

II

Stochastic Calculus

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Typical Setup

Take as given the market price process, $S(t)$, of some underlying asset.

$S(t)$ = price, at t , per unit of underlying asset

Consider a fixed **financial derivative**, e.g. a European call option.

Main Problem: Find the arbitrage free price of the derivative.

We Need:

1. Mathematical model for the underlying price process. (The Black-Scholes model)
2. Mathematical techniques to handle the price dynamics. (The Itô calculus.)

Stochastic Processes

- We model the stock price $S(t)$ as a **stochastic process**, i.e. it **evolves randomly over time**.
- We model S as a **Markov process**, i.e. in order to predict the future only the present value is of interest. All past information is already incorporated into today's stock prices. (Market efficiency).

Stochastic variable

Choosing a **number** at random

Stochastic process

choosing a **curve** (trajectory) at random.

Notation

$$\begin{aligned} X(t) &= \text{any random process,} \\ dt &= \text{small time step,} \\ dX(t) &= X(t + dt) - X(t) \end{aligned}$$

- dX is called the **increment** of X over the interval $[t, t + dt]$.
- For any fixed interval $[t, t + dt]$, the increment dX is a stochastic variable.
- If the increments $dX(s)$ and $dX(t)$, over the disjoint intervals $[s, s + ds]$ and $[t, t + dt]$ are independent, then we say that X has **independent increments**.
- If every increment has a normal distribution we say that X is a **normal**, or **Gaussian** process.

The Wiener Process

A stochastic process W is called a **Wiener process** if it has the following properties

- The increments are normally distributed:
For $s < t$:

$$W(t) - W(s) \sim N[0, \sqrt{t - s}]$$

$$E[W(t) - W(s)] = 0, \quad Var[W(t) - W(s)] = t - s$$

- W has independent increments.
- $W(0) = 0$
- W has continuous trajectories.

Continuous random walk

Important Fact

Theorem:

A Wiener trajectory is, with probability one, **nowhere differentiable**.

Proof. Hard.

Wiener Process with Drift

A stochastic process X is called a Wiener process with **drift** μ and **diffusion coefficient** σ if it has the following dynamics

$$\begin{aligned}dX &= \mu dt + \sigma dW, \\X(0) &= x_0\end{aligned}$$

where x_0 , μ and σ are constants.

Summing all increments over the interval $[0, t]$ gives us

$$X(t) - x_0 = \mu t + \sigma W(t)$$

$$X(t) = x_0 + \mu t + \sigma W(t)$$

The distribution of X is thus given by

$$X(t) \sim N[x_0 + \mu t, \sigma\sqrt{t}]$$

Stochastic Differential Equations

Take as given two functions $\mu(t, x)$ and $\sigma(t, x)$. We say that the process X is an **diffusion** if it has the local dynamics

$$\begin{aligned}dX &= \mu(t, X_t)dt + \sigma(t, X_t)dW, \\X_0 &= x_0\end{aligned}$$

Interpretation:

Over the time interval $[t, t + dt]$, the X -process is driven by two separate terms.

- A locally deterministic velocity $\mu(t, X(t))$.
- An independent Gaussian disturbance term, amplified by the factor $\sigma(t, X(t))$.

How do we make this precise?

Possible Interpretations

$$dX = \mu(t, X_t)dt + \sigma(t, X_t)dW,$$

- (I)** “Divide” formally by dt . Then we obtain the stochastic ODE

$$\frac{dX_t}{dt} = \mu(t, X_t) + \sigma(t, X_t)v_t$$

where

$$v_t = \frac{dW}{dt}$$

is the formal time derivative of the Wiener process W .

This is impossible, since $\frac{dW}{dt}$ does not exist.

- (II)** Write the equation on integrated form as

$$X_t = x_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s$$

How is this interpreted?

$$X_t = x_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s$$

Two terms:

-

$$\int_0^t \mu(s, X_s) ds$$

Riemann integral for each X -trajectory.

-

$$\int_0^t \sigma(s, X_s) dW_s$$

Stochastic integral. This can **not** be interpreted as a Stieljes integral for each trajectory. We need a new theory for this **Itô integral**.

Information

Let the Wiener process W be given.

Def:

\mathcal{F}_t^W = “The information generated by W over the interval $[0, t]$ ”

Def: Let Z be a stochastic variable. If the value of Z is completely determined by \mathcal{F}_t^W , we write

$$Z \in \mathcal{F}_t^W$$

Ex:

For the stochastic variable Z , defined by

$$Z = \int_0^5 W(s) ds,$$

we have $Z \in \mathcal{F}_5^W$.

We do **not** have $Z \in \mathcal{F}_4^W$.

Adapted Processes

Let W be a Wiener process.

Definition:

A process X is **adapted** to the filtration $\{\mathcal{F}_t^W : t \geq 0\}$ if

$$X_t \in \mathcal{F}_t^W, \quad \forall t \geq 0$$

“An adapted process does not look into the future”

Adapted processes are nice integrands for stochastic integrals.

- The process

$$X_t = \int_0^t W_s ds,$$

is adapted.

- The process

$$X_t = \sup_{s \leq t} W_s$$

is adapted.

- The process

$$X_t = \sup_{s \leq t+1} W_s$$

is **not** adapted.

The Itô Integral

We will define the Itô integral

$$\int_a^b g(s) dW(s)$$

for processes $g \in \mathcal{L}^2$, i.e.

- The process g is adapted.
- The process g satisfies

$$\int_a^b E [g^2(s)] ds < \infty$$

This will be done in two steps.

I: Simple Integrand

Definition:

The process g is **simple**, if

- $g \in \mathcal{L}^2$
- There exists deterministic points t_0, \dots, t_n with $a = t_0 < t_1 < \dots < t_n = b$ such that g is piecewise constant, i.e.

$$g(s) = g(t_k), \quad s \in [t_k, t_{k+1})$$

For simple g we define

$$\int_a^b g(s) dW(s) = \sum_{k=0}^{n-1} g(t_k) [W(t_{k+1}) - W(t_k)]$$

FORWARD INCREMENTS!

II: General Case

For a general $g \in \mathcal{L}^2$ we do as follows.

1. Approximate g with a sequence of simple g_n such that

$$\int_a^b E \left[\{g_n(s) - g(s)\}^2 \right] ds \rightarrow 0.$$

2. For each n the integral

$$\int_a^b g_n(s) dW(s)$$

is a well defined stochastic variable Z_n .

3. One can show that the Z_n sequence converges to a limiting stochastic variable.

4. We define $\int_a^b g dW$ by

$$\int_a^b g(s) dW(s) = \lim_{n \rightarrow \infty} \int_a^b g_n(s) dW(s).$$

Properties of the Integral

Theorem:

The following relations hold

-

$$E \left[\int_a^b g(s) dW(s) \right] = 0$$

-

$$E \left[\left(\int_a^b g(s) dW(s) \right)^2 \right] = \int_a^b E [g^2(s)] ds$$

-

$$\int_a^b g(s) dW(s) \in \mathcal{F}_b^W$$

Martingales

Definition: An adapted process is a **martingale** if

$$E[X_t | \mathcal{F}_s] = X_s, \quad \forall s \leq t$$

“A martingale is a process without drift”

Proposition: For $g \in \mathcal{L}^2$, the process

$$X_t = \int_0^t g_s dW_s$$

is a martingale.

Proposition: If X has dynamics

$$dX_t = \mu_t dt + \sigma_t dW_t$$

then X is a martingale **iff** $\mu = 0$.

Stochastic Calculus

General Model:

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

Let the function $f(t, s)$ be given, and define the stochastic process X_t by

$$X_t = f(t, S_t)$$

Problem: What does $df(t, S_t)$ look like?

The answer is given by the **Itô formula**.

A close up of the Wiener process

Consider an “infinitesimal” Wiener increment

$$dW = W(t + dt) - W(t)$$

We know:

$$dW \sim N[0, \sqrt{dt}]$$

$$E[dW] = 0, \quad Var[dW] = dt$$

From this one can show

$$E[(dW)^2] = dt, \quad Var[(dW)^2] = 3(dt)^2$$

Important observation:

1. Both $E[(dW)^2]$ and $Var[(dW)^2]$ are very small when dt is small .
2. $Var[(dW)^2]$ is negligible compared to $E[(dW)^2]$.
3. Thus $(dW)^2$ is **deterministic**.

$$(dW)^2 = dt$$

Multiplication table.

-

$$(dt)^2 = 0$$

-

$$dW \cdot dt = 0$$

-

$$(dW)^2 = dt$$

Deriving the Itô formula

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

$$X_t = f(t, S_t)$$

We want to compute $df(t, S_t)$ (i.e. the change in $f(t, S_t)$)

Make a Taylor expansion of $f(t, S_t)$ including second order terms:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}dS + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial s^2}(dS)^2 + \frac{\partial^2 f}{\partial t\partial s}dt \cdot dS \end{aligned}$$

Plug in the expression for dS , expand, and use the multiplication table!

We get:

$$\begin{aligned}
 df &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial s} [\mu dt + \sigma dW] + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 \\
 &+ \frac{1}{2} \frac{\partial^2 f}{\partial s^2} [\mu dt + \sigma dW]^2 + \frac{\partial^2 f}{\partial t \partial s} dt \cdot [\mu dt + \sigma dW] \\
 &= \frac{\partial f}{\partial t} dt + \mu \frac{\partial f}{\partial s} dt + \sigma \frac{\partial f}{\partial s} dW + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 \\
 &+ \frac{1}{2} \frac{\partial^2 f}{\partial s^2} [\mu^2 (dt)^2 + \sigma^2 (dW)^2 + 2\mu\sigma dt \cdot dW] \\
 &+ \mu \frac{\partial^2 f}{\partial t \partial s} (dt)^2 + \sigma \frac{\partial^2 f}{\partial t \partial s} dt \cdot dW
 \end{aligned}$$

Using the multiplikation table this reduces to:

$$\boxed{
 \begin{aligned}
 df &= \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial s} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt \\
 &+ \sigma \frac{\partial f}{\partial s} dW
 \end{aligned}
 }$$

Itô's formula

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

$$X_t = f(t, S_t)$$

$$df = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial s} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt + \sigma \frac{\partial f}{\partial s} dW$$

Alternatively

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial s} dS + \frac{1}{2} \frac{\partial^2 f}{\partial s^2} (dS)^2,$$

where we use the multiplication table.

A Useful Trick

Problem: Compute $E[X_T]$.

- Use Itô to get

$$dX_t = \mu_t dt + \sigma_t dW_t$$

- Integrate.

$$X_T = x_0 + \int_0^T \mu_t dt + \int_0^T \sigma_t dW_t$$

- Take expectations.

$$E[X_T] = x_0 + \int_0^T E[\mu_t] dt + 0$$

- The problem has been reduced to that of computing $E[\mu_t]$.

The Black-Scholes model

Price dynamics:(Geometrical Brownian Motion)

$$dS = \alpha S dt + \sigma S dW,$$

Simple analysis:

Assume that $\sigma = 0$. Then

$$dS = \alpha S dt$$

Divide by dt !

$$\frac{dS}{dt} = \alpha S$$

Simple ordinary differential equation with solution

$$S_t = s_0 e^{\alpha t}$$

Conjecture: The solution of the SDE above is a randomly disturbed exponential function.

Economic Interpretation

$$\frac{dS}{S} = \alpha dt + \sigma dW$$

Over a small time interval $[t, t + dt]$ this means:

Return = (mean return)
+ $\sigma \times$ (Gaussian random disturbance)

- The asset **return** is a random walk (with drift).
- α = mean rate of return per unit time
- σ = “volatility”

Large σ = large random fluctuations

Small σ = small random fluctuations

We will see that:

$$S(t) = S(0)e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

Stock prices are lognormally distributed.

Returns are normally distributed.

Example GBM

$$dS = \alpha S dt + \sigma S dW$$

We smell something exponential!

Natural Ansatz:

$$\begin{aligned} S(t) &= e^{Z(t)}, \\ Z(t) &= \ln S(t) \end{aligned}$$

Itô on $f(t, s) = \ln(s)$ gives us

$$\frac{\partial f}{\partial s} = \frac{1}{s}, \quad \frac{\partial f}{\partial t} = 0, \quad \frac{\partial^2 f}{\partial s^2} = -\frac{1}{s^2}$$

$$\begin{aligned} dZ &= \frac{1}{S} dS - \frac{1}{2} \frac{1}{S^2} (dS)^2 \\ &= \left(\alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW \end{aligned}$$

Integrate!

$$\begin{aligned} S(t) - S(0) &= \int_0^t \left(\alpha - \frac{1}{2}\sigma^2 \right) d\tau + \sigma \int_0^t dW(s) \\ &= \left(\alpha - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \end{aligned}$$

Using $S = e^Z$ gives us

$$S(t) = S(0)e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

Since $W(t)$ is $N[0, \sqrt{t}]$, we see that $S(t)$ has a lognormal distribution.

The Connection SDE \sim PDE

Given: $\mu(t, x)$, $\sigma(t, x)$, $\Phi(x)$, T

Problem: Find a function F solving the Partial Differential Equation (PDE)

$$\begin{aligned}\frac{\partial F}{\partial t}(t, x) + \mathcal{A}F(t, x) &= 0, \\ F(T, x) &= \Phi(x).\end{aligned}$$

where \mathcal{A} is defined by

$$\mathcal{A}F(t, x) = \mu(t, x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 F}{\partial x^2}(t, x)$$

Assume that F solves the PDE.

Fix the point (t, x) .

Define the process X by

$$\begin{aligned}dX_s &= \mu(s, X_s)dt + \sigma(s, X_s)dW_s, \\X_t &= x,\end{aligned}$$

Apply Ito to the process $F(t, X_t)$!

$$\begin{aligned}F(T, X_T) &= F(t, X_t) \\&+ \int_t^T \left\{ \frac{\partial F}{\partial t}(s, X_s) + \mathcal{A}F(s, X_s) \right\} ds \\&+ \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) dW_s.\end{aligned}$$

By assumption $\frac{\partial F}{\partial t} + \mathcal{A}F = 0$, and $F(T, x) = \Phi(x)$

Thus:

$$\begin{aligned}\Phi(X_T) &= F(t, x) \\ &+ \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) dW_s.\end{aligned}$$

Take expectations.

$$F(t, x) = E_{t,x} [\Phi(X_T)],$$

Feynman-Kač

The solution $F(t, x)$ to the PDE

$$\frac{\partial F}{\partial t} + \mu(t, x) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2} - rF = 0,$$

$$F(T, x) = \Phi(x).$$

is given by

$$F(t, x) = e^{-r(T-t)} E_{t,x} [\Phi(X_T)],$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s)dt + \sigma(s, X_s)dW_s,$$

$$X_t = x.$$