The short-time Fourier transform (STFT) for a signal $x(t)$ is defined as

$$X(t, f) = \int_{-\infty}^{\infty} x(t_1) h^*(t_1 - t) e^{-j2\pi f t_1} dt_1,$$

(1)

where the unit energy window function $h(t)$ centered at time $t$ is multiplied with the signal $x(t)$ before the Fourier transform.

Similar to the ordinary Fourier transform and spectrum we can formulate the spectrogram as

$$S_x(t, f) = |X(t, f)|^2,$$

(2)
The **Wigner distribution** is defined as

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2}) x^*(t-\frac{\tau}{2}) e^{-j2\pi f \tau} d\tau.$$  

(3)

The Wigner distribution is real-valued and time- and frequency-shift invariant. It does, however, produce cross-terms.

**Ambiguity function**

The word “ambiguity” is a bit ambiguous as it then should stand for something that is not clearly defined. There is however, nothing ambiguous about the **ambiguity function**. The ambiguity function is defined as

$$A_x(\nu, \tau) = \int_{-\infty}^{\infty} z(t+\frac{\tau}{2}) z^*(t-\frac{\tau}{2}) e^{-j2\pi \nu t} dt,$$  

(4)

where usually the analytic signal $z(t)$ is used.

**Figure:** The Wigner distribution for the analytic signal of the three-tone sequence.

**Example of the ambiguity function**

**Figure:** A Gaussian function and the representation as Wigner-Ville distribution and ambiguity function (real value).
Example of the ambiguity function

Figure: A Gaussian function and the representation as Wigner-Ville distribution and ambiguity function (real value).

Example of the ambiguity function

Figure: Two Gaussian functions and the representation as Wigner-Ville distribution and ambiguity function (real value).
The ambiguity kernel

Quadratic time-frequency distributions can be formulated as the multiplication of the ambiguity function and the **ambiguity kernel**, 

\[ A^2_Q(\nu, \tau) = A_Q(\nu, \tau) \cdot \phi(\nu, \tau), \]  

giving the filtered ambiguity function. The ambiguity kernel is often defined to preserve the auto-terms at the centre and to suppress the cross-terms away from the centre.

Reduced Interference Distributions

Methods to reduce the cross-terms, also sometimes referred to as **interference terms**, have been proposed and a number of useful kernels, that falls into the so-called Reduced Interference distributions (RID) class can be found. The maybe most applied RID is the Choi-Williams distribution, also called the Exponential distribution (ED), with the ambiguity kernel defined as

\[ \phi_{ED}(\nu, \tau) = e^{-\frac{\nu^2 + \tau^2}{\sigma}}, \]  

where \( \sigma \) is a design parameter. The Choi-Williams distribution also falls into the subclass of **product kernels** which have the advantage in optimization of being dependent of one variable only, i.e., \( x = \nu \tau \).

**Example**: The Wigner and Choi-Williams distributions of two complex sinusoids with Gaussian envelopes located at \( t_1 = 64, f_1 = 0.1 \) and \( t_2 = 128, f_2 = 0.2 \).

**Example**: The Wigner-Ville and Choi-Williams distributions of two complex sinusoids with Gaussian envelopes located at \( t_1 = 64, f_1 = 0.1 \) and \( t_2 = 128, f_2 = 0.1 \).
The Born-Jordan distribution

Other ways of dealing with the cross-terms have been applied, e.g. the Born-Jordan distribution derived by Cohen. The ambiguity kernel for the Born-Jordan distribution, also called the Sinc-distribution, is

$$\phi_{BJ}(\nu, \tau) = \text{sinc}(a\nu\tau) = \frac{\sin(\pi a\nu\tau)}{\pi a\nu\tau},$$  \hspace{1cm} (7)

which also fits into the RID class and is a product kernel. (A lag-windowed form of this distribution is the Zhao, Atlas and Marks (ZAM) distribution.)

Separable kernels

A nice form of simple kernels are the separable kernels defined by

$$\phi(\nu, \tau) = G_1(\nu)g_2(\tau).$$ \hspace{1cm} (8)

This form transfers easily to the time-frequency domain as

$$\Phi(t, f) = g_1(t)G_2(f)$$ \hspace{0.5cm} with \hspace{0.5cm} $$g_1(t) = \mathcal{F}^{-1}\{G_1(\nu)\}$$ \hspace{0.5cm} and \hspace{0.5cm} $$G_2(f) = \mathcal{F}\{g_2(\tau)\}. $$

The quadratic time-frequency formulation becomes

$$W_Q^2(t, f) = g_1(t) * W_2(t, f) * G_2(f),$$ \hspace{1cm} (9)

as $$A_Q^2(\nu, \tau) = G_1(\nu)A_2(\nu, \tau)g_2(\tau).$$ The separable kernel replaces the 2-D convolution of the quadratic time-frequency representation with two 1-D convolutions, which might be beneficial for some signals.

The ambiguity kernel

The ambiguity kernel transfers to the **time-frequency kernel** as

$$\Phi(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\nu, \tau)e^{-i2\pi(\tau f - \nu \nu)} \, d\tau \, d\nu.$$  \hspace{1cm} (10)

The ambiguity kernel $$\phi(\nu, \tau) = 1,$$ transfers to the corresponding time-frequency (non-)smoothing kernel $$\Phi(t, f) = \delta(t)\delta(f)$$ which will give us the Wigner (-Ville) distribution. For any ambiguity kernel, the **smoothed** Wigner-Ville distribution is found as the 2-dimensional convolution

$$W^2(t, f) = W_2(t, f) * \Phi(t, f).$$  \hspace{1cm} (11)

Doppler-independent kernel

If $$G_1(\nu) = 1,$$ \hspace{1cm} (10)

a **doppler-independent** kernel is given as $$\phi(\nu, \tau) = g_2(\tau),$$ and the quadratic time-frequency distribution reduces to

$$W^2_Q(t, f) = W_2(t, f) * G_2(f),$$ \hspace{1cm} (11)

which is a smoothing in the frequency domain. The Doppler-independent kernel is also given the name **Pseudo-Wigner** or **windowed Wigner** distribution.
Lag-independent kernel

The other case is when
\[ g_2(\tau) = 1, \quad (12) \]
giving the \textbf{lag-independent} kernel, \( \phi(\nu, \tau) = G_1(\nu) \) where the
time-frequency formulation gives only a smoothing in the variable \( t \),
\[ W^Q_z(t, f) = g_1(t) \ast W_z(t, f). \quad (13) \]

The marginals again

The Wigner distribution satisfies the so called \textbf{marginals}, i.e.,
\[ \int_{-\infty}^{\infty} W_x(t, f) df = |x(t)|^2, \]
and
\[ \int_{-\infty}^{\infty} W_x(t, f) dt = |X(f)|^2. \]
If the marginals are satisfied the \textbf{total energy condition} is also
automatically satisfied,
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f) dt df = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = E_x, \]
where \( E_x \) is the energy of the signal.

Example

The result of the time-frequency plots for three complex sinusoids
with Gaussian envelopes, located at \( t_1 = 64, f_1 = 0.1 \) and
\( t_2 = 128, f_2 = 0.1 \) and \( t_3 = 128, f_3 = 0.2 \); a) The Wigner distribution; b)
Doppler-independent kernel; c) Lag-independent kernel.

The ambiguity kernel and the marginals

Recalling the time marginal property \( \int_{-\infty}^{\infty} W_x(t, f) df = |z(t)|^2 \) we see that if we put \( \tau = 0 \) in the definition
\[ A_z(\nu, \tau) = \int_{-\infty}^{\infty} z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2}) e^{-i2\pi\nu t} dt, \]
we find
\[ A_z(\nu, 0) = \int_{-\infty}^{\infty} z(t) z^*(t) e^{-i2\pi\nu t} dt = \int_{-\infty}^{\infty} |z(t)|^2 e^{-i2\pi\nu t} dt, \quad (14) \]
which is the Fourier transform of the time marginal. To preserve
the marginals of the time-frequency distribution of a quadratic
\textbf{time-frequency distribution} \( A^Q_z(\nu, \tau) = A_z(\nu, \tau) \cdot \phi(\nu, \tau) \), the
ambiguity kernel must then fulfill
\[ A^Q_z(\nu, 0) = A_z(\nu, 0) \cdot \phi(\nu, 0) = A_z(\nu, 0) \quad (15) \]
giving
\[ \phi(\nu, 0) = 1. \quad (16) \]
The ambiguity kernel and the marginals

With the same reasoning for the frequency marginal property
\[ \int_{-\infty}^{\infty} W_z(t, f) df = |Z(f)|^2 \]
and using \( Z(f) = \int_{-\infty}^{\infty} z(t) e^{-i2\pi ft} dt \) we get
\[ A_z(0, \tau) = \int_{-\infty}^{\infty} Z(f) Z^*(f) e^{i2\pi ft} df = \int_{-\infty}^{\infty} |Z(f)|^2 e^{i2\pi ft} df, \quad (17) \]
which is the inverse Fourier transform of the frequency marginal.

Similarly as above the ambiguity kernel must now fulfill
\[ \phi(0, \tau) = 1. \quad (18) \]
We can also conclude that if the marginals are satisfied then
\[ \phi(0, 0) = 1, \quad (19) \]
which is the claim for the unchanged total energy condition.

Other properties

- For the quadratic class to be real-valued, the kernel must fulfill be Hermitian \( \phi(\nu, \tau) = \phi^*(-\nu, -\tau) \), (Exercise 4.3, hint, consider the complex conjugate of Eq. (4.8)).
- For the time-invariance and frequency-invariance property the kernel \( \phi(\nu, \tau) \) should not be a function of time nor of frequency, (see the supplement).

The Choi-Williams kernel is Hermitian, i.e., as
\[ \phi^{\text{ED}}_z(\nu, \tau) = e^{-\nu^2-\tau^2} = \phi^{\text{ED}}(-\nu, -\tau), \]
and is not a function of time nor of frequency.

Properties of the Choi-Williams kernel

The Choi-Williams distribution
\[ \phi^{\text{ED}}(\nu, \tau) = e^{-\frac{\nu^2+\tau^2}{2}}, \quad (20) \]
satisfies the marginals as
\[ \phi^{\text{ED}}(0, \tau) = \phi^{\text{ED}}(\nu, 0) = 1, \]
and the total energy condition as
\[ \phi^{\text{ED}}(0, 0) = 1. \]

Rihaczek distribution

The Rihaczek distribution (also called the Kirkwood-Rihaczek distribution) is derived from the energy of a complex-valued deterministic signal over finite ranges of \( t \) and \( f \), and if these ranges become infinitesimal, a density function is obtained as
\[ R_z(t, f) = z(t) Z^*(f) e^{-i2\pi ft}, \quad (21) \]
as the energy of a complex-valued signal is
\[ E = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_{-\infty}^{\infty} z(t) Z^*(f) e^{-i2\pi ft} df dt \quad (22) \]
We note that the marginals are satisfied as
$$\int_{-\infty}^{\infty} R_z(t, f) df = |z(t)|^2,$$
(see from Eq. (22)), and
$$\int_{-\infty}^{\infty} R_z(t, f) dt = Z^*(f) \int_{-\infty}^{\infty} z(t) e^{-i2\pi ft} dt = |Z(f)|^2.$$

The ambiguity kernel is
$$\phi(\nu, \tau) = e^{-i\pi \nu \tau}.$$  \hspace{1cm} (23)

From this we also verify that the marginals are satisfied as
$$\phi(\nu, 0) = \phi(0, \tau) = 1.$$

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From this we also verify that the marginals are satisfied as
$$\phi(\nu, 0) = \phi(0, \tau) = 1.$$
Another suggestion of distribution relying on the running Fourier transform,

\[ Z_-(f) = \int_{-\infty}^{t} z(t') e^{-j2\pi ft'} \, dt'. \quad (26) \]

is the Page distribution. Differentiating the squared absolute value with respect to time gives the Page distribution as

\[
P_-(t, f) = \frac{\partial}{\partial t} \left[ \left| \int_{-\infty}^{t} z(t') e^{-j2\pi ft'} \, dt' \right|^2 \right] = 2 \Re \left[ z^*(t) Z_-(f) e^{j2\pi ft} \right].
\]

The main characteristic of this distribution is that the future does not affect the past, i.e., a finite duration sinusoid becomes more and more concentrated and the cross terms corresponding to specific components show up when time increases.

### Summary of distributions

- Doppler-independent or Pseudo-Wigner distribution (separable kernel, smoothing in frequency).
- Lag-independent distribution (separable kernel, smoothing in time).
- Choi-Williams or Exponential distribution (RID and product kernel, smoothing on the diagonal).
- Born-Jordan or Sinc distribution (RID and product kernel).
- Rihaczeck distribution, (complex-valued, relocates cross-terms).
- Levin or Margenau-Hill distribution, (real-valued version of Rihaczek).
- Page distribution, (use the running Fourier transform, so that finite duration signals become more and more concentrated with time).

### The quadratic class

In the 1940s to 60s, a lot of distributions and time-frequency methods were invented. Many of these satisfied the marginals, the instantaneous frequency condition and other properties, e.g., the Rihaczeck distribution, Page distribution and Margenau-Hill or Levin distribution. A formulation was then made by Leon Cohen, the Cohen’s class, which included these and an infinite number of other methods with different kernel functions. Later the quadratic class was defined, also including kernels which do not satisfy the marginals, which was a restriction of the original Cohen’s class. However, the quadratic class is nowadays often referred to as the Cohen’s class. (Leon Cohen - 'Time-Frequency distributions-A review', Proc. of the IEEE, vol. 77, no. 7, July 1989, 941-981)
The quadratic class - Cohen's class

For any ambiguity kernel, the smoothed Wigner-Ville distribution is found as

$$W^Q_z(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{A}^Q_z(\nu, \tau) e^{-j2\pi(\nu t - \nu \tau)} d\tau d\nu$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{A}_z(\nu, \tau) \phi(\nu, \tau) e^{-j2\pi(\nu t - \nu \tau)} d\tau d\nu$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(u + \frac{\tau}{2}) z^*(u - \frac{\tau}{2}) \phi(\nu, \tau) e^{j2\pi(\nu t - \nu u)} d\nu d\tau,$$

(27)

where the last formulation is most used defining the quadratic class. We can note that the Wigner-Ville distribution has the simple ambiguity domain kernel $\phi(\nu, \tau) = 1$.

The doppler distribution

The doppler distribution is the corresponding spectral autocorrelation function which is calculated from the IAF as

$$D_z(\nu, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_z(t, \tau) e^{-j2\pi(f \tau + \nu t)} dt d\tau.$$  

(30)

Replacing $t + \tau/2 = t_1$ and $t - \tau/2 = t_2$ we reformulate the doppler distribution as

$$D_z(\nu, f) = \int_{-\infty}^{\infty} z(t_1) e^{-j2\pi(f + \frac{\nu}{2}) t_1} dt_1 \cdot \int_{-\infty}^{\infty} z^*(t_2) e^{j2\pi(f - \frac{\nu}{2}) t_2} dt_2$$

$$= Z(f + \frac{\nu}{2}) Z^*(f - \frac{\nu}{2}),$$  

(31)

where $Z(f)$ is the Fourier transform of $z(t)$. We now see that the doppler distribution is the frequency dual of the IAF.

The IAF

We can also formulate the quadratic class as

$$W^Q_z(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_z(u, \tau) \rho(t - u, \tau) e^{-j2\pi f \tau} dud\tau,$$  

(28)

where the time-lag kernel is defined as

$$\rho(t, \tau) = \int_{-\infty}^{\infty} \phi(\nu, \tau) e^{j2\pi \nu t} d\nu,$$

and the Instantaneous Autocorrelation Function, (IAF) as

$$r_z(t, \tau) = z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2}).$$  

(29)

(Exercise TF6.1)

The four time-frequency domains

<table>
<thead>
<tr>
<th>IAF, $r_z(t, \tau)$</th>
<th>Ambiguity, $A_z(\nu, \tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{\nu \rightarrow t}$</td>
<td>$F_{\nu \rightarrow t}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$f$</td>
</tr>
</tbody>
</table>

Figure: The four possible domains in time-frequency analysis.
Example: The four domains

![Figure: Two Gaussian signal, one at $f = 0$, $t = 0$ and one shifted in frequency to $f = 0.2$ and in time to $t = 20$ and the representation in all four domains.](image1)

![Figure: A real-valued cosine signal with $f = 0.2$ and Gaussian envelope and the representation in all four domains.](image2)

**The kernel of the spectrogram**

The spectrogram using a window function does also belong to the quadratic class and thereby the window function can also be expressed as an ambiguity kernel $\phi_h(\nu, \tau)$ which is the ambiguity function of a window $h(t)$, i.e.,

$$\phi_h(\nu, \tau) = \int_{-\infty}^{\infty} h(-t - \frac{\tau}{2})h^*(t + \frac{\tau}{2})e^{-i2\pi\nu t} dt.$$  \hspace{1cm} (32)

The spectrogram does not fulfill the marginals as $\phi_h(0, \tau)$ and $\phi_h(\nu, 0)$ never can be one unless the window function $h(t) = \delta(t)$, which is quite a meaningless window.

**Multitaper spectrogram analysis**

The Slepian functions of the Thomson multitapers are well established in spectrum analysis today, and are recognized to give orthonormal spectra, (for white noise) as well as to be the most localized tapers in the frequency domain. In time-frequency analysis the Hermite functions have been shown to give the best time-frequency localization and orthonormality in the time-frequency domain.

It has been shown that the calculation of the two-dimensional convolution between the kernel and the Wigner spectrum estimate of a process realization can be simplified using kernel decomposition and calculating multitaper spectrograms.
Multitaper spectrogram analysis

Replacing $u = (t_1 + t_2)/2$ and $\tau = t_1 - t_2$ in

$$W_z^Q(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_z(u, \tau) \rho(t - u, \tau) e^{-i2\pi f \tau} du \, d\tau,$$

(33)
give

$$W_z^Q(t, f) = \int \int z(t_1) z^*(t_2) \rho(t - \frac{t_1 + t_2}{2}, t_1 - t_2) e^{-i2\pi f (t_1 - t_2)} dt_1 \, dt_2,$$

as

$$r_z(u, \tau) = z(u + \frac{\tau}{2}) z^*(u - \frac{\tau}{2}).$$

(34)

Multitaper spectrogram analysis

The kernel can be expressed as

$$\rho^{\text{rot}}(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k u_k^*(t_1) u_k(t_2).$$

The smoothed Wigner-Ville distribution is now rewritten as a weighted sum of spectrograms, $W_z^Q(t, f) =$

$$= \sum_{k=1}^{\infty} \lambda_k \int \int z(t_1) z^*(t_2) e^{-i2\pi f_1} e^{i2\pi f_2} u_k^*(t - t_1) u_k(t - t_2) dt_1 \, dt_2,$$

(35)

$$= \sum_{k=1}^{\infty} \lambda_k | \int z(t_1) e^{-i2\pi f_1} u_k^*(t - t_1) dt_1 |^2.$$

Multitaper spectrogram analysis

We define a rotated time-lag kernel as

$$\rho^{\text{rot}}(t_1, t_2) = \rho(t_1 + t_2/2, t_1 - t_2),$$

and get

$$W_z^Q(t, f) = \int \int z(t_1) z^*(t_2) \rho^{\text{rot}}(t_1, t_2) e^{-i2\pi f_1} e^{i2\pi f_2} dt_1 \, dt_2.$$

If the kernel $\rho^{\text{rot}}(t_1, t_2)$ satisfies the Hermitian property

$$\rho^{\text{rot}}(t_1, t_2) = (\rho^{\text{rot}}(t_2, t_1))^*,$$

then solving the integral

$$\int \rho^{\text{rot}}(t_1, t_2) u(t_1) dt_1 = \lambda \, u(t_2),$$

results in eigenvalues $\lambda_k$ and eigenfunctions $u_k(t)$ which form a complete set.