1. Your bank offers you a derivative called a Inverse Broken Wing Butterfly. The derivative has maturity $T$ and the following pay-off function:

$$
\begin{align*}
0 & \quad S(T) \leq K \\
K - S(T) & \quad K \leq S(T) \leq K + \Delta, \\
S(T) - K - 2\Delta & \quad K + \Delta \leq S(T) \leq K + 3\Delta, \\
\Delta & \quad K + 3\Delta \leq S(T)
\end{align*}
$$

where $K$ and $\Delta$ are positive constants. In practice $\Delta$ is usually small compared to $K$ and $K$ is usually chosen close to the current stock price. Show how you can replicate this pay-off by using standard contracts. You are allowed to use the stock $S$, European put and call options, and a zero-coupon bond each with maturity $T$.

2. Define the process $M$ for $t \geq 0$ as

$$
M(t) = e^{2t} \left(\cos(W(t))^2 - \frac{1}{2}\right),
$$

where $W$ is a standard Brownian motion. Show that $M$ is a martingale for $t \geq 0$.

3. Consider the following risk-neutral dynamics for the short rate;

$$
\begin{align*}
\text{d}r(u) &= (ae^{-u} + \sigma^2 u) \text{d}u + \sigma \text{d}W(u), \quad t \leq u \leq T, \\
r(t) &= r,
\end{align*}
$$

where $W$ is a standard $Q$ Brownian motion and $a$, $r$ and $\sigma$ are deterministic constants. Find the fair value of the ZCB, $p(u, S)$ for $t \leq u \leq S \leq T$.

4. Solve the PDE for $0 < t < T$

$$
\begin{align*}
rf(t, x) &= \frac{\partial}{\partial t}f(t, x) + rx\frac{\partial}{\partial x}f(t, x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2}f(t, x), \\
f(T, x) &= (x - K)^2,
\end{align*}
$$

where $K$ is a positive constant.

Please turnover!
5. Assume the following 2-dim Black-Scholes model (under $Q$) for the two stocks $S_1$, $S_2$ and the bank account $B$, for $t \leq u \leq T$,

\[
\begin{align*}
    dS_1(u) & = rS_1(u)du + S_1(u)(\sigma_{11}dW_1(u) + \sigma_{12}dW_2(u)), \\
    dS_2(u) & = rS_2(u)du + S_2(u)(\sigma_{21}dW_1(u) + \sigma_{22}dW_2(u)), \\
    dB(u) & = rB(u)du,
\end{align*}
\]

where $W_1$ and $W_2$ are two independent standard $Q$ Brownian motions and where $r$, $\sigma_{12}$, $\sigma_{21}$ are real valued constants and where $a$, $b$, $\sigma_{11}$ and $\sigma_{22}$ are positive constants.

(a) Find the fair value the derivative with maturity $T$ and pay-off:

\[
    \Phi(S_1(T), S_2(T)) = \max(aS_1(T) - bS_2(T), 0),
\]

for $0 < t < T$. (12)

(b) Calculate the replicating portfolio (delta-hedge) for the derivative, that is find a self-financing portfolio consisting of the stocks and the bank account that hedges the derivative. (8)

6. In this problem we want to price and hedge an option on a ZCB. Let $p_1(t)$ be the price at time $t$ of a ZCB with maturity $T_1$ and let $p_2(t)$ be the price at time $t$ of a ZCB with maturity $T_2$. We here want to price an option on the ZCB $p_2$ with maturity $T_1 < T_2$. The easiest way to accomplish this is to express the dynamics of the underlying discounted ZCB

\[
    Z(s) = \frac{p_2(s)}{p_1(s)} = \exp\left( -\int^{T_2}_{T_1} f(s, T) \, dT \right),
\]

under the forward measure $Q^{T_1}$, i.e. the martingale measure which has the ZCB $p_1$ as numeraire. For $t \leq s \leq T_1 \leq T$ the forward rate under $Q^{T_1}$ satisfies the following Gaussian HJM model,

\[
    df(s, T) = \frac{1}{2} \left( \int^{T_1}_{T} \sigma(s, u) \, du \right)^2 ds + \sigma(s, T) \, dW^{Q^{T_1}}(s),
\]

where $\sigma$ is a $1 \times d$-dimensional deterministic function and where $W^{Q^{T_1}}$ is a $d$-dimensional standard $Q^{T_1}$ Brownian motion.

(a) Find the fair value of the derivative with maturity $T_1$ and pay-off

\[
    \Phi(p_2(T_1)) = \max(p_2(T_1) - K, 0) = \max(\varepsilon(T_1) - K, 0),
\]

for $0 < t < T_1$. (12)

(b) Find a replicating portfolio for this contract by trading in $p_1$ and $p_2$. (8)

Good luck