Theorem 1 (Inverse method)**.** *Define for* $u \in (0,1)$ *the* **general inverse**

$$F^{\leftarrow}(u) \stackrel{def}{=} \inf\{x \in \mathbb{R} : F(x) \geq u\}$$

and

draw $U \sim \mathcal{U}(0,1)$
set $X \leftarrow F^{\leftarrow}(U)$
return X

The output X *has distribution function* F*.*

Proof of the inverse method

Now we want to prove that if $U \in \mathcal{U}(0,1)$ then $X = F^{\leftarrow}(U)$ has the correct distribution i.e.

$$\mathbb{P}(F^{\leftarrow}(U) \leq x) = F(x).$$

First we for $x \in \mathbb{R}$ and $u \in (0,1)$ need to establish the relation

$$F(x) \geq u \Longleftrightarrow x \geq F^{\leftarrow}(u).(*)$$

Define the set S_u as $S_u := \{x' \in \mathbb{R} : F(x') \geq u\}$.

$\Rightarrow$: Easy as $F(x) \geq u \Rightarrow x \in S_u$.
$\Rightarrow$: $x \geq \inf S_u = F^{\leftarrow}(u)$.

$\Leftarrow$: A distribution function is (i) Monotonically increasing (ii) Right continuous.

Right continuity of F gives closed left end point $\Rightarrow\ \inf S_u \in S_u$. This gives that the set S_u is of the form $[F^{\leftarrow}(u), \infty)$. So $x \geq F^{\leftarrow}(u)$ implies $x \in S_u$.

For all points x in the set S_u we have that $F(x) \geq u$.
Thus, $x \geq \inf S_u = F^{\leftarrow}(u) \Rightarrow\ F(x) \geq u$.

Now

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(F^{\leftarrow}(U) \leq x) \stackrel{*}{=} \mathbb{P}(F(x) \geq U) = \mathbb{P}(U \leq F(x)) = F(x)$$

which was what to be shown.