Theorem 1 (Inverse method). Define for $u \in (0, 1)$ the general inverse

$$F^-(u) \overset{\text{def}}{=} \inf\{x \in \mathbb{R} : F(x) \geq u\}$$

and

draw $U \sim U(0,1)$
set $X \leftarrow F^-(U)$
return $X$

The output $X$ has distribution function $F$.

Proof of the inverse method

Now we want to prove that if $U \in U(0,1)$ then $X = F^-(U)$ has the correct distribution i.e.

$$P(F^-(U) \leq x) = F(x).$$

First we for $x \in \mathbb{R}$ and $u \in (0, 1)$ need to establish the relation

$$F(x) \geq u \iff x \geq F^-(u).(*)$$

Define the set $S_u$ as $S_u := \{x' \in \mathbb{R} : F(x') \geq u\}$.

$\Rightarrow$: Easy as $F(x) \geq u \Rightarrow x \in S_u$.
$\Rightarrow$: $x \geq \inf S_u = F^-(u)$.

$\Leftarrow$: A distribution function is (i) Monotonically increasing (ii) Right continuous.

Right continuity of $F$ gives closed left end point $\Rightarrow \inf S_u \in S_u$. This gives that the set $S_u$ is of the form $[F^-(u), \infty)$. So $x \geq F^-(u)$ implies $x \in S_u$.

For all points $x$ in the set $S_u$ we have that $F(x) \geq u$.
Thus, $x \geq \inf S_u = F^-(u) \Rightarrow F(x) \geq u$.

Now

$$F_X(x) = P(X \leq x) = P(F^-(U) \leq x) = P(F(x) \geq U) = P(U \leq F(x)) = F(x)$$

which was what to be shown.