1. (10p) Let $X$ and $Y$ have joint density

$$f_{XY}(x, y) = \begin{cases} \frac{c}{xy}, & 1 < y < x, \\ 0, & \text{otherwise}, \end{cases}$$

where $k > 1$ and $c = c(k)$ is a positive constant determined by the value $k$.

Determine for which values of $k$ the expectation of $X$ is finite, and compute for these values the conditional expectation $E\{X|Y = y\}$.

2. (15p) Let $X$ and $Y$ be random variables such that

$$Y | X = x \in N(x, \sigma^2), \quad \text{with } X \in N(0, \tau^2).$$

Determine the characteristic function $\phi_{X,Y}(s,t)$ of vector $(X,Y)$.

3. (15p) Let $X_1$ and $X_2$ be independent Exp(1)-distributed random variables. Show that $X(1)$ and $X(2) - X(1)$ are independent and determine their distributions.

4. (20p) Let $X_1, X_2, \ldots, X_n$ be a sample of $n$ independent observations of random variable $X$ (i.e., $X_1, X_2, \ldots$, are i.i.d. as $X$). Define the empirical distribution function

$$F_n(x) := \frac{\# \text{ observations which are less than } x}{n}.$$ 

Prove that $F_n(x)$ converges in probability to $F_X(x)$ for all $x \in \mathbb{R}$.

5. (20p) Suppose $X_1, X_2, \ldots$, are independent random variables identically distributed with density

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$ 

Let $N$ be the smallest index such that $X_N < 0$, i.e., $N = \min\{k \geq 1 : X_k < 0\}$. Compute the expectation of

$$Y = \sum_{k=1}^N X_k.$$ 

6. (20p) A particle is a subject to hits at the moments which form a Poisson process with intensity $\lambda$. Every hit moves the particle a horizontal, $N(0, \sigma^2)$-distributed distance. All displacements are independent. Let $S(t)$ be the position of particle at time $t$ assuming that $S(0) = 0$. Prove that the limit in distribution of

$$\frac{S(t)}{\sqrt{t}}$$

is a Normal random variable and find its parameters.

Hint: consider the characteristic function.