Lagrange models for asymmetric random waves

LUND UNIVERSITY

Georg Lindgren

Mathematical Statistics, Lund University

georg@maths.lth.se

Abstract

The Lagrange model for ocean waves is a hydrodynamically motivated extension of the Gaussian model – used in ocean engineering since the mid 50'ies. It consists of a Gaussian space and time dependent process for the vertical height of water particles, and an accompanying horizontal displacement process for the horizontal position of the particles. It is shown how by suitable choice of structure, one can obtain waves with realistic crest-trough and front-back asymmetry. It is also shown how one can calculate the exact statistical ditribution of many wave characteristics.

1. Lagrange models

A Lagrange wave, a stochastic version of a Miche wave, describes the vertical and horizontal movements of individual water particle as functions of time t and original location u. In the *stochastic Lagrange model* the vertical and horizontal displacements are correlated random processes. The vertical process, $W(t,u)$, describes elevation above the still water level and is taken as a Gaussian process with mean zero, and so is the horizontal location, $X(t,u)$.

> The height of the surface at location $X(t,u)$ is equal to $W(t,u)$.

The Lagrange model is defined by the covariance functions $r^{ww}(t,u)$, $r^{wx}(t,u)$, $r^{xx}(t,u)$.

THE *free Lagrange wave model* is the stochastic version of Miche waves. The horizontal displacement $X_M(t,u)$ is a linear filtration of $W(t,u)$ with response function

$$H_M(\omega) = i\,\frac{\cosh \kappa h}{\sinh \kappa h}, \qquad (1)$$

TO get front-back asymmetric waves one can replace (1) by a general response function, $H(\omega) = \rho(\omega)\, e^{i\,\theta(\omega)}$, leading to the cross-covariance function

$$r^{wx}(t,u) = \int_0^\infty \cos(\kappa u - \omega t + \theta(\omega))\,\rho(\omega)\,S(\omega)\,d\omega.$$

A physically motivated relation is obtained by letting the horizontal acceleration be, e.g.,

$$\frac{\partial^2 X(t,u)}{\partial t^2} = \frac{\partial^2 X_M(t,u)}{\partial t^2} - \alpha\, W(t,u),$$

with $\alpha > 0$. The response function will then be

$$H(\omega) = i\frac{\cosh \kappa h}{\sinh \kappa h} - \frac{\alpha}{(-i\omega)^2} = \rho(\omega)\, e^{i\,\theta(\omega)}.$$

WE will illustrate the theory on a model with Pierson-Moskowitz (PM) spectrum,

$$S(\omega) = \frac{5H_s^2}{\omega_p(\omega/\omega_p)^5}\, e^{-\frac{5}{4}(\omega/\omega_p)^{-4}}, \qquad 0 < \omega < \omega_c,$$

significant wave height $H_s = 4$m, peak frequency and period, ω_p, $T_p = 12\pi/\omega_p$, and cut off $\omega_c = 32/T_p$.

2. Wave charactersitics

THE *space wave* is the curve $u \mapsto (X(t_0,u), W(t_0,u))$, keeping time $t = t_0$ fixed; defined implicitly through the relation $L(t_0, X(t_0,u)) = W(t_0,u)$. The *time wave* is obtained as measurements of the free water level $L(t,x_0)$ at a fixed location in space with co-ordinate x_0, viz. as the curve $t \mapsto W(t, X^{-1}(t,x_0))$.
Slopes are defined, for time and space, as

$$L_t(t, X(t,u)) = W_t(t,u) - W_u(t,u)\,\frac{X_t(t,u)}{X_u(t,u)}$$

$$L_x(t_0,x) = \frac{W_u(t_0,u)}{X_u(t_0,u)}$$

(SS) Slope in space at level crossings in space:] The distribution of the space slope $L_x(t_0,x)$ observed at the up- or downcrossings of a fixed level v by the space wave $L(t_0,x)$

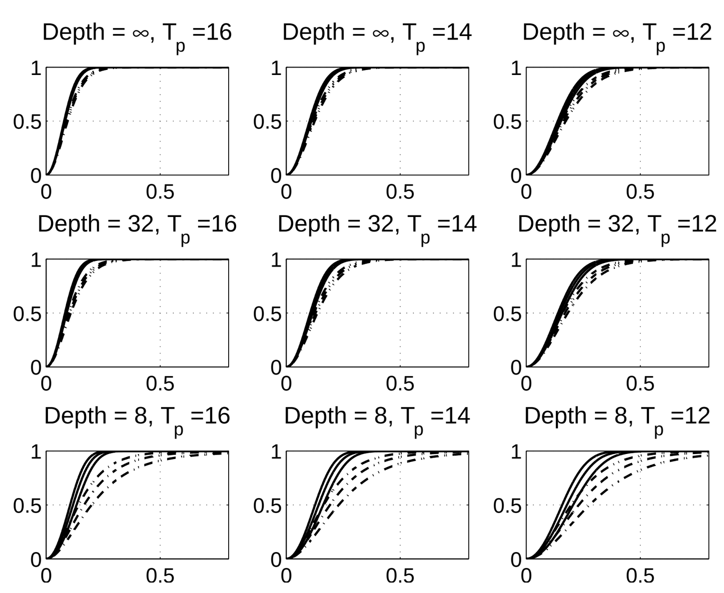

Figure 1: *CDF for slopes at upcrossings (solid) and negative slopes at downcrossings (dash-dotted) of levels -1, 0, 1, for linked Lagrange space waves with $\alpha = 0.4$. Smallest slope at level -1, largest at level +1.*

(TT) Slope in time at level crossings in time:] The distribution of the time slope $L_t(t,x_0)$ observed at the up- or downcrossings of a fixed level v by the time wave $L(t,x_0)$

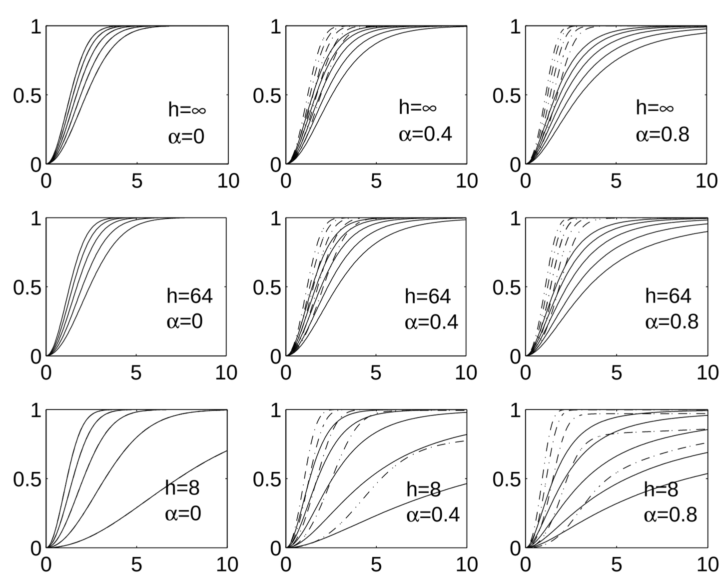

Figure 2: *Cumulative distribution functions for time wave slopes (absolute values) at time wave crossings of different levels. Slope CDF at upcrossings (solid lines) and at downcrossings (dash-dotted lines). Levels v: $[-1,0,1,2,3]\times\sigma$, $4\sigma = H_s$. Largest absolute values correspond to highest level.*

3. Asymmetry in space and time

The distribution of space wave slope at v-level upcrossing is the distribution of the defining ratio, conditioned on that u is a v-upcrossing in the vertical Gaussian process $W(t_0,u)$, with Rayleigh distributed slope. Let R and U be independent standard Rayleigh and normal variables. The notation $X \overset{\mathcal{L}}{=} Y$ means "equal in distribution".

Theorem 1 *The slope of the Lagrange space wave at v-upcrossing has the representation $L_x(t_0,x_k) \overset{\mathcal{L}}{=}$*

$$\frac{R\sqrt{r_{uu}^{ww}}}{1 + v\,\frac{r_{0u}^{wx}}{r^{ww}} + R\frac{r_{uu}^{wx}}{\sqrt{r_{uu}^{ww}}} + U\sqrt{r_{uu}^{xx} - \frac{(r_{0u}^{wx})^2}{r^{ww}} - \frac{(r_{uu}^{wx})^2}{r_{uu}^{ww}}}}.$$

Front-back asymmetry depends on the covariance

$$r_{uu}^{wx} = \int_0^\infty \kappa^2 \cos(\theta(\omega))\, S(\omega)\, d\omega,$$

between the spatial derivatives of the vertical and horizontal processes.
Figure 1 illustrates the asymmetry in case (SS).

A time wave crossing of the level v occurs at time t_k if there is a particle with reference coordinate, u_k, such that $W(t_k,u_k) = v$ and $X(t_k,u_k) = x_0$.
By a remarkable generalization of Rice's formula for the number of level crossings, Mercardier, [4], has given the tool for how to find conditional distributions like the ones we seek. To formulate the theorem, we define

$$D = \big|W_t(0,u)X_u(0,u) - W_u(0,u)X_t(0,u)\big|,$$

and write $q_u(v,x)$ for the density of $\mathbf{V}(0,u) = (W(0,u), X(0,u))$, evaluated at (v,x).

Theorem 2 *The distribution function for slopes at upcrossings of the level v in the Lagrange time wave, is given by,*

$$F_v^{TT+}(y) = \frac{1}{\mathsf{E}(N^+)} \int_{-\infty}^{\infty} g_y^{TT+}(u)\, q_u(v,x_0)\, du,$$

where

$$g_y^{TT+}(u) = \mathsf{E}\big(D \times I^{(T)}(y) \,\big|\, \mathbf{V}(0,u) = (v,x_0)\big),$$
$$I^{(T)}(y) = \mathbf{1}\left\{0 < W_t(0,u) - W_u(0,u)\,\frac{X_t(0,u)}{X_u(0,u)} \le y\right\},$$

and $\mathsf{E}(N^+) = \int_{-\infty}^{\infty} g_\infty^{TT+}(u)\, q_u(v,x_0)\, du.$

Figure 2 illustrates the asymmetry in case (TT).
Theorem 1 and 2 were proved in [1] for the front-back symmetric case, and in [2] and [3] for the asymmetric case.

References

[1] Aberg, S.: Wave intensities and slopes in Lagrangian seas. *Adv. Appl. Probab.* 39 (2007) 1020–1035.

[2] Lindgren, G. and Åberg, S.: First order Lagrange model for asymmetric ocean waves. *J. Offshore Mech. Arct.*, (2009) in press.

[3] Lindgren, G.: Slope distributions in front-back asymmetric stochastic Lagrange waves. Submitted 2009.

[4] Mercardier, C.: Numerical bounds for the distribution of the maxima of some one- and two-parameter Gaussian proceses. *Adv. Appl. Probab.*, 38 (2006) 149-170

Joint workshop, SEAMOCS, PIMS, STINT, QUEST4D, ENCORA, 18-20 March 2009, Preluna, Malta