Valuation of derivative assets
Recapitulation

October 21, 2019
Fundamental economical concepts

- Financial contract/Contingent claim
- Self financing portfolio
- Arbitrage
- Replicating portfolio/Hedge
- Complete market
Portfolio (B: Def 6.2 p. 87)

Let \( \{S_t\}_{t \geq 0} \) be an \( N \)-dimensional price process.

1. A portfolio \( \{h_t\}_{t \geq 0} \) is an \( N \)-dim adapted process.
2. The corresponding value process \( \{V_t^h\}_{t \geq 0} \) is given by

\[
V_t^h = \sum_{i=1}^{N} h_t^i S_t^i
\]

3. A portfolio is **self-financing** if

\[
V_{t+\Delta}^h - V_t^h = \sum_{i=1}^{N} h_t^i (S_{t+\Delta}^i - S_t^i) \quad \text{discrete time}
\]

\[
dV_t^h = \sum_{i=1}^{N} h_t^i dS_t^i \quad \text{continuous time}
\]
Contingent Claim (B: Def 7.4 p. 94)

Let \( \{F^S_t\}_{t \geq 0} \) be the filtration generated by the asset process \( S \).

A **contingent claim** with maturity \( T \) is any \( F^S_T \)-measurable r.v. \( X \).

\( X \) is a **simple claim** if \( X = \Phi(S_T) \), where \( \Phi \) is called a contract function.
Important financial contracts

Forward $\Phi(S_T) = S_T - K$

European call $\Phi(S_T) = \max(S_T - K, 0)$

European put $\Phi(S_T) = \max(K - S_T, 0)$

Asian arithmetic call $\Phi = \max\left(\frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K, 0\right)$
or $\Phi = \max\left(\frac{1}{T} \int_{0}^{T} S_t \, dt - K, 0\right), \quad 0 \leq t_1 < t_2 < \cdots < t_n \leq T$

Asian arithmetic put $\Phi = \max(K - \frac{1}{n} \sum_{i=1}^{n} S_{t_i}, 0)$
or $\Phi = \max(K - \frac{1}{T} \int_{0}^{T} S_t \, dt, 0)$

Asian geometric call $\Phi = \max\left(\exp\left(\frac{1}{n} \sum_{i=1}^{n} \ln(S_{t_i})\right) - K, 0\right)$
or $\Phi = \max\left(\exp\left(\frac{1}{T} \int_{0}^{T} \ln(S_t) \, dt\right) - K, 0\right)$

Asian geometric put $\Phi = \max(K - \exp\left(\frac{1}{n} \sum_{i=1}^{n} \ln(S_{t_i})\right), 0)$
or $\Phi = \max(K - \exp\left(\frac{1}{T} \int_{0}^{T} \ln(S_t) \, dt\right), 0)$
An **arbitrage opportunity** is a self-financing portfolio $h$ with value process $V^h$ such that

i) $V^h_0 = 0$,

ii) $\mathbb{P}(V^h_t \geq 0) = 1$,

iii) $\mathbb{P}(V^h_t > 0) > 0$,

for some $t > 0$.

If there does not exist any arbitrage opportunities on a market, the market is called **free of arbitrage**.
Locally risk-free assets

A self-financing portfolio \( h \) is **locally risk-free** if

\[
dV_t^h = k(t)V_t^h \, dt,
\]

where \( k \) is an adapted process.

**Proposition 7.6 (B: p. 97)** If \( h \) is locally risk-free then \( k(t) \) should equal the short rate \( r(t) \) for all \( t \) in order to avoid arbitrage opportunities.

Remember that this is only relevant if \( V^h(0) \neq 0 \).
Let $X$ be a contingent claim with maturity $T$. If there exists a self-financing portfolio $h$ such that

$$\mathbb{P}(V_T^h = X) = 1,$$

then $h$ is a **hedge** or a **replicating portfolio** for the contingent claim $X$. 

---

**Hedge/Replicating portfolio**

(B: Def 8.1 p. 115)
Arbitrage principle

If a contingent claim has a replicating portfolio then the value of the contingent claim must equal the value of the replicating portfolio at all times to avoid arbitrage.

Motivation: If the values differ sell the expensive one and buy the cheap one, put the positive difference into the bank account. At maturity the replicating portfolio exactly off sets the contingent claim, but we still have the positive difference which now has earned interest rate in the Bank account.
Complete and incomplete markets

If all contingent claims on an arbitrage free market have replicating portfolios then the market is said to be **complete** otherwise the market is called **incomplete**.
Tools and concepts from stochastic calculus

- Martingales
- Ito’s formula
- Ito isometry
- Feynman-Kac representation
- Change of measure (Girsanov transformation)
- Change of numeraire
Martingales

Discrete time
Let \( \{\mathcal{F}_t\}_{t=0,1,\ldots} \) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). A process \( \{X_t\}_{t=0,1,\ldots} \) is a martingale w.r.t \( \mathcal{F}_t \) if

i) \( X_t \) is adapted to \( \mathcal{F}_t \) for \( t = 0, 1, \cdots \),

ii) \( \mathbb{E}^\mathbb{P}[|X_t|] < \infty \), for \( t = 0, 1, \cdots \),

iii) \( \mathbb{E}^\mathbb{P}[X_{t+1}|\mathcal{F}_t] = X_t \) for \( t = 0, 1, \cdots \).

Continuous time
Let \( \{\mathcal{F}_t\}_{t\geq 0} \) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). A process \( \{X_t\}_{t\geq 0} \) is a martingale w.r.t \( \mathcal{F}_t \) if

i) \( X_t \) is adapted to \( \mathcal{F}_t \) for \( t \geq 0 \),

ii) \( \mathbb{E}^\mathbb{P}[|X_t|] < \infty \), for \( t \geq 0 \),

iii) \( \mathbb{E}^\mathbb{P}[X_{t+1}|\mathcal{F}_s] = X_s \) for all \( s \leq t \) for each \( t \geq 0 \).
Itô stochastic integral

**Prop 4.4 p 45 + Cor 4.8 p. 48**
Let \( \{\mathcal{F}_t\}_{t \geq 0} \) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( \{W_t\}_{t \geq 0} \) is a Brownian motion w.r.t. \( \mathcal{F}_t \). Suppose that \( g_t \) is adapted to \( \mathcal{F}_t \) and that

\[
\int_0^t \mathbb{E}_\mathbb{P}[g_s^2] \, ds < \infty, \quad 0 \leq t \leq T
\]

then the Itô stochastic integral \( \int_0^t g_s \, dW_s, \quad 0 \leq t \leq T \) is well-defined and the following properties hold

i) \( I_t \) is a martingale w.r.t. \( \mathcal{F}_t \), for \( 0 \leq t \leq T \),

ii) \( \mathbb{E}_\mathbb{P}[I_t] = 0 \), for \( 0 \leq t \leq T \),

iii) \( \mathbb{E}_\mathbb{P}[I_t^2] = \int_0^t \mathbb{E}_\mathbb{P}[g_s^2] \, ds \), for \( 0 \leq t \leq T \) (Itô isometry).
One dimensional Ito formula

Suppose \( X_t \) satisfies the SDE

\[
\begin{align*}
\frac{dX_t}{dt} &= \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \\
X_0 &= x_0.
\end{align*}
\]

If \( Y_t = f(t, X_t) \) where \( f(t, x) \) is once continuously differentiable w.r.t \( t \) and twice continuously differentiable w.r.t \( x \) then \( Y_t \) satisfies the SDE

\[
\begin{align*}
\frac{dY_t}{dt} &= \left( f_t(t, X_t) + f_x(t, X_t)\mu(t, X_t) + f_{xx}(t, X_t)\sigma(t, X_t)^2/2 \right) \, dt \\
&\quad + f_x(t, X_t)\sigma(t, X_t) \, dW_t \\
Y_0 &= f(0, x_0),
\end{align*}
\]

where \( f_t(t, X_t) = \left. \frac{\partial f(t,x)}{\partial t} \right|_{x=X_t} \), \( f_x(t, X_t) = \left. \frac{\partial f(t,x)}{\partial x} \right|_{x=X_t} \)

and \( f_{xx}(t, X_t) = \left. \frac{\partial^2 f(t,x)}{\partial x^2} \right|_{x=X_t} \).
How to remember Ito’s formula

Take a second order Taylor expansion of \( f(t, x) \) in \( t \) and \( x \)

\[
f(t + dt, X_t + dX_t) - f(t, X_t) = f'_t(t, X_t) dt + f'_x(t, X_t) dX_t \\
+ f''_{tt}(t, X_t) \frac{dt^2}{2} + 2f''_{tx}(t, X_t) \frac{dt \, dX_t}{2} + f''_{xx}(t, X_t) \frac{dX_t^2}{2}
\]

plugin in what \( dX_t \) is, use the rules of calculation below:

**Box algebra** (multiplication table)

<table>
<thead>
<tr>
<th>×</th>
<th>( dt )</th>
<th>( dW_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dt )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( dW_t )</td>
<td>0</td>
<td>( dt )</td>
</tr>
</tbody>
</table>

The underlined terms could therefore be deleted!
Suppose that $X_t = (X^i_t, \cdots, X^N_t)'$ satisfies the N-dimensional SDE

$$\begin{align*}
d X_t &= \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \\
X_0 &= x_0,
\end{align*}$$

where $\mu$ is a $N$-dim column vector and $\sigma$ an $N \times d$-dim matrix and $W_t$ is $d$-dim Brownian motion.

If $Y_t = f(t, X_t) : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ where $f(t, x)$ is once continuously differentiable w.r.t $t$ and twice continuously differentiable w.r.t $x$ then $Y_t$ satisfies the SDE

$$\begin{align*}
d Y_t &= \mu_f(t, X_t) \, dt + \sigma_f(t, X_t) \, dW_t \\
Y_0 &= f(0, x_0),
\end{align*}$$
Multi dimensional Ito formula (cont)

where

\[
\begin{align*}
\mu_f &= \left( \frac{\partial f(t,x)}{\partial t} + \sum_{i=1}^{N} \mu^i(t, X_t) \frac{\partial f(t,x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij}(t,x) \frac{\partial^2 f(t,x)}{\partial x_i \partial x_j} \right) |_{x=X_t}, \\
\sigma_f &= \sum_{i=1}^{N} \frac{\partial f(t,x)}{\partial x_i} \sigma_i(t, X_t) |_{x=X_t}, \\
\text{and} \\
C(t,x) &= \sigma(t,x) \sigma(t,x)^*. 
\end{align*}
\]
How to remember Ito’s formula M-dim version

Assume \( f : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R} \) and that all BM are independent

Take a second order Taylor expansion of \( f(t, x) \) in \( t \) and \( x \)

\[
f(t + dt, X_t + dX_t) - f(t, X_t) = f'_t(t, X_t) \, dt + (\nabla_x f)(t, X_t) \, dX_t + \frac{f''_{tt}(t, X_t) \, dt^2}{2} + 2(\nabla_x f'_t)(t, X_t) \frac{dt \, dX_t}{2} + \frac{dX^*_t(\nabla^2_x f)(t, X_t) \, dX_t}{2}
\]

plugin in what \( dX_t \) is, use the rules of calculation below:

**Box algebra** (multiplication table) \( (i \neq j) \)

<table>
<thead>
<tr>
<th>( \times )</th>
<th>( dt )</th>
<th>( dW_i(t) )</th>
<th>( dW_j(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dt )</td>
<td>0</td>
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</tr>
<tr>
<td>( dW_i(t) )</td>
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<td>( dt )</td>
<td>0</td>
</tr>
<tr>
<td>( dW_j(t) )</td>
<td>0</td>
<td>0</td>
<td>( dt )</td>
</tr>
</tbody>
</table>

The underlined terms could therefore be deleted!
Geometric Brownian motion

Suppose that $X$ satisfies the SDE:

$$
\begin{align*}
\mathrm{d}X(u) &= \mu X(u) \mathrm{d}u + \sigma X(u) \mathrm{d}W(u), \quad u > t \\
X(t) &= x.
\end{align*}
$$

This SDE has the solution:

$$
X(u) = xe^{\left(\mu - \frac{\sigma^2}{2}\right)(u-t) + \sigma(W(u) - W(t))}.
$$

Marginal distribution:

$$
X(u) \overset{d}{=} xe^{\left(\mu - \frac{\sigma^2}{2}\right)(u-t) + \sigma \sqrt{u-t} G},
$$

where $G \in \mathcal{N}(0, 1)$. 

Recapitulation
Geometric Brownian motion time varying case

Suppose that $X$ satisfies the SDE:

$$
\begin{align*}
    dX(u) &= \mu(u)X(u)\,du + \sigma(u)X(u)\,dW(u), \quad u > t \\
    X(t) &= x.
\end{align*}
$$

This SDE has the solution:

$$
X(u) = xe^{\int_t^u \left( \mu(s) - \frac{\sigma^2(s)}{2} \right) \,ds + \int_t^u \sigma(s) \,dW(s)}.
$$

Marginal distribution:

$$
X(u) \overset{d}{=} xe^{\int_t^u \left( \mu(s) - \frac{\sigma^2(s)}{2} \right) \,ds + \sqrt{\int_t^u \sigma(s)^2 \,ds} G},
$$

where $G \in N(0, 1)$. 

Suppose that $X$ satisfies the SDE:

$$
\begin{align*}
\frac{dX(u)}{du} &= \kappa (\theta - X(u)) \, du + \sigma \, dW(u), \quad u > t \\
X(t) &= x.
\end{align*}
$$

This SDE has the solution:

$$
X(u) = \theta \left(1 - e^{-\kappa(u-t)}\right) + e^{-\kappa(u-t)} x + \int_t^u e^{-\kappa(u-s)} \sigma \, dW(s).
$$

Marginal distribution:

$$
X(u) \overset{d}{=} \theta \left(1 - e^{-\kappa(u-t)}\right) + e^{-\kappa(u-t)} x + \sqrt{\frac{\sigma^2}{2\kappa}} \left(1 - e^{-2\kappa(u-t)}\right) G,
$$

where $G \in \mathcal{N}(0,1)$. 
Assume that $F$ is a solution to the boundary value problem

$$\frac{\partial F(t, x)}{\partial t} = -\mu(t, x) \frac{\partial F(t, x)}{\partial x} - \frac{1}{2} \sigma(t, x)^2 \frac{\partial^2 F(t, x)}{\partial x^2} + rF(t, x)$$
$$F(T, x) = \Phi(x)$$

and assume that $E \left[ \int_0^T \sigma(t, X_s)^2 \left( \frac{\partial F(s, x)}{\partial x} \right)^2_{x=X_s} ds \right] < \infty$. Then $F$ has the representation

$$F(t, x) = \exp(-(T-t)r)E[\Phi(X_T) \mid X_t = x],$$

where

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad t \leq s \leq T,$$
$$X_t = x$$
Black-Scholes equation (B: Thm 7.7 p. 101)

Assume that we have a market consisting of a risky asset $S$ and a bank-account $B$, where the corresponding $\mathbb{P}$-dynamics are given as

\[
\begin{align*}
\text{d}S_t &= S_t \mu(t, S_t) \text{d}t + S_t \sigma(t, S_t) \text{d}W_t, \\
S_0 &= s_0 \\
\text{d}B_t &= rB_t \text{d}t, \\
B_0 &= 1.
\end{align*}
\]

Assume that $F(t, s)$ is the price at time $t$ of a simple claim with maturity $T$ of the form $F(T, s) = \Phi(s)$. Then the price $F$ is a solution to the boundary value problem

\[
\begin{align*}
\frac{\partial F(t, s)}{\partial t} &= -rs \frac{\partial F(t, s)}{\partial s} - \frac{1}{2}s^2 \sigma(t, s)^2 \frac{\partial^2 F(t, s)}{\partial s^2} + rF(t, s) \\
F(T, s) &= \Phi(s).
\end{align*}
\]
Replicating portfolio or Hedge

Let $X$ be a simple claim with contract function $\Phi$ with a price function $F$ that satisfies the Black-Scholes equation on the previous slide. A portfolio $h = (h^S, h^B)$ with value process

$$V^h_t = h^S_t S_t + h^B_t B_t,$$

where

$$h^S_t = \frac{\partial F(t, s)}{\partial s} \bigg|_{s=S_t}$$

$$h^B_t = \frac{\partial F(t,s)}{\partial t} \bigg|_{s=S_t} + \frac{1}{2} s^2 \sigma^2(t, s) \frac{\partial^2 F(t,s)}{\partial s^2} \bigg|_{s=S_t} = \frac{F(t, S_t) - h^S_t S_t}{B_t} r B_t,$$

is a replicating portfolio for the claim $X$. 
Girsanov transformation (Å: Thm 9.8 p. 225)

Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\{W^\mathbb{P}_t\}_{t \geq 0}$ is a $(d\text{-dim})$ Brownian motion w.r.t. $\mathcal{F}_t$. Let $g_t$ be a $(d\text{-dim})$ process adapted to $\mathcal{F}_t$ for $t \in [0, T]$ which satisfies

$$
\mathbb{E}^\mathbb{P}\left[ \exp\left(\frac{1}{2} \int_0^T |g_t|^2 \, dt\right) \right] < \infty, \quad \text{(Novikov condition)}.
$$

Define the process $L_t$ by

$$
L_t = \exp\left( - \int_0^t g_s^* \, dW^\mathbb{P}_s - \frac{1}{2} \int_0^t |g_s|^2 \, ds \right), \quad 0 \leq t \leq T.
$$

Define a new probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\mathbb{Q}(A) = \mathbb{E}^\mathbb{P}[1_A L_T]$ for $A \in \mathcal{F}_T$.

Then $W^\mathbb{Q}_t = W^\mathbb{P}_t + \int_0^t g_s \, ds$ is a standard $(d\text{-dim})$ $\mathbb{Q}$-BM on $[0, T]$. 

Recapitulation
The new dynamics after change of measure

Suppose that the market have the $\mathbb{P}$-dynamics

\[
\begin{align*}
    dS_t &= S_t \mu(t, S_t) \, dt + S_t \sigma(t, S_t) \, dW^\mathbb{P}_t, \\
    S_0 &= s_0 \\
    dB_t &= r(t) B_t \, dt, \\
    B_0 &= 1.
\end{align*}
\]

Using the Girsanov kernel $g_t$ we get the $\mathbb{Q}$-dynamics

\[
\begin{align*}
    dS_t &= S_t (\mu(t, S_t) - \sigma(t, S_t) g_t) \, dt + S_t \sigma(t, S_t) \, dW^\mathbb{Q}_t, \\
    S_0 &= s_0 \\
    dB_t &= r(t) B_t \, dt, \\
    B_0 &= 1.
\end{align*}
\]
Arbitrage and completeness

If there exists at least one function $g_t$ which solves the equation

$$\mu(t, S_t) - \sigma(t, S_t)g_t = r(t), \ \forall t \in [0, T]$$

which also satisfies the Novikov condition then the market defined in the previous slide is free of arbitrage.

If this solution also is unique then the market is free of arbitrage and complete. This is the same as saying that there exist a unique martingale measure $Q$.

Note that the above is also true in the multi-dimensional setting, where $\mu$ is a $n$-dim vector, $g$ is a $d$-dim vector and $\sigma$ an $n \times d$-dim matrix provided that we change the equation to

$$\mu(t, S_t) - \sigma(t, S_t)g_t = r(t)1_n, \ \forall t \in [0, T]$$
Risk neutral valuation (B: Thm 7.8 p. 103)

Using the Feynman-Kac representation on the Black-Scholes equation we get that

$$F(t, s) = \exp(-(T - t)r) \mathbb{E}^Q [\Phi(S_T) \mid S_t = s],$$

where $S_t$ has the $Q$-dynamics

$$dS_t = rS_t \, ds + S_t \sigma(t, S_t) \, dW_t.$$
Black-Scholes formula (B: Prop 7.10 p. 105)

Assume that we have a market consisting of a risky asset $S$ and a bank-account $B$, where the corresponding $\mathbb{P}$-dynamics are given as

\[
\begin{align*}
    dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t, \\
    S_0 &= s_0 \\
    dB_t &= r B_t \, dt, \\
    B_0 &= 1.
\end{align*}
\]

Then the price $\Pi_E^c(t, K, T, s)$ of a European call option with maturity $T$, strike $K$ and $S_t = s$ is given as

\[
\begin{align*}
    \Pi_E^c(t, K, T, s) &=
    s N(d_1(t, s)) - e^{-r(T-t)} K N(d_2(t, s)) \\
    d_1(t, s) &= \frac{1}{\sigma \sqrt{T-t}} \left( \ln \left( \frac{s}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T-t) \right) \\
    d_2(t, s) &= d_1(t, s) - \sigma \sqrt{T-t}
\end{align*}
\]
A numeraire is a positive self-financing portfolio on the market. The usual numeraire is the bank account $B$, which corresponds to the martingale measure $\mathcal{Q}$. If we change numeraire we should change the martingale measure accordingly. Suppose $N_t$ is a numeraire on the market then the corresponding martingale measure $\mathcal{Q}^N$ is given by

$$\mathcal{Q}^N(A) = \mathbb{E}^\mathcal{Q} \left[ 1_A \frac{N_T B_0}{B_T N_0} \right], \text{ for } A \in \mathcal{F}_T,$$

that is the LR-process $L_t = (B_0 N_t)/(B_t N_0)$. Since $N_t/B_t$ is a $\mathcal{Q}$-martingale we get that $N_t$ has the $\mathcal{Q}$-dynamics

$$dN_t = r(t)N_t \, dt + N_t \sigma_N(t, N_t) \, dW_t^\mathcal{Q},$$

for some function $\sigma_N(t, N_t)$. 

Change of numeraire
This gives that $L_t$ has the $Q$-dynamics

$$dL_t = L_t \sigma_N(t, N_t) \, dW^Q_t,$$

so that the Girsanov kernel $g$ changing from $Q$ to $Q^N$ is given by $g_t = -\sigma_N(t, N_t)^*$ so that

$$W^Q_N = W^Q_t - \int_0^t \sigma_N(t, N_s)^* \, ds$$

is a standard $Q^N$-BM. So if an asset $X_t$ has drift $\mu(t, X_t)$ and diffusion term $\sigma(t, X_t)$ under $Q$ the drift under $Q^N$ is

$$\mu^Q_N = \mu(t, X_t) + \sigma(t, X_t) \sigma_N(t, N_t)^*$$

while $\sigma$, as usual, is unchanged.
Interest rates

- Basic contracts and their relations
- Market models for basic contracts
- Short rate models
- Term structure equation
- Affine term structures
- Forward rate models (HJM-framework)
Basic contracts and their relations

Zero coupon bond (ZCB) This contract pays one unit of currency at the future time $T$. Price at time $t \leq T$ denoted $p(t, T)$.

Coupon bond This contract pays out coupons with values $c_i, i = 1, \cdots, n$ at times $T_i, i = 1, \cdots, n$ and a final payment $K$ (face value) at time $T_n$. Price at time $t \leq T_1$ denoted $p(t) = Kp(t, T_n) + \sum_{i=1}^{n} c_ip(t, T_i)$.

SPOT LIBOR This contract gives the stochastic interest rate $L[T, S]$, (the rate decided at time $T$), over the future time interval $[T, S]$. If we put in $A$ units of currency at time $T$ we get out $A(1 + (S - T)L[T, S])$ at time $S$. We have that $(1 + (S - T)L[T, S]) = 1/p(T, S)$. 
Forward rate agreement (FRA)

The **Forward rate agreement (FRA)** contract swaps the floating LIBOR rate $L[T, S]$ to the fixed rate $K$ over the future time period $[T, S]$. Payoff $(S - T)(L[T, S] - K)$. Price at time $t$ for $t \leq T$

$$\Pi_t^{FRA}([T, S], K) = (S - T)p(t, S) \left(\frac{p(t, T) - p(t, S)}{(S - T)p(t, S)} - K\right)$$

**Forward LIBOR rate** $L_t[T, S]$ The fixed rate $K$ that makes $\Pi_t^{FRA}([T, S], K) = 0$ i.e. $L_t[T, S] = \frac{p(t, T) - p(t, S)}{(S - T)p(t, S)}$.

The **SWAP** contract swaps the floating LIBOR rates $L[T_{i-1}, T_i]$ to the fixed rate $K$ over the future time periods $[T_{i-1}, T_i]$, $i = \alpha + 1, \cdots, \beta$. Payoff $\sum_{i=\alpha+1}^{\beta} (T_i - T_{i-1})(L[T_{i-1}, T_i] - K)$. Price at time $t$ for $t \leq T_\alpha$

$$
\Pi_t^{SWAP}(\bar{T}, K) = A_{\alpha, \beta}(t) \left( \frac{p(t, T_\alpha) - p(t, T_\beta)}{A_{\alpha, \beta}(t)} - K \right),
$$

where

$$
A_{\alpha, \beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)
$$

**SWAP-rate**

$$
S_{\alpha, \beta}(t) = \frac{p(t, T_\alpha) - p(t, T_\beta)}{A_{\alpha, \beta}(t)}
$$
LIBOR market model

Under the forward measure $\mathbb{Q}^S$ (the $p(t, S)$ numeraire measure) the LIBOR rate $L_t[T, S]$ is a martingale.

Standard model:

$$dL_t[T, S] = L_t[T, S] \sigma(t) \, dW(t),$$

where $W$ is a d-dim $\mathbb{Q}^S$-BM and $\sigma$ is a deterministic function (d-dim).

Black 76 type formula for caplets (B: Prop 27.6 p 421)

$$\Pi^{\text{caplet}}(t, T, K) = (S - T)p(t, S) (L_t[T, S]N(d_1) - K N(d_2)),$$

where

$$d_1 = \left( \ln \left( \frac{L_t[T, S]}{K} \right) + \frac{\Sigma^2(t, T)}{2} \right) / \Sigma(t, T)$$
$$d_2 = d_1 - \Sigma(t, T)$$

$$\Sigma^2(t, T) = \int_t^T |\sigma(s)|^2 \, ds$$
SWAP market model

Under the SWAP measure $\mathbb{Q}^{\alpha,\beta}$ (the $A_{\alpha,\beta}(t)$ numeraire measure) the SWAP rate $S_{\alpha,\beta}(t)$ is a martingale.

Standard model:

$$dS_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)\sigma_{\alpha,\beta}(t)\,dW(t),$$

where $W$ is a d-dim $\mathbb{Q}^S$-BM and $\sigma_{\alpha,\beta}$ is a d-dim deterministic function.

Black 76 type formula for swaptions (B: Prop 27.16 p 433)

$$\Pi^{SWAPtion}(t, T, K) = A_{\alpha,\beta}(t) \left( S_{\alpha,\beta}(t)N(d_1) - K N(d_2) \right),$$

where

$$d_1 = \left( \ln\left( \frac{S_{\alpha,\beta}(t)}{K} \right) + \frac{\Sigma_{\alpha,\beta}(t, T_\alpha)}{2} \right) / \Sigma_{\alpha,\beta}(t, T_\alpha),$$

$$d_2 = d_1 - \Sigma_{\alpha,\beta}(t, T_\alpha),$$

$$\Sigma_{\alpha,\beta}(t, T_\alpha) = \int_t^{T_\alpha} |\sigma_{\alpha,\beta}(s)|^2 \, ds.$$
Short rate, forward rate and the ZCB

The bank account is given by

$$dB_t = r(t)B_t \, dt, \quad B_0 = 1,$$

where $r(t)$ is called continuously compounded short rate or just short rate. **The forward rate** $f(t, T) = -\frac{\partial \ln(p(t, T))}{\partial T}$, where $p(t, T)$ is the ZCB-price. **The short rate** is then given as $r(t) = f(t, t)$.

**The ZCB** is then given as $p(t, T) = \exp(-\int_t^T f(t, s) \, ds)$. We also have that $p(t, T) = E^Q[\exp(-\int_t^T r(s) \, ds)|\mathcal{F}_t]$. These two prices should coincide in order to avoid arbitrage.
Suppose that $r(t)$ has the $\mathbb{Q}$-dynamics

$$dr(t) = \mu(t, r(t)) \, dt + \sigma(t, r(t)) \, dW_t$$

then the price of the ZCB $p(t, T) = F(t, r, T)$ should for all times of maturity $T$ satisfy

$$\begin{align*}
\frac{\partial F(t, r, T)}{\partial t} &= -\mu(t, r) \frac{\partial F(t, r, T)}{\partial r} - \frac{1}{2} \sigma(t, r)^2 \frac{\partial^2 F(t, r, T)}{\partial r^2} + rF(t, r, T) \\
F(T, r, T) &= 1.
\end{align*}$$
Some common short rate models under $\mathbb{Q}$

(B: p. 375)

- **Vasicek**
  
  $dr(t) = (b - ar(t)) \, dt + \sigma \, dW_t, \quad (a, b, \sigma > 0)$.

- **Cox-Ingersoll-Ross (CIR)**
  
  $dr(t) = a(b - r(t)) \, dt + \sigma \sqrt{r(t)} \, dW_t, \quad (a, b, \sigma > 0)$.

- **Ho-Lee**
  
  $dr(t) = \Theta(t) \, dt + \sigma \, dW_t$.

- **Hull-White (extended Vasicek)**
  
  $dr(t) = (\Theta(t) - a(t)r(t)) \, dt + \sigma(t) \, dW_t, \quad (\Theta(t), a(t), \sigma(t) > 0)$.

- **Hull-White (extended CIR)**
  
  $dr(t) = (\Theta(t) - a(t)r(t)) \, dt + \sigma(t) \sqrt{r(t)} \, dW_t, \quad (\Theta(t), a(t), \sigma(t) > 0)$.
Affine term structure (B Prop 24.2 p. 379)

For all the models in the previous slide we have that the ZCB price have an affine term structure that is

\[ p(t, T) = \exp(A(t, T) - B(t, T)r(t)), \]

where \( A, B \) are deterministic functions which do not depend on \( r \).

This is in fact true for all short rate models of the form

\[ dr(t) = \alpha(t)r(t) + \beta(t) \, dt + \sqrt{\gamma(t)r(t) + \delta(t)} \, dW_t. \]

The functions \( A \) and \( B \) satisfy the following system of ordinary differential equations (ODE:s)

\[ B'_t(t, T) = -\alpha(t)B(t, T) + \frac{1}{2}\gamma(t)B^2(t, T) - 1, \quad B(T, T) = 0 \]

\[ A'_t(t, T) = \beta(t)B(t, T) - \frac{1}{2}\delta(t)B^2(t, T), \quad A(T, T) = 0 \]
The HJM framework

The forward rate $f(t,T)$ has $Q$-dynamics

$$\begin{align*}
\text{d}f(t,T) &= \alpha(t,T) \text{d}t + \sigma(t,T) \text{d}W_t \\
f(0,T) &= f^*(0,T),
\end{align*}$$

where $f^*(0,T)$ is the observed forward rate.

**HJM drift condition , (B: 25.2 p. 390)**

Under $Q$ the drift term $\alpha$ and the diffusion term $\sigma$ for $T \geq t$ must satisfy the relation

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)^* \text{d}s$$

If $\sigma$ is a deterministic function then the forward rate will be a Gaussian process.