Lunds universitet
Matematisk statistik

Written exam in probability (5p) MAS203
1998–10–29 kl 9-14.00

Complete answers are required for all problems at the same level of detail. Only paper delivered by the department is allowed. In particular, your own paper is not allowed as scratch paper. Each sheet of paper should contain your full name. Each solution should start on top a new page. Pens or pencils in red colour are not allowed. Allowed tools: the sheet delivered by the department containing a table of distributions, and a hand calculator.

"VG" requires at least 75p and "G" requires at least 50p.

1. (10p) Let X and Y be independent $\mathcal{N}(0,1)$-distributed random variables. Find the distribution of $\sqrt{3}X + Y$ conditional on $X - \sqrt{3}Y = a$, where $a \in \mathbf{R}$.

2. (15p) Let X_1, X_2 and X_3 be independent $U(0,1)$-distributed random variables. Compute for any fixed $0 \leq x < 1$
$$\mathbf{P}\{X_{(3)} < \frac{2}{3} \mid X_{(1)} = x\},$$
where $X_{(1)}$ and $X_{(3)}$ denote the order statistics.

3. (15p) Assume that θ_k, $k = 1, 2, \ldots$, are independent random variables uniformly distributed on $[0, \pi]$. Find the limit in probability of
$$Y_n = \sin\left(\frac{n}{\sum_{k=1}^{n} \sin\theta_k}\right) \qquad \text{as} \quad n \to \infty.$$

4. (20p) A gardener has planted in April, 1999 one seed of an exotic plant. The plant from one seed can have only 1 or 2 flowers in the first summer with probabilities 1/2 and 1/4, correspondingly, otherwise it dies without blossoming. Each flower produces correspondingly 2 or 3 seeds with probabilities 1/2. During a winter the plant dies, but each seed survives with a probability 1/5.

   (10p) What is the expected number of the exotic flowers the gardener will have in the summer of 2000?

   (10p) What is the probability of the extinction of the population of the plants from this unique seed?

5. (20p) Let X_n be $Po\,(n^2)$-distributed random variable. Find the limiting distribution of
$$Y_n = \sqrt{X_n} - \frac{n^2}{\sqrt{X_n}} \qquad \text{as} \quad n \to \infty.$$

6. (20p) Assume that the locations of the gas-stations along the road form a Poisson process with intensity λ. At any gas-station one can get gas immediately (i.e. without waiting for others to be served) with probability 1/3.
   Sven is in a hurry. He got gas at one station and passed 200 km without seeing any other gas-station. He decides to turn back if he does not see along the following 100 km of drive a gas-station where he can be served immediately. Determine the value of λ so that with a probability at least 1/2 Sven will not turn back.