Recall that for squared error loss, the minimizer is
\[ \hat{v}(y) = E_0(\delta_0(X) | Y = y), \]
and the MRE is
\[ \hat{\delta}(x) = \delta_0(x) - E(\delta_0(X) | Y = y), \]
for an arbitrary fixed equivariant estimator.

**Theorem 8 (Pitman estimator)** Assume \( X_1, \ldots, X_n \) is a i.i.d. sample distributed according to location family, let \( Y = (X_1 - X_n, \ldots, X_{n-1} - X_n) \), and let \( L(\xi, d) = (\xi - d)^2 \). Then the MRE estimator \( \hat{\delta}(x) = \delta_0(x) - E_0(\delta_0(X) | Y) \) is given by
\[
\hat{\delta}(x) = \frac{\int uf(x_1 - u, x_n - u) \, du}{\int f(x_1 - u, \ldots, x_n - u) \, du}
\]
and called the Pitman estimator of \( \xi \).

**Proof.** Let \( \delta_0(x) = x_n \) and note that this is an equivariant estimator. Let \( y_1 = x_1 - x_n, \ldots, y_{n-1} = x_{n-1} - x_n, y_n = x_n \) which in matrix formulation is \( Y = AX \) with matrix \( A \) having Jacobian \( |A| = 1 \). Then the joint density of \( Y = (Y_1, \ldots, Y_n) \) is (by the change of variable formula)
\[
p_Y(y_1, \ldots, y_n) = f(y_1 + y_n, \ldots, y_{n-1} + y_n, y_n),
\]
and the conditional density of \( \delta_0(X) = X_n = Y_n \) given \( y = (y_1, \ldots, y_{n-1}) \) is
\[
f(y_1 + y_n, \ldots, y_{n-1} + y_n, y_n) / \int f(y_1 + t, \ldots, y_{n-1} + t, t) \, dt.
\]
Therefore
\[
E(\delta_0(X_n) | y) = \frac{\int tf(y_1 + t, \ldots, y_{n-1} + t, t) \, dt}{\int f(y_1 + t, \ldots, y_{n-1} + t, t) \, dt} = \frac{\int tf(x_1 - x_n + t, \ldots, x_{n-1} - x_n + t, t) \, dt}{\int f(x_1 - x_n + t, \ldots, x_{n-1} - x_n + t, t) \, dt}
\]
which with change variable \( u = x_n - t \) becomes
\[
\hat{\delta}(x) = \delta_0(x) - E(\delta_0(X) | y) = \frac{\int uf(x_1 - u, \ldots, x_n - u) \, du}{\int f(x_1 - u, \ldots, x_n - u) \, du},
\]
which ends the proof. \( \square \)
Example 21 Assume $X_1, \ldots, X_n$ are i.i.d. $\text{Un} (\xi - b / 2, \xi + b / 2)$ with $b$ assumed known and $\xi$ unknown. The joint density is

$$f(x_1 - \xi, \ldots, x_n - \xi) = \begin{cases} \frac{1}{b^n} & \text{if } \xi - \frac{b}{2} \leq x_{(1)} \leq x_{(n)} \leq \xi + \frac{b}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

where $x_{(1)} \leq \ldots \leq x_{(n)}$ is the ordered sample. Assume we have quadratic loss function $\rho(u) = u^2$. Then the MRE is then given by the Pitman estimator as

$$\hat{\delta}(x) = \frac{\int_{x_{(1)} - b/2}^{x_{(1)} + b/2} ub^{-n} du}{\int_{x_{(1)} - b/2}^{x_{(1)} + b/2} b^{-n} du} = \frac{\frac{u^2 x_{(1)} + b/2}{2}}{\frac{x_{(1)} - b/2}{2}} = \frac{1}{2} (x_{(1)} + x_{(n)}).$$

5.3 Randomized estimators and equivariance

Randomized estimators $\tilde{\delta}(X)$ based on a sample $X$ can be obtained using a deterministic rule $\delta$

$$\tilde{\delta}(X) = \delta(X, W)$$

with $W$ a r.v. that is independent of $X$ and with known distribution (i.e. whose distribution is not a function of the unknown parameter $\theta$).

One can define invariance of location family distributions and loss functions as before

$$f(x'; \xi') = f(x, \xi),$$
$$L(\xi', d') = L(\xi, d),$$

under transformations

$$x' = x + a,$$
$$\xi' = \xi + a,$$
$$d' = d + a.$$ 

One defines a randomized estimator to be equivariant if

$$\delta(X + a, W) = \delta(X, W) + a,$$
for all \(a\). As before one can show that bias, variance and risk are all constant for such estimators.

It is easily seen that the set of equivariant estimators is given by

\[
\{\delta(x, w) = \delta_0(x, w) + u(x, w) : u(x + a, w) = u(x, w), \ \forall x, w, a\}
\]

and with \(\delta_0\) a fixed equivariant estimator.

Again one can show that the condition \(u(x + a, w) = u(x, w), \forall x, w, a\), holds if and only if \(u\) is a function of \(y = (x_1 - x_n, \ldots, x_n - x_n)\), so that \(\delta(x, w)\) is equivariant if and only if

\[
\delta(x, w) = \delta_0(x, w) - v(y, w).
\]

Finally the MRE estimator is obtained by minimizing

\[
E_0(\rho \{\delta_0(X, W) - v(Y, W)\} \mid Y = y, W = w).
\]

But, start with a nonrandomized equivariant estimator \(\delta_0(X)\). Then

\[
E_0(\rho(\delta_0(X) - v(Y, W)) \mid Y = y, W = w) = E_0(\rho(\delta_0(X) - v(Y, W)) \mid Y = y),
\]

since \(y\) is a function of \(x\) and \(X\) and \(W\) are independent. Then the minimizing function \(\hat{v}\) will not depend on \(w\), and therefore it will be nonrandomized.

**Example 22** Assume \(\rho\) is quadratic loss function. Then

\[
\hat{v} = \arg\min_v E_0(\rho(\delta_0(X) - v(Y, W)) \mid Y = y, W = w)
\]

is given by

\[
\hat{v}(y, w) = E_0(\delta_0(X) \mid Y = y, W = w)
= E_0(\delta(X) \mid Y = y)
\]

and is not a function of \(w\). \(\square\)

Thus, starting with a nonrandomized equivariant estimator \(\delta_0(X)\), the MRE estimator \(\delta_0(X) - \hat{v}(Y) = \delta(X)\) is nonrandomized.

**5.3.1 Sufficiency and equivariance**

Assume \(\mathcal{P}\) is a location model family for distributions and that \(T\) is a sufficient statistic for \(\xi\). Then any estimator \(\delta(X)\) of \(\xi\) can be seen as a randomized estimator based on \(T\), since there is always a randomized estimator \(\tilde{\delta}(T) = \delta(X)\) and thus

\[
\delta(X) = \tilde{\delta}(T, W)
\]

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for a deterministic rule \( \hat{\delta} \), and with \( W \) a r.v. independent of \( T \) and with known distribution (so in particular not a function of \( \xi \)). Now assume that \( T = (T_1, \ldots, T_r) \) and equivariant, so that

\[
T(x + a) = T(x) + a.
\]

Then one sees that the distribution of \( T \) is a location family (show this!). But since \( \delta_0(X) = T(X) \) is equivariant and nonrandomized, one has that

\[
\hat{v}(y, w) = \arg\min_{v \in \mathbb{R}^{n-1}} \mathbb{E}_0(\rho(\delta_0(T) - v(Y, W)) | Y = y, W = w)
\]

is a function only of \( y \). Therefore the MRE estimator is given by

\[
\hat{\delta}(T) = \delta_0(T) - \hat{v}(Y)
\]

is a function only of \( T \).

This reasoning shows one connection between sufficiency and equivariance, namely that for a location family, if there is a sufficient statistic that is also equivariant, the the MRE estimator can be found to depend only on \( T \).

Are MRE estimators unbiased?

**Lemma 6** Assume we have squared error loss. Then

a) If \( \delta(X) \) is an equivariant estimator with bias \( b \), then \( \delta(X) - b \) is equivariant, unbiased and has smaller risk that \( \delta(X) \).

b) The (unique) MRE estimator is unbiased.

c) If a UMVU estimator exists and is equivariant then it is MRE.

**Proof.** a) Clearly \( \delta(X) - b \) is equivariant and unbiased. Then since bias, risk, and variance does not depend on \( \xi \)

\[
\mathbb{E}_0((\delta(X) - b)^2) = \text{Var}_0(\delta(X) - b) + \text{bias}(\delta(X) - b)^2 = \text{Var}_0(\delta(X))
\]

which is (uniformly in \( \xi \)) smaller than

\[
\mathbb{E}_0((\delta(X))^2) = \text{Var}_0(\delta(X)) + \text{bias}(\delta(X))^2.
\]

b) If \( \delta(X) \) is the MRE and it is not unbiased, so if it would have a bias \( b > 0 \), then \( \delta(X) - b \) would be unbiased, equivariant, with smaller risk which contradicts that \( \delta(X) \) is the MRE.

c) An unbiased minimum risk estimator that is equivariant is by b) the MRE. \( \square \)