Theorem 1 (Kolmogorov 1933) Let $\mathcal{X}$ be the set of all real-valued functions defined on a set $T$, let $\mathcal{A}$ be the algebra of subsets of $\mathcal{X}$ which is generated by the sets $\{x: x(t) \leq \xi\}$, $t \in T$, $\xi$ real, and let $P$ be a nonnegative finitely additive function defined on $\mathcal{A}$ with $P(\mathcal{X}) = 1$.

If all functions $F_t$, $t \in T$ defined by

$$F_t(\xi) = P(\{x: x(t) \leq \xi\})$$

are right-continuous with $\lim_{\xi \to \infty} F_t(\xi) = 1$ and $\lim_{\xi \to -\infty} F_t(\xi) = 0$, then $P$ can be extended in a unique way to a probability measure defined on the $\sigma$-algebra generated by $\mathcal{A}$.

Proof

By the Extension Theorem, it suffices to prove that if $A_1 \supset A_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then $P(A_n) \to 0$. The idea is to approximate the $A_n$ with somewhat smaller $K_n$ which act like compact sets, so that there is an $n_0$ for which $\bigcap_{n=1}^{n_0} K_n = \emptyset$.

Let $t_1, t_2, \ldots$ be the $t$-values involved. By introducing conditions of the type $-\infty < x(t_i) < \infty$, we can write

$$A_n = \bigcup_{j=1}^{j_n} \bigcap_{i=1}^{i_n} \{x: x(t_i) \in I_{nij}\}$$

where $i_1 \leq i_2 \leq \ldots$, and the $I$ are intervals of the type $[a, b]$, $]-\infty, b]$, $]a, \infty[$, or $]-\infty, \infty[$. Upon changing the intervals to somewhat smaller ones with endpoints, respectively, $(a', b' = b)$, $(a', b' = b)$, $(a', b')$, and $(a', b')$, we get

$$B_n \subset K_n \subset A_n$$

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*Centre for Mathematical Sciences, Lund University, Lund, Sweden.
†Homepage: http://www.maths.lth.se/matstat/staff/bengtr
where $B_n \in A$ has $a'_{nij} < x(t_i) \leq b'_{nij}$ and $K_n$ has $a'_{nij} \leq x(t_i) \leq b'_{nij}$ instead of $x(t_i) \in I_{nij}$. It follows from the assumptions on the $F_t$ and the lemma below that, for an arbitrarily small $\varepsilon > 0$, this can be done so that

$$P(A_n) - P(B_n) < \varepsilon/2^n.$$ 

Now suppose each $\cap_{n=1}^k K_n$ were nonempty, that is, there were a sequence in $\mathfrak{X}$ with $x_k \in \cap_{n=1}^k K_n$. But, sooner or later, each sequence $x_1(t_i), x_2(t_i), \ldots$ enters a bounded set, so, by the Bolzano-Weierstrass theorem, it has a convergent subsequence, and with the Cantor diagonal method, a subsequence can be extracted which converges for all $t_i$. Define

$$x(t) = \begin{cases} \lim_{k' \to \infty} x_{k'}(t_i) & \text{if } t = t_i \\ 0 & \text{otherwise.} \end{cases}$$

Since each interval in $K_n$ is closed, this $x$ belongs to $K_n$ for all $n$, which gives a contradiction. Therefore, $\cap_{n=0}^{n_0} K_n$ and, consequently, $\cap_{n=1}^{n_0} B_n$ are empty for $n_0$ sufficiently large. Finally, the lemma gives

$$P(A_{n_0}) - P(\cap_{n=1}^{n_0} A_n) \leq P(\cap_{n=1}^{n_0} B_n) + \sum_{n=1}^{n_0} (P(A_n) - P(B_n)) < 0 + \sum_{n=1}^{n_0} \varepsilon/2^n < \varepsilon$$

if $n_0$ is large, which proves the theorem.

\begin{lemma}
If $P$ is a nonnegative finitely additive set function, and $A_1 \supset B_1, A_2 \supset B_2, \ldots, A_n \supset B_n$, then

$$P(\cup_{k=1}^n A_k) - P(\cup_{k=1}^n B_k) \leq \sum_{k=1}^n (P(A_k) - P(B_k))$$

and

$$P(\cap_{k=1}^n A_k) - P(\cap_{k=1}^n B_k) \leq \sum_{k=1}^n (P(A_k) - P(B_k)).$$

\end{lemma}

\begin{proof}
The first inequality follows from $(\cup_{j=1}^n B_j)^c \subset B_k^c$ and Boole’s inequality:

$$P((\cup_{k=1}^n A_k)(\cup_{j=1}^n B_j)^c) \leq P((\cup_{k=1}^n A_k)B_k^c) \leq \sum_{k=1}^n P(A_k B_k^c).$$

For the second one $\cap_{k=1}^n A_k \subset A_j$ is used.
\end{proof}