Suggestions for solutions. Answers are underlined.

1. Make the transformation

\[
\begin{pmatrix}
  U \\
  V
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 1 \\
  1 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
  X_1 \\
  X_2 \\
  X_3
\end{pmatrix} = A
\begin{pmatrix}
  X_1 \\
  X_2 \\
  X_3
\end{pmatrix} \sim \mathcal{N}(0, AA^T) = \mathcal{N}\left(0, \begin{pmatrix} 3 & 3 & 3 \\
  3 & 5 & 5 \\
  3 & 5 & 5 \end{pmatrix}\right).
\]

Now, using results on conditional distributions for multivariate normal vectors (see lecture notes) gives immediately

\[
U | V = 1 \sim \mathcal{N}\left(0 + \frac{3}{5} (1 - 0), 3 - \frac{3^2}{5}\right) = \mathcal{N}\left(\frac{3}{5}, \frac{6}{5}\right).
\]

2. Developing the parenthesis yields

\[
\frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k^2 - 2X_k\bar{X}_n + \bar{X}_n^2) = \frac{1}{n-1} \left( \sum_{k=1}^{n} X_k^2 - 2\bar{X}_n \sum_{k=1}^{n} X_k + n\bar{X}_n^2 \right)
\]

\[
= \frac{n}{n-1} \left( \frac{1}{n} \sum_{k=1}^{n} X_k^2 - \bar{X}_n^2 \right).
\]

By the law of large numbers we have that $\bar{X}_n \xrightarrow{P} \mu$ as $n$ tends to infinity, which in its turn implies that $\bar{X}_n^2 \xrightarrow{P} \mu^2$. In addition,

\[
\frac{1}{n} \sum_{k=1}^{n} X_k^2 \xrightarrow{P} \mathbb{E}[X_1^2] = \sigma^2 + \mu^2.
\]

Thus, using results on convergence of sums and products of $\mathbb{P}$-convergent sequences, we obtain, since the ratio $n/(n-1)$ tends to 1 as $n$ tends to infinity,

\[
\frac{n}{n-1} \left( \frac{1}{n} \sum_{k=1}^{n} X_k^2 - \bar{X}_n^2 \right) \xrightarrow{P} 1 \cdot (\sigma^2 + \mu^2 - \mu^2) = \sigma^2.
\]

This completes the proof.

3. Note that there is a binomial experiment at each round. Hence, recall that the expectation and variance of a binomial variable $Z \sim \text{Bin}(n, p)$ are given by $\mathbb{E}[Z] = np$ and $\text{Var}[Z] = np(1-p)$, respectively. Let $Y$ denote the number of balloons hit by Herr Bom/Mister Missit in the first round. Then, since there are $20 - Y$ balloons left in the second round and the probability of hitting each of these is still 0.6,

\[
\mathbb{E}[X | Y] = Y + (20 - Y) \cdot 0.6 = 12 + 0.4Y.
\]

Moreover,

\[
\text{Var}[X | Y] = (20 - Y) \cdot 0.6 \cdot 0.4 = 4.8 - 0.24Y.
\]

\footnote{In the following the notation $X \overset{d}{=} Y$ means that $X$ and $Y$ have the same distribution.}
Now, using results on conditional expectations and variances from the lectures combined with the fact that $Y \overset{d}{=} \text{Bin}(20, 0.6)$,

$$
\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[12 + 0.4 Y] = 12 + 0.4 \mathbb{E}[Y] = 12 + 20 \cdot 0.24 = 16.8.
$$

In addition,

$$
\text{Var}[X] = \mathbb{E}[\text{Var}[X|Y]] + \text{Var}[\mathbb{E}[X|Y]] = \mathbb{E}[4.8 - 0.24 Y] + \text{Var}[12 + 0.4 Y] \\
= 4.8 - 0.24 \mathbb{E}[Y] + 0.4^2 \text{Var}[Y] = 4.8 - 0.24 \cdot 20 \cdot 0.6 + 0.4^2 \cdot 20 \cdot 0.24 = 2.688.
$$

4. We apply the transformation theorem to the mapping

\[ \begin{cases} X_1 = \sqrt{-2 \log U_1 \cos(2\pi U_2)}, \\ X_2 = \sqrt{-2 \log U_1 \sin(2\pi U_2)}. \end{cases} \]

To find the inverse, write

\[ X_1^2 + X_2^2 = -2 \log U_1 \cos^2(2\pi U_2) - 2 \log U_1 \sin^2(2\pi U_2) = -2 \log U_1 \iff U_1 = \exp \left( -\frac{1}{2} \{X_1^2 + X_2^2\} \right). \]

In addition,

\[ \frac{X_2}{X_1} = \frac{\sin(2\pi U_2)}{\cos(2\pi U_2)} = \tan(2\pi U_2) \iff U_2 = \begin{cases} \frac{1}{2\pi} \arctan \frac{X_1}{X_2} & \text{for } X_1 > 0, X_2 > 0, \\ \frac{1}{2\pi} \arctan \frac{X_1}{X_2} + 1 & \text{for } X_1 > 0, X_2 < 0, \\ \frac{1}{2\pi} \arctan \frac{X_1}{X_2} + \frac{1}{2} & \text{for } X_1 < 0, \end{cases} \]

yielding the Jacobian determinant

\[ J = \left| -x_1 \exp(-\{x_1^2 + x_2^2\}/2) \quad -x_2 \exp(-\{x_1^2 + x_2^2\}/2) \right| \]
\[ = -\frac{1}{2\pi} \exp(-\{x_1^2 + x_2^2\}/2) \cdot \left( -\frac{1}{2\pi} \right) = -\frac{1}{2\pi} \exp \left( -\frac{1}{2} \{x_1^2 + x_2^2\} \right). \]

Finally, since $f_{U_1, U_2}(u_1, u_2) \equiv 1$ for all $0 \leq u_1, u_2 \leq 1$, we obtain, by applying the transformation theorem, the joint density

\[ f_{X_1, X_2}(x_1, x_2) = f_{U_1, U_2}(u_1(x_1, x_2), u_2(x_1, x_2)) |J| = |J| = \frac{1}{2\pi} \exp \left( -\frac{1}{2} \{x_1^2 + x_2^2\} \right) \]

as well as the marginals $f_{X_1}(x_1) = 1/\sqrt{2\pi} \exp(-x_1^2/2)$ and $f_{X_2}(x_2) = 1/\sqrt{2\pi} \exp(-x_2^2/2)$. This shows that $X_1 \overset{d}{=} \mathcal{N}(0, 1)$, $X_2 \overset{d}{=} \mathcal{N}(0, 1)$ and, since $f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$, that $X_1$ and $X_2$ are independent.

5. The key result for solving this problem is that $(T_k|X(1) = n) \overset{d}{=} \beta(k, n + 1 - k)$ for any Poisson process occurrence time $T_k$, $1 \leq k \leq n$ (see the lecture notes). Thus, by looking at the expression of the expected value for the $\beta$-distribution in the list of distributions, $\mathbb{E}[T_k|X(1) = n] = k/(n + 1)$.

(a) Using the result above we conclude that $\mathbb{E}[T_2|X(1) = 2] = 2/3$ and $\mathbb{E}[T_1|X(1) = 2] = 1/3$. This gives immediately

$$
\mathbb{E}[T_2 - T_1|X(1) = 2] = \mathbb{E}[T_2|X(1) = 2] - \mathbb{E}[T_1|X(1) = 2] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.
$$

\[ \text{Here } \log \text{ denotes the natural logarithm.} \]
(b) By shifting time-scale from ‘hours’ to ‘working days’ we obtain a new Poisson process \( \{X(t)\}_{t \geq 0} \) with intensity \( \tilde{\lambda} = 6\lambda \). Then, conditioning on the number of customers arrived at the end of the day,

\[
\mathbb{E}[T] = \sum_{n=0}^{\infty} \mathbb{E}[T|X(1) = n] \mathbb{P}(X(1) = n) = \sum_{n=0}^{\infty} \left( 1 - \mathbb{E}[T_n|X(1) = n] \right) \mathbb{P}(X(1) = n).
\]

Plugging the expressions \( \mathbb{E}[T_n|X(1) = n] = n/(n + 1) \) and \( \mathbb{P}(X(1) = n) = \exp(-\tilde{\lambda})\tilde{\lambda}^n/n! \) into the formula above yields

\[
\mathbb{E}[T] = \sum_{n=0}^{\infty} \left( 1 - \frac{n}{n+1} \right) \exp(-\tilde{\lambda})\frac{\tilde{\lambda}^n}{n!} = \exp(-\tilde{\lambda}) \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n}{(n+1)!}.
\]

We are done if we can compute the series on the right hand side. However, juggling around with indices gives

\[
\exp(-\tilde{\lambda}) \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n}{(n+1)!} = \exp(-\tilde{\lambda})\tilde{\lambda}^{-1} \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^{n+1}}{(n+1)!} = \exp(-\tilde{\lambda})\tilde{\lambda}^{-1} \sum_{n=1}^{\infty} \frac{\tilde{\lambda}^n}{n!} = \exp(-\tilde{\lambda})\tilde{\lambda}^{-1} \left( \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n}{n!} - 1 \right).
\]

Now, note that \( \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n}{n!} = \exp(\tilde{\lambda}) \). The right hand side can hence be expressed as

\[
\exp(-\tilde{\lambda})\tilde{\lambda}^{-1} \left( \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n}{n!} - 1 \right) = \exp(-\tilde{\lambda})\tilde{\lambda}^{-1}(\exp(\tilde{\lambda}) - 1) = \tilde{\lambda}^{-1}(1 - \exp(-\tilde{\lambda})).
\]

Finally, replacing \( \tilde{\lambda} \) by \( 6\lambda \) gives the answer \( \mathbb{E}[T] = (1/6\lambda)(1 - \exp(-6\lambda)) \).