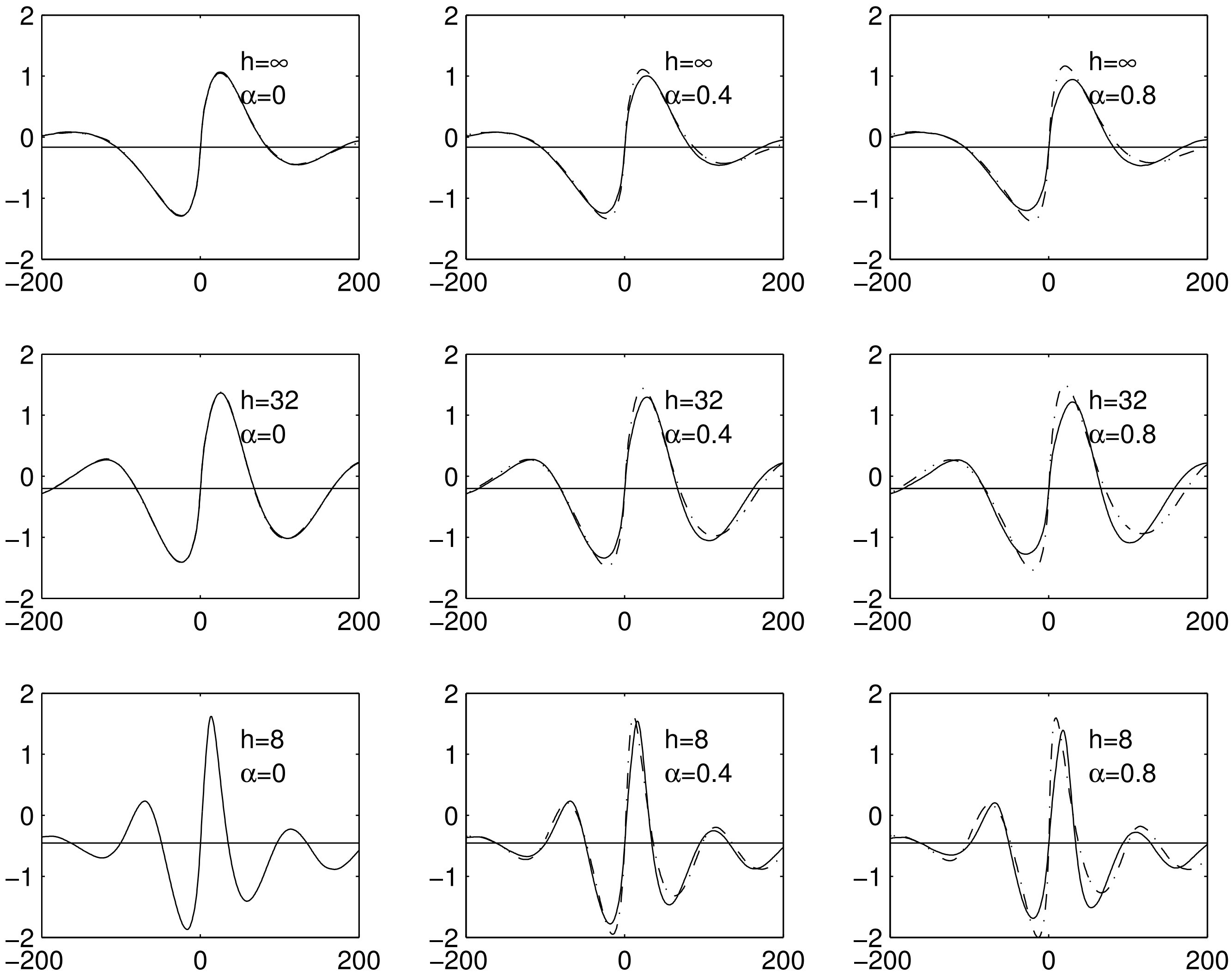
$$\frac{aR}{1 + bv + cR + dU}$$

where $a, b, c, d \neq 0$ depend on the correlations between the W- and the X-process, and R, U are independent standard Rayleigh and normal variables

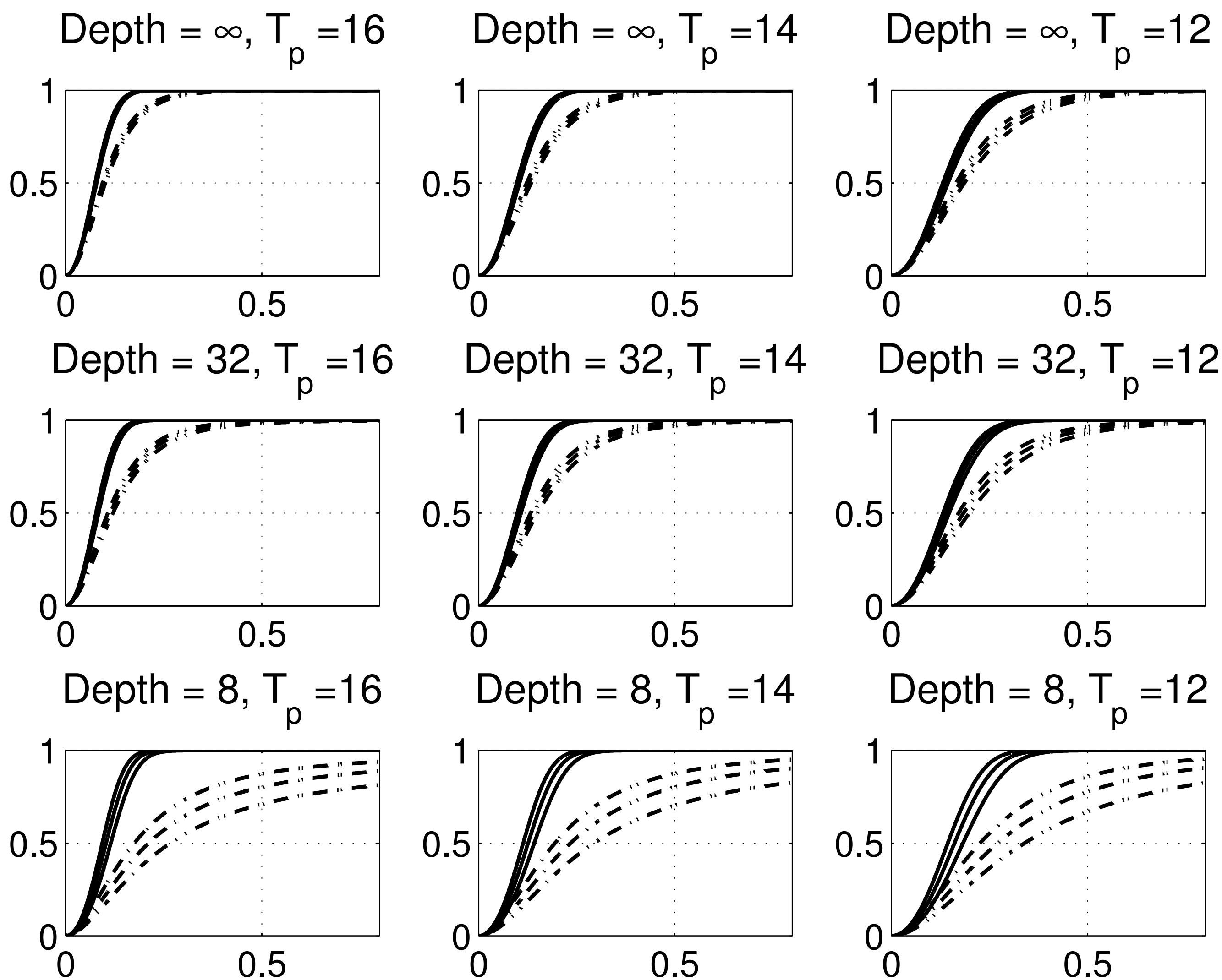
$c = 0$ if $\alpha = 0$; gives front-back symmetry. Lindgren and Aberg: 2008

Space waves with front-back asymmetry

Averaged up- (solid) and down- (dashed) crossings waves



CDFs of slopes as function of level (space)



TT, ST, VT: Slopes of Lagrange time waves

With $X(t,0)^{-1} = \{u; X(t,u) = 0\}$, the time wave at location 0 is

$$L(t,0) = W(t, X(t,0)^{-1})$$

The time wave has a crossing of level v at location 0 at time t if, simulaneously, $X(t,u) = 0$ and $W(t,u) = v$.

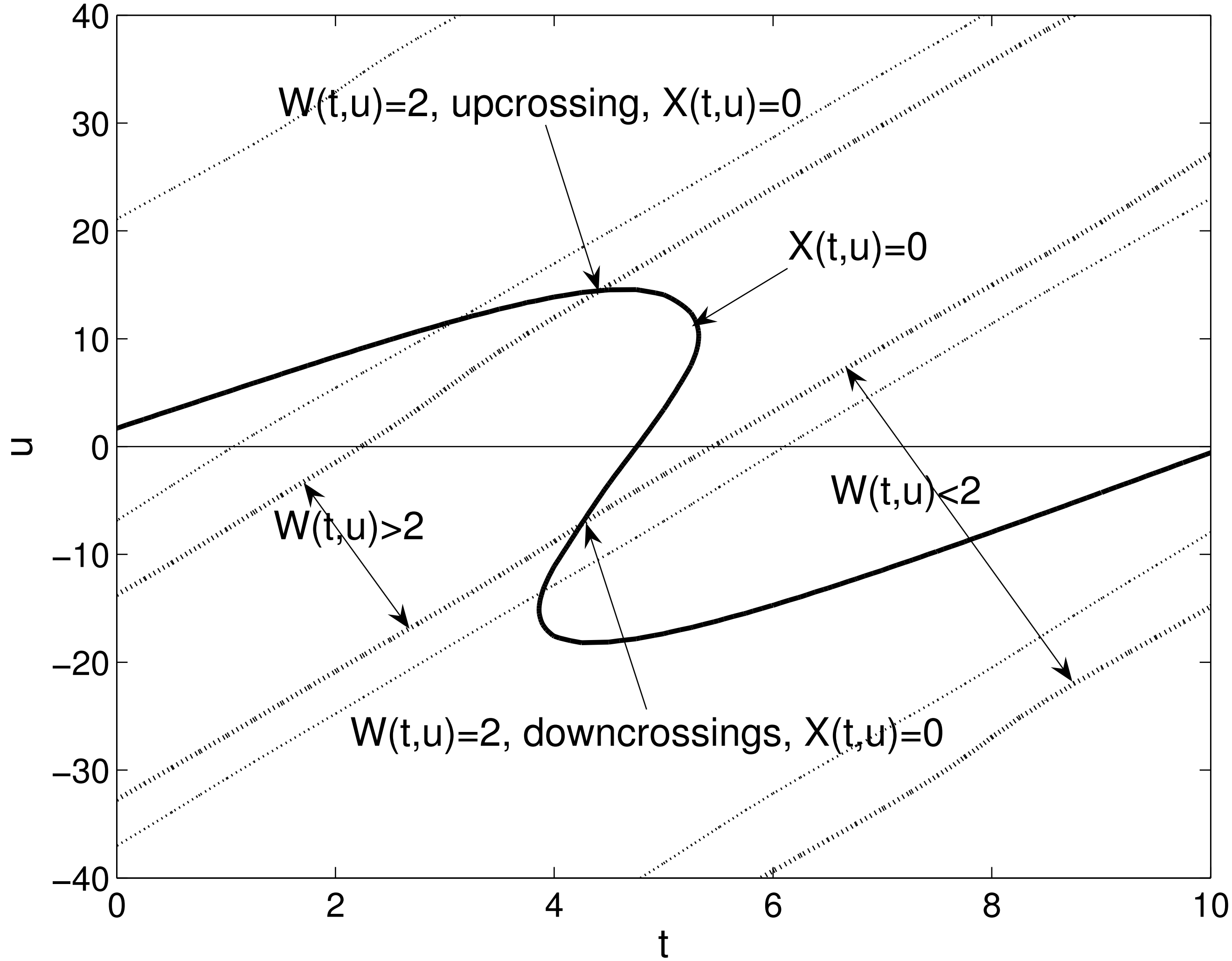
- The time slope at the crossing is

$$W_t(t,u) - W_u(t,u) \cdot \frac{X_t(t,u)}{X_u(t,u)}$$

- The space slope at the crossing is

$$W_u(t,u)/X_u(t,u))$$

Crossing event in time wave



Crossing intensity, Mercardier, Azaïs

Case TT: With

$$N(y) = \#\{(t,u) \in [0,1] \times \mathbb{R}; W(t,u) = v, X(t,u) = 0, \\ W_t(t,u) - W_u(t,u)\,\frac{X_t(t,u)}{X_u(t,u)} \leq y\},$$

$$E(N(y)) = \int_{-\infty}^{\infty} g_y(u)\, f_{W(0,u),X(0,u)}(v,0)\, du,$$

where $f_{W(0,u),X(0,u)}(v,0)$ is a Gaussian density function, and

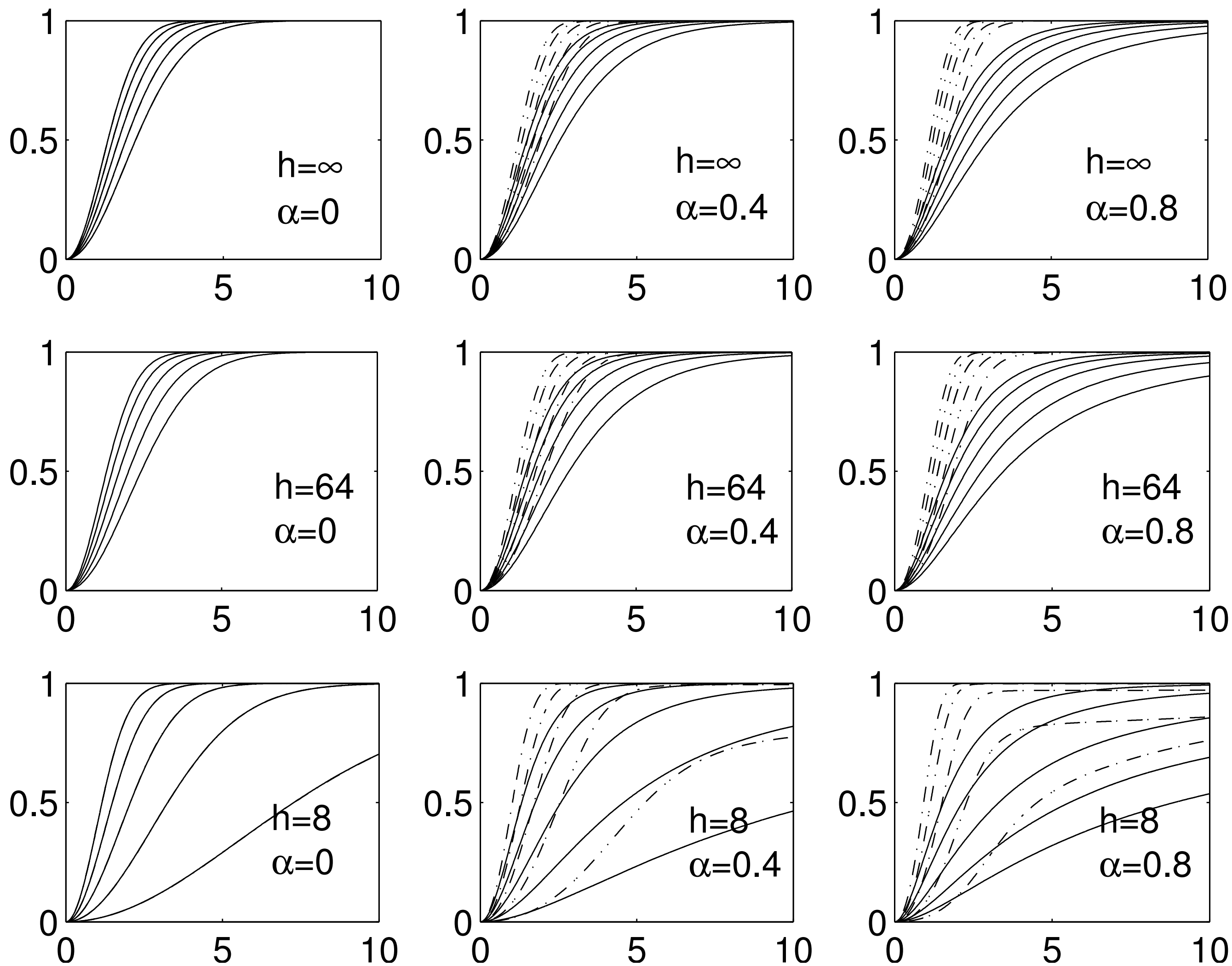
$$g_y(u) = E\Big(\Big|W_t(0,u)X_u(0,u) - W_u(0,u)X_t(0,u)\Big| \\ \times \mathbf{1}\Big(W_t(0,u) - W_u(0,u)\,\frac{X_t(0,u)}{X_u(0,u)} \leq y\Big) \\ \Big|\; W(0,u) = v, X(0,u) = 0\Big)$$

CDF for slopes in Lagrange time waves

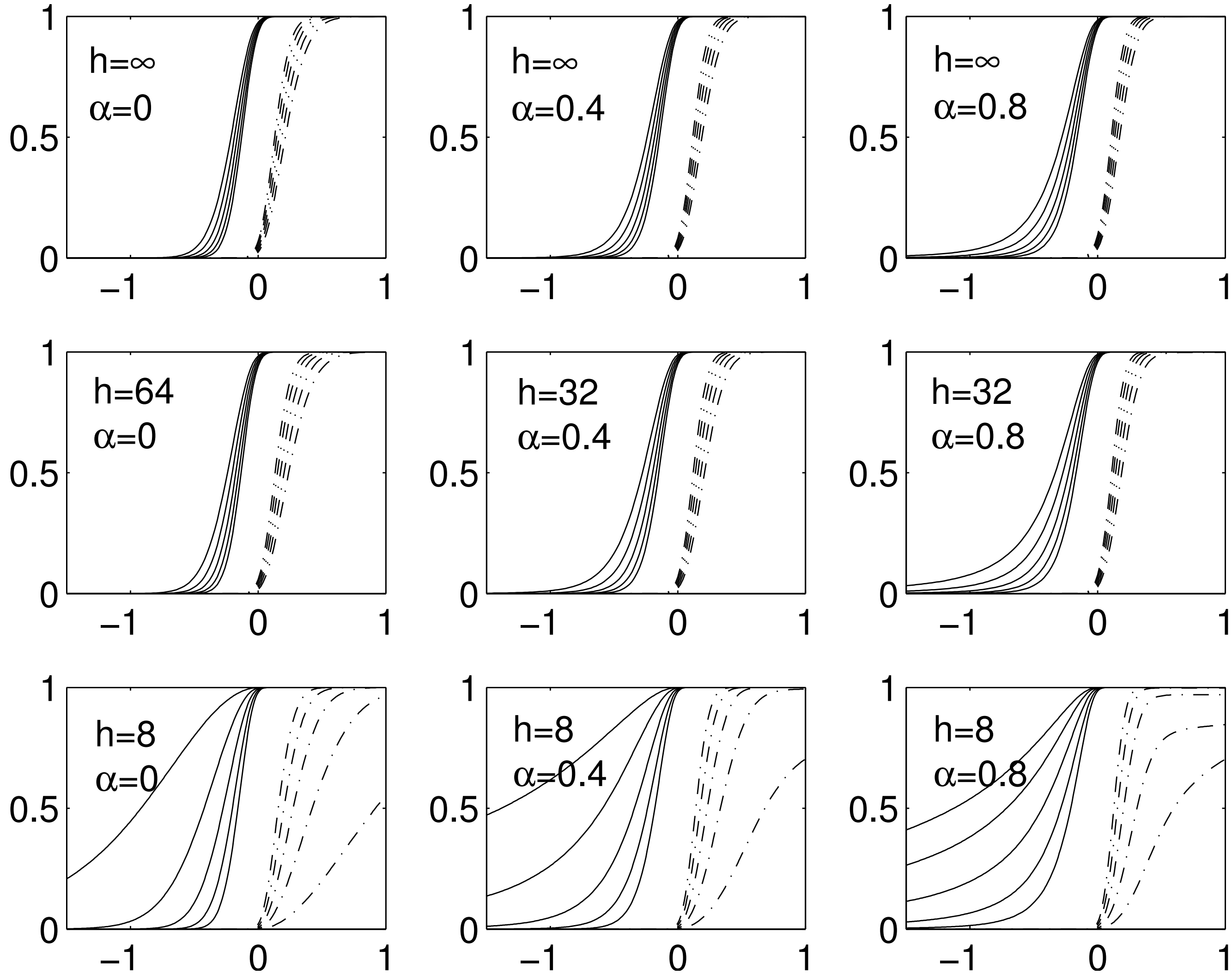
The ratio $E(N(y))/E(N(+\infty))$ is the CDF of the slope or other interesting wave quantity at level crossings for the time wave.

No explicit formula for $g_y(u)$ exists - easy to make Monte Carlo simulation with a six-dimensional Gaussian distribution

TT: CDF for time slopes at time crossings



ST: CDF for space slopes at time crossings



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