Lunds universitet
Matematisk statistik

Written exam in probability (5p) MAS203
2000–11–10 kl 9-14.00

Complete and clearly written answers are required for all problems at the same level of detail. Only paper delivered by the department is allowed. In particular, your own paper is not allowed as scratch paper. Each sheet of paper should contain your full name. Each solution should start on top of a new page. Pens or pencils in red colour are not allowed. Allowed tools: the sheet delivered by the department containing a table of distributions, and a hand calculator.

"VG" requires at least 75p and "G" requires at least 50p.

1. (15p) A plant produces Y offspring every year independent of others plants. Let Y be uniformly distributed over the numbers $\{0, 1, 2\}$. Given that in the beginning there are 3 plants in the garden, what is the probability that the population of these plants will survive for an arbitrary long time?

2. (15p) Assume that N_k, $k = 1, 2, \ldots$ are independent random variables distributed as $\Gamma(k, 1)$, respectively. Let $X_1, X_2, \ldots$ be independent $Exp(1)$-distributed random variables independent of N_k, $k = 1, 2, \ldots$. Find the limit in probability of

$$Y_k = \cos\left(\frac{\sum_{i=1}^{k}(X_i - 1)}{N_k}\right) \quad \text{as } k \to \infty.$$

3. (15p) Let $X_1 \sim \Gamma(k, 1)$ and $X_2 \sim \Gamma(k + n, 1)$ be independent random variables. Assuming $k > 0$ and $n > 0$ show that X_1/X_2 and $X_1 + X_2$ are independent.

4. (15p) Let T be the time of the winner of the New York marathon. Assume that the next two best times are distributed uniformly over $[T, T + \delta]$, where $\delta = 30$ min. Given that the second best time is $T + \delta/3$ find the probability that the third best time is less then $T + \delta/2$.

5. (20p) Let $X_1, X_2, \ldots$ be independent $U(0, 1)$-distributed random variables. Let

$$Z_n := \max_{1 \le k \le n} X_k - \min_{1 \le k \le n} X_k.$$

   Find out which limits exist as $n \to \infty$ (*a.s.*, in probability, or in distribution) and determine them.

6. (20p) The traffic light works with a constant rate: it shows green for 5 minutes and then it shows red or yellow for the following 5 min, and so on. The cars arrive at the traffic light at the moments which form a Poisson process with intensity 2 cars every minute. If the light is green the car passes, otherwise it stops. If 10 or less cars are waiting for the green, we assume that this does not disturb traffic for the next green period. If more than 10 cars are waiting for the green light, then only 10 can pass and the rest have to wait for the next turn of green.

   Just as the traffic light turns off green, you see out of the window that there are no cars at the traffic light at all. You arrive in your car to the same traffic light exactly 15 minutes later. What is the expected number of the cars waiting in front of you?