SOME PROBLEMS WITH SOLUTIONS
FMSN25/MA5M24 VALUATION OF DERIVATIVE ASSETS

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Chapter 1

Problems

1.1 Martingales

1.1.1 Assume that the process \( \{S_t\}_{t \geq 0} \) follows the standard Black & Scholes model and that \( \gamma \in \mathbb{R} \). Find \( \gamma \neq 1 \) such that \( \{S_t e^{-\eta t}\}_{t \geq 0} \) will be a \( \mathbb{Q} \)-martingale.

1.1.2 Show that the process \( X(t) = e^{t/2} \cos(W_t) \), where \( W_t \) is a standard Brownian motion, is a martingale for \( t \geq 0 \).

1.2 Static replication of derivatives

1.2.1 We have a derivative with maturity \( T \) and pay-off

\[
\Phi(S_T) = \max(K - |K - S_T|, 0).
\]

Find a static hedge for this derivative using the asset \( S \) and European options on \( S \).

1.2.2 Assume that \( X \) is a derivative with maturity \( T \) having the following pay-off function:

\[
\begin{align*}
  &K && S_T \leq K \\
  &2K - S_T && K \leq S_T \leq 2K \\
  &0 && S_T \geq 2K
\end{align*}
\]

Express the price for \( 0 < t < T \) of \( X \) using European put and call options, the stock \( S \) and the bank-account \( B \).

1.3 PDE:s and Feynman-Kac formula

1.3.1 Solve the PDE
1.4 Pricing of derivatives and hedging

1.4.1 Consider a derivative with pay-off \( \Phi(S_T) = \max(K - (S_T)^2, 0) \) in the Black-Scholes market.

(a) Find the price-formula for the derivative for \( 0 \leq t \leq T \).

(b) Calculate the replicating portfolio (delta-hedge) for the derivative, that is find a self-financing portfolio consisting of the stock and the bank account that hedges the derivative.

1.5 Change of measure and completeness

1.5.1 A simple extension to the binomial model is the trinomial model (TM) where we also allow the stock price to remain unchanged with some probability. More precisely we have that

\[
P(S_{k+1} = uS_k) = p_u, \quad P(S_{k+1} = S_k) = p_1, \quad P(S_{k+1} = dS_k) = p_d, \quad k = 0, 1, \ldots, (N-1)
\]

in the \( N \)-period case where \( u > 1 \) and \( d < 1 \) and a bank account \( B \) with \( B_k = \exp(kr) \). Even though it can provide a better approximation of the Black-Scholes model for a finite number of periods, TM is seldom used. The main reason for this is that TM gives an incomplete market. We are here going to study TM in the one period setting with

\[
u = 6/5, \quad d = 3/5, \quad S_0 = 1 \quad \text{and with zero interest rate.}
\]

(a) Show that TM is incomplete by showing that the risk-neutral probabilities \( q_u, q_1, q_d \) are not unique.

(b) If we add another asset to the market it is possible to make the market complete. Suppose that we add the asset \( X \) where \( X \) is the derivative with pay-off \( S_t^1 \) at time 1 and further suppose that the price of \( X \) at time zero is equal to 26/25. If the price of \( X \) is given by the market we can view this as a calibration of our model to fit the market price. This will then allow us to price other derivatives on the asset \( S \). Show that the market is complete by finding the unique risk-neutral probabilities.

(c) Use the probabilities from (b) to price the derivative \( Y \), with maturity \( T = 1 \) and pay-off \( \max(S_1 - 9/10, 0) \). Moreover find the replicating portfolio for this derivative using the assets \( S, X \) and \( B \).

(d) Note that we cannot have an arbitrary price of \( X \) at time zero. Show that all prices outside the interval \([1, 27/25]\) will give arbitrage in the sense that there are no risk-neutral probabilities compatible with those prices.

1.5.2 Assume that we have a market consisting of one risky asset \( S^{(1)} \) and one bank account \( S^{(0)} \) with the \( \mathbb{P} \)-dynamics:

\[
dS^{(1)}_t = \mu_1 S^{(1)}_t \, dt + \sigma_1 S^{(1)}_t \, dW^{(1)}_t + \sigma_1 S^{(1)}_t \, dW^{(2)}_t
\]

\[
dS^{(0)}_t = rS^{(0)}_t \, dt,
\]

where \( W^{(1)} \) and \( W^{(2)} \) are two independent standard BM:s. Using the meta-theorem on this model we get that it is free of arbitrage but incomplete, i.e. the martingale measure exists but is not unique.

(a) However, show that all simple claims \( X \) with maturity \( T \) of the form \( \Phi(S^{(1)}_T) \) will have a price-formula that does not depend on the choice of martingale measure.

(b) Show that the claim \( Y = 1_{W^{(2)}_T > K} \), that is a contract which pays one unit of currency at time \( T \) if \( W^{(2)}_T > K \), will have a price-formula that do depend on the choice of martingale measure.

(c) Show that if we add the asset \( S^{(2)} \) with \( \mathbb{P} \) dynamics:

\[
dS^{(2)}_t = \mu_2 S^{(2)}_t \, dt + \sigma_2 S^{(2)}_t \, dW^{(2)}_t
\]

to the market, that the market will be free of arbitrage and complete by finding the unique Girsanov kernel \( (g_1, g_2) \).

(d) Price the derivative in (b) using this unique martingale measure.
1.6 Change of numeraires

1.6.1 Assume the following 2-dim Black-Scholes model (under $\mathbb{Q}$) for the two stocks $S_1$ and $S_2$,

\[
\begin{align*}
    &dS_1(t) = rS_1(t)dt + S_1(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) \\
    &dS_2(t) = rS_2(t)dt + S_2(t)(\sigma_{21}dW_1(t) + \sigma_{22}dW_2(t))
\end{align*}
\]

where $W_1$ and $W_2$ are two independent standard $\mathbb{Q}$ Brownian motions and where $r$, $\sigma_{11}$, $\sigma_{12}$, $\sigma_{21}$ and $\sigma_{22}$ are positive constants. Price the derivative with maturity $T$ and pay-off:

$$
\Phi(S_1(T), S_2(T)) = \max(S_2(T) - S_1(T), 0),
$$

for $0 < t < T$. (Hint: the key is to find the right numeraires).

1.6.2 In a realistic situation the short interest rate $r$ is not a deterministic constant. What one wants to do is to use observed prices of Zero Coupon bonds (ZCB) as a discounting factor when pricing derivatives. The way to accomplish this is to express the dynamics of the underlying stock $S$ under the forward measure $\mathbb{Q}^F$, i.e., the martingale measure which has the ZCB as numeraire. Under the measure $\mathbb{Q}^F$ we have that the discounted stock process $Z(t) = S(t)/\rho(t, T)$ should be a martingale, where $\rho(t, T)$ is the price at time $t$ of a ZCB with maturity $T$. Note that we assume the following model for $Z(t)$ under $\mathbb{Q}^F$:

$$
dZ(t) = Z(t)\nu(t, T)dW^{Q_T}(t), \quad 0 \leq t \leq T,
$$

where $W^{Q_T}$ is a standard 2-dim $\mathbb{Q}^T$ Brownian motion and where $\nu(t, T)$ is a positive deterministic function (a $1 \times 2$ row vector). Note that by definition we have $Z(T) = S(T)/\rho(T, T) = 1$.

(a) Now assume that under the usual martingale measure $\mathbb{Q}$, we have the following model for the ZCB and the stock,

\[
\begin{align*}
    &dP(t, T) = r(t)P(t, T)dt + P(t, T)(T - t)\gamma dW_1(t), \quad 0 \leq t \leq T, \\
    &dS(t) = r(t)S(t)dt + \sigma(t)dW_2(t),
\end{align*}
\]

where $W_1$ and $W_2$ are two independent standard $\mathbb{Q}$ Brownian motions and where $\sigma$ and $\gamma$ are positive constants. Calculate the volatility function $\nu(t, T)$ implied by this model. Recall that the volatilities do not change when we change the measure.

(b) Price a standard European call option with strike $K$ and maturity $T$ for this model, for $t$ such that $0 < t < T$. You should express the price in a Black-Scholes type of formula. Remember to check that your price simplifies to the standard Black-Scholes formula if the interest rate is constant equal to $r$ and $|\nu(t, T)| \equiv \sigma$.

1.6.3 You are thinking of investing in some stocks. Company A and B are competing on the same market. So you think if you should buy stock A or stock B. Assume that you have the following model (under $\mathbb{Q}$) for the stocks and the bank account.

\[
\begin{align*}
    &dB(t) = rB(t)dt, \\
    &dS_A(t) = rS_A(t)dt + \sigma_A(t)dW_1(t), \\
    &dS_B(t) = rS_B(t)dt + \sigma_B(t)(dW_1(t) + \sqrt{1 - \rho^2}dW_2(t)),
\end{align*}
\]

where $r$, $\sigma_A$, $\sigma_B > 0$ and $-1 \leq \rho < -0.5$ with $W_1$ and $W_2$ being independent standard Brownian motions. The negative $\rho$ should give the effect that one stock will go up and while the other stock goes down. For this setup to be realistic, one should assume that the initial prices of stock A and B are equal and that the volatilities also are roughly the same. You can use this assumption in the discussion of (d), but in (a)-(c) solve the general problem.

(a) Someone at your bank offers you a derivative with maturity $T$ years and with pay-off

$$
\Phi(S_A(T), S_B(T)) = \max(S_A(T), S_B(T)),
$$

which take care of your problem to decide which stock to buy. What is the fair price of this contract at time $t = 0$ and $0 < t < T$?

(b) You talk to a friend who tell you to buy stock A now, keep it for $T$ years, and to buy a spread option with maturity $T$ years and pay-off:

$$
\Phi(S_A(T), S_B(T)) = (S_B(T) - S_A(T))^+.
$$

Verify that this setup is equivalent to the derivative offered by your bank.
1.7.1 Volatility

1.6.4 Two friends, we can call them Anna and Belle, meet at a café on a Saturday. They are amused to see that their two favourite stocks, $S_A$ and $S_B$ respectively, happened to have the same closing price on the previous day. To celebrate this they decide to make a bet. If stock $S_A$ has a higher closing price than $S_B$ at time $T$ Anna will receive $S_A(T)$ from Belle. If the closing prices at time $T$ are equal, neither Anna nor Belle will receive any money. Note that no money is paid until time $T$. Your task is now to investigate if this is a fair bet, by looking at the bet as a financial derivative. In order to do this we need a model for the stocks. Assume that the stocks under the martingale measure $\mathbb{Q}$ follow a two dimensional Black-Scholes model of the following form:

$$
\begin{align*}
\text{d}S_A(t) &= rS_A(t)\text{d}t + \sigma_A S_A(t)\text{d}W_1(t) \\
\text{d}S_B(t) &= rS_B(t)\text{d}t + \sigma_B S_B(t) \left( \text{d}W_1(t) + \sqrt{1-\rho^2}\text{d}W_2(t) \right), \\
\text{d}B(t) &= rB(t)\text{d}t,
\end{align*}
$$

where $W_1$ and $W_2$ are two independent $\mathbb{Q}$-BM:s and where $-1 < \rho < 1$, $\sigma_A$, $\sigma_B > 0$ and $r > 0$.

(a) Calculate the value of the derivative (bet) from Anna’s point of view on the date of issue $t = 0$, i.e. the day Anna and Belle decide to make the bet. Moreover explain how this is related to the value from Belle’s point of view. Is the bet fair?

(b) What is the value at time $t$ for $0 < t < T$ for arbitrary positive values of $S_A(t)$ and $S_B(t)$ from Anna’s point of view?

(c) Find a replicating portfolio for the bet from Anna’s and Belle’s points of view respectively.

(d) Do you think it is a good idea to hedge the bet? Motivate your answer properly.

1.7 Volatility

1.7.1 VIX: On Chicago Board Options Exchange (CBOE) there is an index called VIX, which measures expected volatility on the SP500 index. The VIX is calculated using market prices of European put and call options on the SP500 index. Let $\Pi^P(t, t+\tau, S(t), K)$ and $\Pi^C(t, t+\tau, S(t), K)$ be prices of a European put and a European call options respectively given at time $t$ with maturity $t + \tau$ with current stock price $S(t)$ and strike $K$. Let further $F_t^{t+\tau}$ be the forward level at time $t$ of the stock price at time $t + \tau$, i.e. $F_t^{t+\tau} = \mathbb{E}^Q[S(t + \tau) | \mathcal{F}_t]$. Having access to a wide range of option prices over different strikes we can get an almost model-free estimate of the expected forward volatility.

$$
VIX(t, \tau) = \left( \frac{2}{\tau} \sum_{i=1}^{NP-1} \varepsilon^+ \Pi^P_i(t, t + \tau, S(t), K_i) \frac{K_{i+1} - K_i}{K_i^2} \mathbb{I}(K_i < F_t^{t+\tau}) \right. \\
+ \frac{2}{\tau} \sum_{j=1}^{NC-1} \varepsilon^+ \Pi^C_j(t, t + \tau, S(t), K_j) \frac{K_{j+1} - K_j}{K_j^2} \mathbb{I}(K_j > F_t^{t+\tau}) \bigg)^{1/2},
$$

where $NP$ and $NC$ are the number of available put and call options with maturity $t + \tau$ respectively.\(^\dagger\) The empirically calculated VIX index squared $VIX(t, \tau)^2$ is an approximation of an integral

$$
\begin{align*}
VIX(t, \tau)^2 &\approx \frac{2}{\tau} \left( \int_{F_t^{t+\tau}}^\infty \varepsilon^+ \Pi^P_i(t, t + \tau, S(t), K) \frac{dK}{K^2} + \int_{F_t^{t+\tau}}^\infty \varepsilon^+ \Pi^C_j(t, t + \tau, S(t), K) \frac{dK}{K^2} \right) \\
&= \frac{2}{\tau} \mathbb{E}^Q \left[ \int_{0}^{F_t^{t+\tau}} (K - S(t + \tau))^+ \frac{dK}{K^2} + \int_{F_t^{t+\tau}}^{\infty} (S(t + \tau) - K)^+ \frac{dK}{K^2} \bigg| \mathcal{F}_t \right] \\
&= \frac{2}{\tau} \mathbb{E}^Q \left[ - \ln \left( \frac{S(t + \tau)}{F_t^{t+\tau}} \right) \bigg| \mathcal{F}_t \right].
\end{align*}
$$

\(^\dagger\)This is a simplified version of the real index. See [https://www.cboe.com/micro/vix/vixwhite.pdf](https://www.cboe.com/micro/vix/vixwhite.pdf) for more details.
Now assume that we have a market with the following \( Q \)-dynamics:
\[
\begin{align*}
    dB(t) &= rB(t)dt, \\
    dS(t) &= rS(t)dt + \sigma(t)S(t)dW(t),
\end{align*}
\]
where \( r \) is the risk free rate, \( \sigma \) a deterministic function and \( W \) is a standard \( Q \)-Brownian motion. This is the model you shall use in the rest of the problem.

(a) Calculate \( F_t^i \doteq E^Q[S(t + \tau)|\mathcal{F}_t] \).
(b) Evaluate the expectation
\[
\frac{2}{\tau}E^Q\left[-\ln\left(\frac{S(t + \tau)}{F_t^i}\right) \mid \mathcal{F}_t\right].
\]
(c) Show that
\[
E^Q\left[\int_0^{F_t^i} (K - S(t + \tau)) + \frac{dK}{K^2} + \int_{F_t^i}^{\infty} (S(t + \tau) - K) + \frac{dK}{K^2} \mid \mathcal{F}_t\right] = E^Q\left[-\ln\left(\frac{S(t + \tau)}{F_t^i}\right) \mid \mathcal{F}_t\right].
\]

1.7.2 (VIX cont.) Assume that \( S \) follows the Heston model i.e. we have a market with the following \( Q \)-dynamics:
\[
\begin{align*}
    dB(t) &= rB(t)dt, \\
    dS(t) &= rS(t)dt + \sqrt{V(t)}S(t)dW_1(t) + \sqrt{1 - \rho^2}dW_2(t), \\
    dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_1(t),
\end{align*}
\]
where \( r \) is the risk free rate, \( \sigma, \kappa, \theta, \rho \) are positive deterministic constants, \(-1 \leq \rho \leq 1\) and \( W_1 \) and \( W_2 \) are independent standard \( Q \)-Brownian motions. Evaluate the expectation
\[
\frac{2}{\tau}E^Q\left[-\ln\left(\frac{S(t + \tau)}{F_t^i}\right) \mid \mathcal{F}_t\right],
\]
where \( F_t^i \doteq E^Q[S(t + \tau)|\mathcal{F}_t] \).

1.8 Simple interest rate contracts and Martingale models for the short rate

1.8.1 Calculate the price of a Zero Coupon bond with maturity \( T \) at time \( t \) where \( 0 < t < T \) in the following Ho-Lee model for the short rate,
\[
\begin{align*}
    dr_t &= \Theta(s)ds + \sigma dW_t, \text{ for } s \geq t \\
    r_t &= r
\end{align*}
\]
where \( \Theta(s) = \alpha e^{-\gamma s} + \sigma^2s \) and where \( \sigma, r, \epsilon \) are positive constants.

1.8.2 A floating rate bond with face value \( A \) is like a coupon bond with face value \( A \) which has coupon rates equal to the floating LIBOR rates
\[
L_{T_{i-1}T_i}[T_{i-1}, T_i] = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{p(T_{i-1}, T_i)} - 1\right)
\]
over time intervals \([T_{i-1}, T_i], i = 1, \ldots, n\). So this contracts pays out \( A \) at time \( T_n \) and the coupons \( c_i = A(T_i - T_{i-1})L_{T_{i-1}T_i}[T_{i-1}, T_i], i = 1, \ldots, n \). The value of coupon \( i \) is not known until time \( T_{i-1} \). Find the value of this contract at time \( t = T_0 \).

1.9 The HJM framework

1.9.1 Consider the following HJM model for the forward rate \( f(t, T), \ T \geq 0; \)
\[
\begin{align*}
    df(t, T) &= \sigma^2(T - t)dt + \sigma dW_t^Q, \ 0 \leq t \leq T \\
    f(0, T) &= f^*(0, T), \ T \geq 0
\end{align*}
\]
where \( W \) is a standard BM and \( f^*(0, T) \) is the observed forward rate on the e market.
(a) Find the ZCB price \( p(t, T) \) for this model.

(b) Calculate the \( Q^T \)-dynamics, that is the dynamics under the numeraire measure where \( p(t, S) \) is used as a numeraire, for the process \( X_t = (1 + (S - T)L_r(T, S)) \) for \( 0 \leq t \leq T \), where \( L_r(T, S) = (p(t, T) - p(t, S))/(S - T) \).

(c) Use the dynamics obtained in (b) to price the Caplet \( \Phi_S \) for \( 0 \leq t \leq T \). That is the derivative with maturity \( S \) and pay-off
\[
(S - T) \max(L_r(T, S) - K, 0),
\]
where \( K \) is a positive constant.

1.9.2 You get an urgent phone call from a friend who has found a programming error in the bank’s accounting system.

Instead of paying out a three month spot LIBOR-contract over the time interval \( [T_1, T_2] \) at time \( T_2 \) where the rate is decided at time \( T_1 \), the system assigns the rate over the coming interval \( [T_2, T_3] \), i.e. the pay-off is incorrectly set to \( Y = 1 + (T_2 - T_1)L(T_1, T_3) \), where \( T_2 - T_1 = T_3 - T_2 = 0.25 \). The cash-flow is still at time \( T_2 \). For the correct pay-off \( X \) we have that the fair value at time \( T_1 \) is one regardless of which model we use i.e. \( \pi^k(T_1) = 1 \). You will however need at model to find \( \Pi^k(T_1) \), i.e. the fair value of \( Y \) at time \( T_1 \). So we now assume that we have the following \( Q^T \)-dynamics (dynamics under the numeraire measure corresponding to the numeraire \( p(t, T_3) \)) for the forward rate for \( T_1 \leq t \leq u \leq T_3 \) as:

\[
\begin{align*}
df(t, u) &= -\sigma(t)^2(T_3 - u)dt + \sigma(t)dW(t), \\
f(T_1, u) &= a + b(1 - e^{-(u - T_1)}),
\end{align*}
\]
where \( a, a + b > 0 \). Note that for arbitrary \( t, S_1, S_2 \) satisfying \( 0 \leq t \leq S_1 < S_2 \) we have

\[
1 + (S_2 - S_1)L(t, S_1, S_2) = p(t, S_1)/p(t, S_2)
\]
where \( p(t, S_1) \) and \( p(t, S_2) \) are ZCB values at time \( t \) for ZCBs with maturity \( S_1 \) and \( S_2 \) respectively.

(a) Your task is now to find out the fair value of the pay-off \( Y \) at time \( T_1 \) using the model above.

(b) You should now figure out if the fair value calculated in a) is larger or smaller than one (i.e. the fair value of \( Y \)). To simplify things in b) we assume that \( a = 0.02, b = -0.01 \) and \( \sigma(t) \equiv \sigma \) where \( \sigma = 0.1 \).

(c) So would the bank on average win or lose on the programming error given the result you have obtain for the fair value in b)?

1.10 Dividend paying stocks

1.10.1 Continuous dividends

Most real stock pay out dividends. One common model for a dividend paying stock is to assume that dividends are payed out in a continuous cash flow. More precisely, by holding a stock, \( S \) over the time interval \( [t, T] \) we receive the following dividend cash flow

\[
D(T) - D(t) = \int_t^T qS(u)du,
\]
where the positive constant \( q \) is called dividend yield. The model for the stock and bank account \( B \) and dividends under \( Q \) is given by

\[
\begin{align*}
dS(u) &= \alpha S(u)du + \sigma S(u)dW^Q(u), \\
 dB(u) &= rB(u)du, \quad B(0) = 1 \\
 dD(u) &= qS(u)du, \quad D(0) = 0
\end{align*}
\]
where \( \alpha \) is the drift, \( \sigma > 0 \) the volatility, \( W^Q \) is a standard \( Q \)-BM, \( r \) the interest rate and \( q \geq 0 \) is the dividend yield. So the stock behaves like a geometric BM as in the BS standard model. However we cannot have \( \alpha = r \) as in the BS case (unless \( q = 0 \)).

(a) How should we choose \( \alpha \) here? The key observation here is that we want an arbitrage free market. So let \( t \) and \( T \) be to arbitrary times satisfying \( t < T \). Buying the stock at time \( t \) and selling it at time \( T \) we get the following discounted cash flow

\[
S(T) B(t) + \int_t^T \frac{B(t)}{B(u)} dD(u) = S(T) \frac{B(t)}{B(T)} + \int_t^T q \frac{B(t)}{B(u)} q(u)du.
\]
This cost us $S(t)$ at time $t$ so for no arbitrage we must have
\[ S(t) = \mathbb{E}_{Q} \left[ S(T) \frac{B(t)}{B(T)} + \int_{t}^{T} B(t) \frac{B(u)}{B(T)} \delta S(u) du \right], \]
for all $0 < t < T$. No use this to find the correct value of $x$.

(b) Given the result in a), use the fundamental theorems of asset pricing to check if this market is free of arbitrage and complete.

(c) Now use the model with the $x$-value calculated in a) to find the value at time $t$ of a contract where we receive the stock at time $T$, where $T > t$.

(d) Find a replicating portfolio for the contract in c). Notice that for dividend paying stocks a simple buy and hold strategy will not work, since holding the stock you receive dividends whereas getting the stock at time $T$ give you no dividends between $t$ and $T$. A portfolio consisting of holdings in the stock and the bank account with value process $V(t) = a(t)S(t) + b(t)B(t)$ is self-financing if
\[
\frac{dV(t)}{V(t)} = a(t) \frac{dS(t)}{S(t)} + a(t) \frac{dB(t)}{B(t)} + b(t) \frac{dB(t)}{B(t)}.
\]

Hint: Look at the delta-hedge and check that it will be self-financing.

(e) Now use the model with the $x$-value calculated in a) to price a standard European call option and compare the result to the standard BS-formula. Conclusions? Remember that holding an European call option does not give you any dividends.

(f) Find a replicating portfolio for the contract in c).

1.10.2 Discrete dividends

Most real stock pay out dividends. Another common model for a dividend paying stock is to assume that dividends flow paid out. More precisely, by holding a stock, $\alpha$ is the dividend fraction. $\delta$ is the stochastic term in the stock dynamics. $\sigma$ is the volatility, $r$ the interest rate and $\delta \geq 0$ is the dividend fraction. So the stock behaves like a geometric BM as in the BS standard model between the downward jumps. For this model we need to use a generalised version of the Itô formula which gives us the following
\[
\frac{df(t, S(t))}{f(t, S(t))} = \left( f'(t, S(t)) + \frac{\sigma^2}{2} f''(t, S(t)) + r f'(t, S(t)) \right) \delta S(t) + \sigma f'(t, S(t)) \frac{dW^Q(t)}{f(t, S(t))} + \frac{df(t, S(t))}{f(t, S(t))} \delta N(t).
\]

Applying this to the function $f(t, S(t)) = S(t)$ we of course get back the dynamics for $S$ given above (but please do check this).

(a) How should we choose $\alpha$ here? The key observation here is that we want an arbitrage free market. So let $t$ and $T$ be to arbitrary times satisfying $t < T$. Buying the stock at time $t$ and selling it at time $T$ we get the following discounted cash flow
\[
S(T) \frac{B(t)}{B(T)} + \int_{t}^{T} \frac{B(t)}{B(u)} \delta S(u) du = S(T) \frac{B(t)}{B(T)} + \int_{t}^{T} \delta \frac{B(t)}{B(u)} S(u) du.
\]
This cost us $S(t)$ at time $t$ so for no arbitrage we must have

$$S(t) = \mathbb{E}_t^Q \left[ S(T) \frac{B(t)}{B(T)} + \int_t^T \frac{B(t)}{B(u)} S(u-) dN(u) | \mathcal{F}_t \right],$$

for all $0 < t < T$. No use this to find the correct value of $\alpha$.

(b) Given the result in a), use the fundamental theorems of asset pricing to check if this market is free of arbitrage and complete.

(c) Now use the model with the $\alpha$-value calculated in a) to find the value at time $t$ of a contract where we receive the stock at time $T$, where $T > t$.

(d) Find a replicating portfolio for the contract in b). Notice that for dividend paying stocks a simple buy and hold strategy will not work, since holding the stock you receive dividends whereas getting the stock at time $T$ give you no dividends between $t$ and $T$. A portfolio consisting of holdings in the stock and the bank account with value process $V(t) = a(t)S(t) + b(t)B(t)$ is self-financing if

$$dV(t) = a(t) dS(t) + a(t) dD(t) + b(t) dB(t).$$

Hint: Look at the delta-hedge and check that it will be self-financing.

(e) Now use the model with the $\alpha$-value calculated in a) to price a standard European call option and compare the result to the standard BS-formula. Conclusions? Remember that holding an European call option does not give you any dividends.

(f) Find a replicating portfolio for the contract in e).
Chapter 2

Solutions

Sol: 1.1 Martingales

Sol: 1.1.1 Using that $S_t$ has $Q$-dynamics

\[
    dS_t = rS_t dt + \sigma S_t dW_t
\]

and applying Itô’s formula to the process $X_t = (S_t)^\gamma e^{-rt}$ we get that

\[
    dX_t = (-rX_t + \gamma rX_t + \frac{1}{2} \sigma^2 \gamma (\gamma - 1) X_t)dt + \gamma \sigma X_t dW_t
\]

\[
    = X_t (r(\gamma - 1) + \frac{1}{2} \sigma^2 \gamma (\gamma - 1)) dt + \gamma \sigma X_t dW_t
\]

For $X_t$ to be a martingale we need that the drift is zero which gives that $\gamma$ should solve the equation

\[(\gamma - 1)(r + \frac{1}{2} \sigma^2 \gamma) = 0.\]

We see right away that it has the solutions $\gamma = 1$ and $\gamma = -\frac{2r}{\sigma^2}$, and since we wanted a solution different from $\gamma = 1$ we see that the requested solution is $\gamma = -\frac{2r}{\sigma^2}$. We should also check that $X$ satisfies the required moment conditions.

Plugging in $\gamma = -\frac{2r}{\sigma^2}$ we get the following dynamics for $X$:

\[
    dX_t = -\frac{2r}{\sigma} X_t dW_t,
\]

which has the solution

\[
    X_t = X_0 \exp\left(-\frac{2r^2}{\sigma^2} t - \frac{2r}{\sigma} W_t\right).
\]

Now we have (assuming $X_0 = S_0^\gamma > 0$)

\[
    E[|X_t|] = E[X_t] = E[X_0 \exp\left(-\frac{2r^2}{\sigma^2} t - \frac{2r}{\sigma} W_t\right)] = X_0 \exp\left(-\frac{2r^2}{\sigma^2} t + \frac{2r^2}{\sigma^2} t\right) = X_0 < \infty,
\]

which shows that $X$ is a martingale under $Q$.

Sol: 1.1.2 Let $X_t = f(t, W_t) = \epsilon^{t/2} \cos(W_t)$. We now calculate $dX_t$ with Itô’s formula:

\[
    dX_t = df(t, W_t) = \left\{ \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right\} dt + \frac{\partial f}{\partial x} dW_t
\]

\[
    = \left\{ \frac{1}{2} \epsilon^{t/2} \cos(W_t) - \frac{1}{2} \epsilon^{t/2} \cos(W_t) \right\} dt - \epsilon^{t/2} \sin(W_t) dW_t
\]

\[
    = -\epsilon^{t/2} \sin(W_t) dW_t
\]
This process has no drift term moreover we have that the diffusion part has finite second moment using the Itô isometry, i.e.
\[
E\left[ \int_0^t -e^{t/2} \sin(W_s)dW_s \right] = \int_0^t E[e^{s} \sin(W_s)dW_s] ds \leq \int_0^t E[e^{s}]ds = e^t - 1 < \infty.
\]
This gives that \( X(t) \) is a Martingale.

**Alternative solution:**
We can directly calculate \( E[X_t|\mathcal{F}_s] \) for \( s < t \) as
\[
E[X_t|\mathcal{F}_s] = E[e^{t/2} \cos(W_t)|\mathcal{F}_s] = E[e^{(t-s)/2} \cos(W_t - W_s) e^{s/2}|\mathcal{F}_s]
= e^{s/2} \cos(W_s) e^{(t-s)/2} + \sin(W_s) e^{s/2} 0
= e^{s/2} \cos(W_s) X_s.
\]
Finally we need to establish that \( E[|X_t|] < \infty \) which is easily done since \( |X_t| < e^{t/2} < \infty \).

---

**Sol: 1.2  Static replication of derivatives**

**Sol: 1.2.1** The easiest way to find a hedge for the pay-off is to draw a picture.

The solid line correspond to the original pay-off and the dashed lines correspond to the stock, and one long European call option with strike \( 2K \) and two short European call options with strike \( K \) respectively (top to bottom). So we can replicate the pay-off using one stock, and one long European call option with strike \( 2K \) and two short European call option with strike \( K \). To do things properly we should check that
\[
S - 2 \max(S - K, 0) + \max(S - 2K, 0) = \max(K - |S - K|, 0)
\]
for all \( S \geq 0 \). We start with the left hand side
\[
S - 2 \max(S - K, 0) + \max(S - 2K, 0) = \begin{cases} S & 0 \leq S \leq K \\ 2K - S & K \leq S \leq 2K \\ 0 & S \geq 2K \end{cases}
\]
and then move on to the right hand side
\[
\max(K - |S - K|, 0) = \begin{cases} S & 0 \leq S \leq K \\ 2K - S & K \leq S \leq 2K \\ 0 & S \geq 2K \end{cases}
\]
which shows the equivalence between the two pay-offs.
Sol: 1.2.2 Let $\Pi(t)$ be the price of the derivative $X$ for $t \leq T$, moreover let $P_K(t)$ and $C_K(t)$ be the price at time $t$ of a put option and call option respectively both with maturity $T$ and strike price $K$. The price $\Pi(t)$ is given by any of the following equivalent portfolios (there are more possible equivalent portfolios).

$$
\Pi(t) = P_{2K}(t) - P_K(t) = K \frac{B(t)}{B(T)} - C_K(t) + C_{2K}(t) = S(t) + P_K(t) - 2C_K(t) + C_{2K}(t)
$$

Looking at the payoffs at time $T$ show that the portfolios have the same payoff as the contract $X$, i.e.

$$
\begin{cases}
K & S(T) \leq K \\
2K - S(T) & K \leq S(T) \leq 2K \\
0 & S(T) \geq 2K
\end{cases}
$$

Sol: 1.3 PDE:s and Feynman-Kac formula

Sol: 1.3.1 According to Feynman-Kac’s representation theorem the PDE is solved by

$$
f(t, x) = \mathbb{E}[e^{X_T}|X_t = x],
$$

where $X$ has the dynamics

$$
\begin{align*}
\text{d}X_t &= (\mu - \frac{\sigma^2}{2})\text{d}t + \sigma \text{d}W_t, \quad t \leq s \leq T \\
X_t &= x.
\end{align*}
$$

It is straightforward to see (at least it should be) that $X$ is is a BM with drift $\mu - \frac{\sigma^2}{2}$ and standard deviation $\sigma$ starting at $x$ at time $t$, i.e.

$$
X_T = x + (\mu - \frac{\sigma^2}{2})(T - t) + \sigma(W_T - W_t).
$$

Looking at $\exp(X_T)$ we see that it has exactly the same distribution as the geometric BM in the standard Black-Scholes model. We thus get that

$$
f(t, x) = \mathbb{E}[e^{X_T}|X_t = x] = \mathbb{E}[e^{\mu(T-t) + \sigma(W_T - W_t)}] = e^{\mu(T-t) + \frac{\sigma^2}{2}(T-t)} = e^{\mu(T-t)}.
$$

We should also check the obtained solution fulfills the PDE and the boundary condition. We start with the last task

$$
\frac{\partial f(t, x)}{\partial t} + \left( \mu - \frac{\sigma^2}{2} \right) \frac{\partial f(t, x)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2} = -\mu f(t, x) + (\mu - \frac{\sigma^2}{2})f(t, x) + \frac{\sigma^2}{2}f(t, x) = 0,
$$

which verifies that the obtained solution is correct.

Sol: 1.4 Pricing of derivatives and hedging

Sol: 1.4.1 (a) Under $Q$ we have that

$$
S_T = S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)}.
$$
According to the risk-neutral-valuation-formula we have that the price of the derivative, \( P_t = F(t, S_t) \), is given by

\[
F(t, S_t) = e^{-r(T-t)}E^Q \left[ \Phi(S_T) | \mathcal{F}_t \right] = e^{-r(T-t)}E^Q \left[ \Phi(S_T) | S_t \right] = e^{-r(T-t)}E^Q \left[ \max(K - (rT)^{1/2} \sqrt{\sigma^2(T-t)} + \sigmaW_t + W_t, 0) \right] E^Q \left[ (K - (rT)^{1/2} \sqrt{\sigma^2(T-t)} + \sigmaW_t + W_t)1_{K < (rT)^{1/2} \sqrt{\sigma^2(T-t)} + \sigmaW_t + W_t} | S_t \right] = e^{-r(T-t)}E^Q \left[ (K - S_t^2 e^{2(r-\frac{\sigma^2}{2})(T-t) + 2\sigma \sqrt{T-t}Z}) \Phi(S_T) | S_t \right]
\]

where \( Z \sim N(0, 1) \). Expressing the expectation as an integral we get

\[
F(t, S_t) = e^{-r(T-t)} \int_{-\infty}^{d_1} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz - S_t^2 \left( e^{(r-\sigma^2)(T-t)} \int_{-\infty}^{d_1} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial z} \left( e^{\frac{z^2}{2}} \right) dz \right)
\]

where \( N(d_1) = \frac{\log(S_t/K) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \).

(b) Let \( b = (b_b, b_h) \) be a self-financing portfolio with value process \( V(t) = b_b(t)B_t + b_h(t)S_t \). By the self-financing condition we get that the dynamics of \( V \) is given by

\[
dV(t) = b_b(t)dB_t + b_h(t)dS_t.
\]

We now want to choose \( b_b \) and \( b_h \) such \( V \) and the derivative with price process \( P \) has the same dynamics. The delta-hedge gives that we should choose \( b_h(t) \) as

\[
b_h(t) = \frac{\partial}{\partial S_t} F(t, S_t)
\]

and since we should have \( V_t = F(t, S_t) \) we get that

\[
b_h(t) = \frac{F(t, S_t) - b_b(t)S_t}{B_t} = \frac{F(t, S_t) - S_t \frac{\partial}{\partial S_t} F(t, S_t)}{B_t}.
\]

Using the \( F \) obtained in (a) we get that

\[
b_h(t) = -2Se^{(r+\sigma^2)(T-t)}N(d_t) + 2\sigma \sqrt{T-t} \]

\[
b_h(t) = e^{-r(T-t)}KN(d_t) + S_t^2 e^{(r+\sigma^2)(T-t)}N(d_t) - 2\sigma \sqrt{T-t}.
\]

Alternative derivation: If you have forgotten the delta-hedge you should look at following to refresh your memory.

Applying Ito's formula to \( P_t = F(t, S_t) \) we get that

\[
dP_t = \left( \frac{\partial}{\partial t} F(t, S_t) + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2}{\partial S_t^2} F(t, S_t) \right) dt + \left( \frac{\partial}{\partial S_t} F(t, S_t) \right) dB_t + \left( \sigma \sqrt{T-t} \right) dW_t
\]

\[
dP_t = \left( \frac{\partial}{\partial t} F(t, S_t) + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2}{\partial S_t^2} F(t, S_t) \right) \frac{rB_t}{rB_t} dt + \left( \frac{\partial}{\partial S_t} F(t, S_t) \right) dB_t + \left( \sigma \sqrt{T-t} \right) dW_t.
\]
Sol: 1.5 Change of measure and completeness

Sol: 1.5.1 The mathematics in this solution is only simple linear algebra but the financial implications are still noteworthy.

(a) The condition for the risk neutral probabilities are that

\[ S_0 = E^Q[S_t | S_0] = q_d dS_0 + q_1 S_0 + q_u uS_0 \]

and that the risk neutral probabilities sum to one. (We must also have that the probabilities are positive and smaller than one.) We thus obtain the following linear system of equations for the risk neutral probabilities:

\[ \begin{bmatrix} 1 & 1 & 1 \\ 3/5 & 1 & 6/5 \end{bmatrix} \begin{bmatrix} q_d \\ q_1 \\ q_u \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

So we have two equations and three unknowns so there exists multiple (infinitely many) solutions. In principle we are done here since now we have that the risk neutral probabilities are not unique. We should check that at least two solutions satisfy that they are non-negative and smaller than one. We however need the full solution in (d) so we solve it now. We start by finding all solutions and then we make restrictions so that the probabilities are positive and smaller than one. Three unknowns and two equations gives us a one parametric family of solutions. We put \( q_d = \vartheta \) and aim to express the other probabilities using \( \vartheta \). Plugging this into the equations above we obtain

\[ \begin{bmatrix} 1 & 1 & 6/5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_u \end{bmatrix} = \begin{bmatrix} 1 - \vartheta \\ 1 - 3\vartheta/5 \end{bmatrix}. \]

This has the solution \( [q_1, q_u] = [1 - 3\vartheta, 2\vartheta] \). We here immediately see that \( 0 \leq \vartheta \leq 1/3 \) for this to be probabilities. For financial reasons we do not want the cases \( \vartheta = 0 \) (implies \( S = B \) with prob 1) and \( \vartheta = 1/3 \) (which gives us back the binomial model with prob 1). Taking this into consideration gives us the solutions \( [q_d, q_1, q_u] = [\vartheta, 1 - 3\vartheta, 2\vartheta], 0 < \vartheta < 1/3 \). We can thus by changing \( \vartheta \) calibrate to different market situations.

(b) Since we all ready have the full solution all we need to do here is to find \( \vartheta \) so that the price of the derivative match, i.e. we calibrate our model to fit the market price. We have that \( E^Q[S_t^2 | S_0 = 1] = 26/25 \) which gives that

\[ \vartheta(3/5)^2 + (1 - 3\vartheta)^2 + 2\vartheta(6/5)^2 = 26/25, \]

\[ 1 + \vartheta(9 - 75 + 72)/25 = 26/25, \]

\[ 1 + 6\vartheta/25 = 26/25 \]

\[ 6\vartheta = 1 \]

\[ \vartheta = 1/6. \]

So we have a unique risk neutral measure with \( [q_d, q_1, q_u] = [1/6, 1/2, 1/3] \).
(c) We now use the probabilities from (b) to price the derivative
\[
\mathbb{E}^Q[\max(S_t - 9/10, 0)|S_0 = 1] = 0 \cdot 1/6 + (1 - 9/10) \cdot 1/2 + (6/5 - 9/10) \cdot 1/3
\]
\[
= 1/20 + 1/10 = 3/20.
\]
To find the hedge portfolio we must solve system of linear equations just as in the binomial model but now we have three assets and three equations. So we should find \( h_X, h_S \) and \( h_B \) (which are the portfolio weights for \( X = S^2, S \) and \( B \) respectively) such that
\[
h_X S_t^2 + h_S S_t + h_B B_t = \max(S_t - 9/10, 0),
\]
for all possible outcomes of \( S_t \). This gives rise to following system of equations
\[
\begin{bmatrix}
9/25 & 3/5 & 1 \\
1 & 1 & 1 \\
36/25 & 6/5 & 1
\end{bmatrix}
\begin{bmatrix}
h_X \\
h_S \\
h_B
\end{bmatrix} = \begin{bmatrix}
0 \\
1/10 \\
3/10
\end{bmatrix}
\]
We easily solve this by Gauss elimination obtaining
\[
\begin{bmatrix}
h_X \\
h_S \\
h_B
\end{bmatrix} = \begin{bmatrix}
5/4 \\
-7/4 \\
3/5
\end{bmatrix}.
\]
(d) From (a) and (b) we get that
\[
\mathbb{E}^Q[S_t^2|S_0 = 1] = 1 + 6\vartheta/25, \quad 0 < \vartheta < 1/3.
\]
This is an increasing function of \( \vartheta \) thus we get the upper and lower bounds for the price by plugging in the allowed upper and lower bounds for \( \vartheta \). This gives that
\[
1 < \mathbb{E}^Q[S_t^2|S_0] < 27/25,
\]
which was to be shown. So it is not possible to calibrate the model to prices outside this interval, since that would imply negative probabilities and or probabilities larger than one.

Sol: 1.5.2 (a) Regardless of the choice of martingale measure \( Q \) we have that \( S_t^{(1)} \) will have the \( Q \)-dynamics
\[
dS_t^{(1)} = r S_t^{(1)} dt + S_t^{(1)} \left( \sigma_{11} dW_t^{(1),Q} + \sigma_{12} dW_t^{(2),Q} \right),
\]
where \( W_t^{(1),Q} \) and \( W_t^{(2),Q} \) are standard BM:s under \( Q \). We therefore get that
\[
S_T = S_0 e^{(r - \frac{\sigma_{11}^2 + \sigma_{12}^2}{2}) T - \frac{1}{2} \sigma_{11}^2 T} \Phi \left( \frac{\sigma_{11} \sqrt{T} - \sigma_{12} \sqrt{T} - \sqrt{T - \sigma_{11}^2 T - \sigma_{12}^2 T}}{\sqrt{2 \pi}} \right)
\]
which has the same distribution as
\[
S_0 e^{(r - \frac{\sigma_{11}^2 + \sigma_{12}^2}{2}) T - \frac{1}{2} \sigma_{11}^2 T} \Phi \left( \frac{\sigma_{11} \sqrt{T} - \sigma_{12} \sqrt{T} - \sqrt{T - \sigma_{11}^2 T - \sigma_{12}^2 T}}{\sqrt{2 \pi}} \right),
\]
where \( Z \in N(0, 1) \).

Thus the price at time \( t, \Pi_t \), of a simple claim with maturity \( T \) and pay-off \( \Phi(S_T) \) is given by
\[
\Pi_t = e^{-r(T - t)} \int_{-\infty}^{\infty} \Phi \left( S_0 e^{(r - \frac{\sigma_{11}^2 + \sigma_{12}^2}{2}) (T - t) + \sigma_{11} \sigma_{12} T - \frac{1}{2} \sigma_{11}^2 T} \right) e^{-\frac{z^2}{2}} dz
\]
regardless of the choice of \( Q \), which was to be shown.
(b) The dynamics of \( W_t^{(2)} \) under \( Q \) is given by
\[
dW_t^{(2)} = dW_t^{(2),Q} - g_2(t) dt
\]
where \( g_1(t), g_2(t) \) are any functions satisfying the equation
\[
\mu_t - \sigma_{11} g_1(t) - \sigma_{12} g_2(t) = r, \quad \text{for}, \quad 0 \leq t \leq T
\]
and the Novikov condition
\[
\mathbb{E}^P \left[ e^{\int_0^T g_1^2(t) + g_2^2(t) dt} \right] < \infty.
\]
Two possible choices are e.g. \((g_1(t), g_2(t)) = ((\mu_1 - r)/\sigma_{11}, 0)\) and \((g_1(t), g_2(t)) = (0, (\mu_1 - r)/\sigma_{12})\). These two choices will give different prices to the derivative \(\Pi_{g_i} \geq K\), more precisely

\[
e^{-r(T-t)}\mathbb{Q} \left( \frac{W_i(t) - K}{\sqrt{T-t}} \right)
\]

and

\[
e^{-r(T-t)}\mathbb{Q} \left( \frac{W_i(t) - \frac{\mu_i - r}{\sigma_{12}}(T-t)}{\sqrt{T-t}} \right)
\]

respectively.

(c) We get that \((g_1(t), g_2(t))\) should solve the following system of linear equations

\[
-\sigma_{11}g_1(t) - \sigma_{12}g_2(t) = r - \mu_1
\]

\[
-\sigma_{22}g_2(t) = r - \mu_2
\]

which has the unique solution (provided that \(\sigma_{11}\sigma_{22} \neq 0\))

\[
g_2(t) = \frac{\mu_2 - r}{\sigma_{22}}, \quad g_1(t) = \frac{\mu_1 - r - \frac{\mu_2 - r}{\sigma_{12}}}{\sigma_{11}},
\]

and since the Girsanov kernel is unique we get that the market is free of arbitrage and complete.

(d) Using the result from (c) we get that the price of the derivative is given as

\[
e^{-r(T-t)}\mathbb{Q} \left( \frac{W_i(t) - \frac{\mu_i - r}{\sigma_{12}}(T-t)}{\sqrt{T-t}} \right).
\]

\[\blacksquare\]

Remark: Note that usually we say that a derivative or contingent claim should only be a function of the (trajectory of the) underlying asset \((S_t)\) up to maturity. The derivative in (a) is such a contract and there it does not matter that we have added an extra Brownian motion. However the contract in (b) is not a function of \(S_t\) only, here we need more information to know the value at maturity. This ambiguity regarding completeness is the reason why it is called a meta-theorem (see Åberg p. 111 and Björk p. 122) and not a theorem. The moral of this is perhaps that we should use a model description that only uses as many driving Brownian motions as we actually need to represent the model.

Sol: 1.6 Change of numeraires

Sol: 1.6.1 We want to price the derivative \(X\) with maturity \(T\) and pay-off:

\[
\Phi(S_1(T), S_2(T)) = \max(S_2(T) - S_1(T), 0).
\]

According to the risk-neutral valuation formula the price at time \(t\), \(\Pi_X(t)\), is given by

\[
\Pi_X(t) = e^{-r(T-t)}\mathbb{Q}^t \left[ \max(S_2(T) - S_1(T), 0) | \mathcal{F}_t \right].
\]

There are now some different approaches to calculate this expectation using change of numeraire techniques. The perhaps easiest approach is to use \(S_1\) as a numeraire this leads to

\[
\Pi_X(t) = S_1(t)\mathbb{Q}^t \left[ 1 \left/ S_1(t) \right| \max(S_2(T) - S_1(T), 0) | \mathcal{F}_t \right] = S_1(t)\mathbb{Q}^t \left[ \max(S_2(T) - S_1(T), 0) | \mathcal{F}_t \right],
\]

where \(\mathbb{Q}^{S_1}\) is the numeraire measure for \(S_1\). This can now be seen as a European call option on the ratio \(S_2/S_1\) with strike 1. Moreover since \(S_2/S_1\) is a ratio between a traded asset and a numeraire it is automatically a martingale under \(\mathbb{Q}^{S_1}\). Using that the volatilities does not change when we change measure we can by calculate the volatilities under \(\mathbb{Q}\) as

\[
\left( \frac{\partial}{\partial S_1} S_1(t) \right) S_1(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) + \left( \frac{\partial}{\partial S_2} S_1(t) \right) S_1(t)(\sigma_{21}dW_1(t) + \sigma_{22}dW_2(t))
\]

\[
= S_1(t) \sigma_1(t) \left( \frac{S_2(t)}{S_1(t)} \right) \left( (\sigma_{21} - \sigma_{11})dW_1(t) + (\sigma_{22} - \sigma_{12})dW_2(t) \right)
\]
This gives that the dynamics for $S_t/S_1$ under $Q^2_1$ for $s \geq t$ is given by
\[
\begin{align*}
\frac{dS_t}{S_1(t)} &= \frac{S_1(t)}{S_1(t)} \left( (\sigma_{21} - \sigma_{11})dW^Q_1(t) + (\sigma_{22} - \sigma_{12})dW^Q_2(t) \right), \\
\frac{S_t}{S_1(t)} &= \frac{S_1(t)}{s_1(t)}
\end{align*}
\]
where $W^Q_1(t)$ and $W^Q_2(t)$ are independent standard $Q^2_1$ Brownian motions. We can now solve this SDE to obtain
\[
\frac{S_2(T)}{S_1(T)} = \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \left( (\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2 \right) (T - t) + (\sigma_{21} - \sigma_{11})(W^Q_1(T) - W^Q_1(t)) \\
+ (\sigma_{22} - \sigma_{12})(W^Q_2(T) - W^Q_2(t)) \right)
\]
This has the same distribution as
\[
\frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma\sqrt{T - t}G \right),
\]
where $G$ is a standard Gaussian random variable and where
\[
\sigma = \sqrt{(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2}.
\]
We can therefore calculate the price $P_X(t)$ as
\[
\begin{align*}
P_X(t) &= \frac{s_2(t)}{s_1(t)} \int_{-\infty}^{\infty} \max \left( \frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma\sqrt{T - t}g \right) - 1, 0 \right) \frac{\exp(-g^2/2)}{\sqrt{2\pi}} \, dg \\
&= \frac{s_2(t)}{s_1(t)} \int_{-d}^{d} \frac{s_2(t)}{s_1(t)} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma\sqrt{T - t}g - \frac{g^2}{2} \right) \frac{\exp(-g^2/2)}{\sqrt{2\pi}} \, dg \\
&= \frac{s_2(t)}{s_1(t)} \int_{-d}^{d} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} (g - \sigma\sqrt{T - t})^2 \right) \, dg - \frac{s_1(t)}{s_1(t)} \int_{-d}^{d} \frac{\exp(-g^2/2)}{\sqrt{2\pi}} \, dg \\
&= \frac{s_2(t)}{s_1(t)} N(d + \sigma\sqrt{T - t}) - s_1(t) (1 - N(-d)) \\
&= \frac{s_2(t)}{s_1(t)} N(d + \sigma\sqrt{T - t}) - s_1(t) N(d),
\end{align*}
\]
where
\[
d = \frac{\ln(s_2(t)/s_1(t)) - \sigma^2(T - t)/2}{\sigma\sqrt{T - t}},
\]
and where $N$ is the distribution function of the standard Gaussian distribution. \textbf{Alternative solution:} We can also use both $S_1$ and $S_2$ as numeraires which gives that
\[
P_X(t) = S_2(t)E^{Q^2_1} \left[ I(S_2(T) > S_1(T)) \big| F_t \right] - S_1(t)E^{Q^2_1} \left[ I(S_2(T) > S_1(T)) \big| F_t \right].
\]
This can now be rewritten as
\[
P_X(t) = S_2(t)E^{Q^2_1} \left[ I \left( \frac{S_2(T)}{S_2(T)} < 1 \right) \big| F_t \right] - S_1(t)E^{Q^2_1} \left[ I \left( \frac{S_2(T)}{S_2(T)} > 1 \right) \big| F_t \right].
\]
Now $S_t/S_2$ is a $Q^2_1$-martingale and $S_t/S_1$ is a $Q^2_1$-martingale. By using the same type of argument and calculations as in the first solution we obtain that $S_1(T)/S_2(T)$ under $Q^2_1$ has the same distribution as
\[
\frac{s_1(t)}{s_2(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma\sqrt{T - t}G \right),
\]
and that $S_2(T)/S_1(T)$ under $Q^2_1$ has the same distribution as
\[
\frac{s_2(t)}{s_1(t)} \exp \left( -\frac{1}{2} \sigma^2(T - t) + \sigma\sqrt{T - t}G \right),
\]
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where $\sigma$ and $G$ are as defined in the first solution. Straightforward calculation now give that
\[
P_i(t) = s_i(t)N(d + \sigma\sqrt{T-t}) - s_i(t)N(d),
\]
where $d$ are defined as in the first solution.

**Alternative solution 2:**
We can also calculate the dynamics for $S_1$ and $S_2$ under both $\mathbb{Q}^{G_1}$ and $\mathbb{Q}^{G_2}$. For this we need to find two two-dimensional Girsanov kernels, where we also need to look at dynamics of the bank-account to get the right Girsanov kernels. This is however as seen from the calculations above an unnecessary detour. For the sake of completeness we supply the appropriate Girsanov kernels:
\[
\begin{align*}
\dot{g}^{1*}_{1} &= -\sigma_{11}, & \dot{g}^{1*}_{2} &= -\sigma_{12} \\
\dot{g}^{2*}_{1} &= -\sigma_{21}, & \dot{g}^{2*}_{2} &= -\sigma_{22}.
\end{align*}
\]

By the same type of calculations as in the first two solutions we finally arrive at the same answer. ■

**Sol: 1.6.2**

(a) By using that $Z(t) = S(t)/P(t, T)$, for $0 \leq t \leq T$, is a $\mathbb{Q}^{G}$ martingale and that volatilities do not change when we change measure, we see that we can calculate the volatility function $\nu(t, T)$ using the diffusion part of the $\mathbb{Q}$-dynamics for $S(t)/P(t, T)$. This gives that
\[
Z(t)\nu(t, T) = \begin{cases}
\frac{\partial \mathcal{S}}{\partial \mathcal{P}}(t, T) & S(t)\gamma(T-t) dW_1(t) + \frac{\partial \mathcal{S}}{\partial \mathcal{S}}(t, T) S(t) \sigma dW_2(t) \\
-Z(t)\gamma(T-t) dW_1(t) + Z(t)\sigma dW_2(t) \\
= Z(t) [ -\gamma(T-t) \sigma] \begin{bmatrix}
\frac{dW_1(t)}{dW_2(t)} 
\end{bmatrix}
\end{cases}
\]

This gives that $\nu(t, T) = [-\gamma(T-t) \sigma]$. 

(b) Using that $S(T) = Z(T)$ we can view the contract as written on the process $Z$ instead of $S$. So the price of the European call option can thus be calculated as
\[
\Pi(t) = p(t, T)\mathbb{E}^{\mathbb{Q}^{G}} \left[ \max(Z(T) - K, 0) | \mathcal{F}_t \right].
\]

To calculate this we must find the distribution of $Z(T)$ under $\mathbb{Q}^{G}$. Solving the SDE for $Z$ under $\mathbb{Q}^{G}$ gives that
\[
Z(T) = Z(t) \exp \left( -\frac{1}{2} \int_t^T [\nu(s, T)]^2 ds + \int_t^T \nu(s, T) dW_s^{\mathbb{Q}^{G}}(s) \right).
\]

This now has the same distribution as
\[
Z(t) \exp \left( -\frac{1}{2} \Sigma(t, T)^2 (T-t) + \Sigma(t, T)\sqrt{T-t} \right),
\]

where
\[
\Sigma(t, T) = \sqrt{\int_t^T \gamma^2(T-s) + \sigma^2 ds} = \sqrt{\gamma^2(T-t)^2 / 2 + \sigma^2 (T-t) / T-t},
\]

and where $G$ is standard Gaussian random variable. By exactly the same calculations as in the derivation of the standard Black-Scholes formula but with $\sigma$ replaced by $\Sigma(t, T)$ and $e^{-r(T-t)}$ replace by $p(t, T)$ we obtain
\[
\Pi(t) = p(t, T)Z(t)N(d + \Sigma(t, T)\sqrt{T-t}) - p(t, T)KN(d) \\
= S(t)N(d + \Sigma(t, T)\sqrt{T-t}) - p(t, T)KN(d),
\]

where
\[
d = \frac{\ln(Z(t)/K) - \Sigma(t, T)^2 (T-t) / 2}{\Sigma(t, T)\sqrt{T-t}} = \frac{\ln(S(t)/K) - \ln(P(t, T)) - \Sigma(t, T)^2 (T-t) / 2}{\Sigma(t, T)\sqrt{T-t}},
\]

and where $N$ is the distribution function of the standard Gaussian distribution.
To check that we get back the original formula if \( r(t) \equiv r \) and \( \rho(t, T) \equiv \sigma \), we notice that this imply that \( \gamma = 0 \) and \( p(t, T) = \exp(-r(T - t)) \). This gives that \( \Sigma(t, T) \equiv \sigma \) and \( -\ln(p(t, T)) = r(T - t) \). Now plugging this into our price gives

\[
\Pi(t) = S(t)N(d + \sigma \sqrt{T - t} - e^{-r(T - t)} KN(d),
\]

where

\[
d = \frac{\ln(S(t)/K) + r(T - t) - \sigma^2(T - t)/2}{\sigma \sqrt{T - t}},
\]

which we recognize as the ordinary Black-Scholes formula.

**Sol:** 1.6.3 (a) We start by examining the pay-off:

\[
\begin{align*}
\max(S_A(T), S_B(T)) &= S_A(T)I(S_A(T) \geq S_B(T)) + S_B(T)I(S_B(T) > S_A(T)) \\
&= S_A(T)I \left( \frac{S_B(T)}{S_A(T)} \leq 1 \right) + S_B(T)I \left( \frac{S_A(T)}{S_B(T)} < 1 \right).
\end{align*}
\]

After this rewriting we are ready to attack the problem with RNVE. Let \( \Pi(t, S_A(t), S_B(t)) \) be the value of the contract at time \( t \). Thus we have

\[
\begin{align*}
\Pi(t, S_A(t), S_B(t)) &= \mathbb{E}^a \left[ \frac{B(t)}{B(T)} \max(S_A(T), S_B(T)) | \mathcal{F}_t \right] \\
&= \mathbb{E}^a \left[ \frac{B(t)}{B(T)} S_A(T)I \left( \frac{S_B(T)}{S_A(T)} \leq 1 \right) | \mathcal{F}_t \right] \\
&\quad + \mathbb{E}^a \left[ \frac{B(t)}{B(T)} S_B(T)I \left( \frac{S_A(T)}{S_B(T)} < 1 \right) | \mathcal{F}_t \right].
\end{align*}
\]

Now we apply the change of numeraire technique to simplify the calculations:

\[
\begin{align*}
\Pi(t, S_A(t), S_B(t)) &= \mathbb{E}^{Q^{S_A}} \left[ \frac{S_A(t)}{S_A(T)} S_A(T)I \left( \frac{S_B(T)}{S_A(T)} \leq 1 \right) | \mathcal{F}_t \right] \\
&\quad + \mathbb{E}^{Q^{S_A}} \left[ \frac{S_B(t)}{S_B(T)} S_B(T)I \left( \frac{S_A(T)}{S_B(T)} < 1 \right) | \mathcal{F}_t \right] \\
&= S_A(t)\mathbb{E}^{Q^{S_A}} \left[ I \left( \frac{S_A(T)}{S_B(T)} \leq 1 \right) | \mathcal{F}_t \right] + S_B(t)\mathbb{E}^{Q^{S_A}} \left[ I \left( \frac{S_A(T)}{S_B(T)} < 1 \right) | \mathcal{F}_t \right] \\
&= S_A(t)\mathbb{E}^{Q^{S_A}} \left[ \frac{S_B(T)}{S_A(T)} \leq 1 | \mathcal{F}_t \right] + S_B(t)\mathbb{E}^{Q^{S_A}} \left[ \frac{S_A(T)}{S_B(T)} < 1 | \mathcal{F}_t \right].
\end{align*}
\]

We can now use that \( S_B(t)/S_A(t) \) is a \( Q^{S_A} \) martingale and that \( S_A(t)/S_B(t) \) is a \( Q^{S_B} \) martingale, since they are both ratios of traded assets and numeraires. We then get the following \( Q^{S_A} \)-dynamics for \( S_B(t)/S_A(t) \) (using Ito and the MG-property)

\[
\begin{align*}
\frac{dS_B(t)}{S_A(t)} &= \frac{S_B(t)}{S_A(t)} \left( (\sigma_B \rho - \sigma_A)dW_1^{Q^{S_A}}(t) + \sigma_B \sqrt{1 - \rho^2} dW_2^{Q^{S_A}}(t) \right) \\
&= \frac{S_B(t)}{S_A(t)} \sqrt{\sigma_B^2 - \sigma_A^2} dW_2^{Q^{S_A}}(t) \\
&= \frac{S_B(t)}{S_A(t)} \sigma dW_2^{Q^{S_A}}(t),
\end{align*}
\]

where \( \sigma = \sqrt{\sigma_B^2 - \sigma_A^2} \) and \( W^{Q^{S_A}} \) is a standard \( Q^{S_A} \) BM. So then we get that

\[
\frac{S_B(T)}{S_A(T)} \triangleq \frac{S_B(t)}{S_A(t)} e^{-\frac{\sigma^2T}{2} + \sigma \sqrt{T - t} G},
\]

where \( G \) is standard Gaussian random variable. Using the same type of arguments we get the following distribution for \( S_A(t)/S_B(t) \) under \( Q^{S_B} \):

\[
\frac{S_A(T)}{S_B(T)} \triangleq \frac{S_A(t)}{S_B(t)} e^{-\frac{\sigma^2T}{2} - \sigma \sqrt{T - t} G},
\]

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where $G$ is standard Gaussian random variable and where $\bar{\sigma} = \sqrt{(\sigma_B^2 - \sigma_A^2 + \sigma^2_{\bar{\sigma}}}$ as before. Putting this together we obtain

$$
P(t, S_A(t), S_B(t)) = S_A(t)Q_{S_A} \left( \frac{S_A(t)}{S_A(t)} e^{-\frac{\sigma_A^2}{2}(T-t) + \bar{\sigma}\sqrt{T-t}G} \leq 1|F_t \right)$$

$$+S_B(t)Q_{S_B} \left( \frac{S_B(t)}{S_B(t)} e^{-\frac{\sigma_B^2}{2}(T-t) + \bar{\sigma}\sqrt{T-t}G} < 1|F_t \right)$$

$$= S_A(t)Q_{S_A} \left( G \leq \frac{\ln \left( \frac{S_A(t)}{S_A(t)} \right) + \bar{\sigma} \frac{T-t}{\sqrt{T-t}}}{\bar{\sigma}\sqrt{T-t}} \right)$$

$$+S_B(t)Q_{S_B} \left( G < \frac{\ln \left( \frac{S_B(t)}{S_B(t)} \right) + \bar{\sigma} \frac{T-t}{\sqrt{T-t}}}{\bar{\sigma}\sqrt{T-t}} \right)$$

$$= S_A(t)N \left( \ln \left( \frac{S_A(t)}{S_A(t)} \right) + \frac{\sigma_A^2}{2} T \right) \bar{\sigma}\sqrt{T-t} \right) + S_B(t)N \left( \ln \left( \frac{S_B(t)}{S_B(t)} \right) + \frac{\sigma_B^2}{2} T \right) \bar{\sigma}\sqrt{T-t} \right)$$

As a preparation for (d) we also look explicitly at the price at time zero

$$P(0, S_A(0), S_B(0)) = S_A(0)N \left( \ln \left( \frac{S_A(0)}{S_A(0)} \right) + \frac{\sigma_A^2}{2} T \right) \bar{\sigma}\sqrt{T-t} \right) + S_B(0)N \left( \ln \left( \frac{S_B(0)}{S_B(0)} \right) + \frac{\sigma_B^2}{2} T \right) \bar{\sigma}\sqrt{T-t} \right)$$

(b) The easiest way of showing this is to re-write the value at time $T$ for (b)

$$\Phi_{\bar{\sigma}}(S_A(T), S_B(T)) = S_A(T) + (S_B(T) - S_A(T))^+$$

$$= S_A(T) + (S_B(T) - S_A(T))I(S_A(T) < S_B(T))$$

$$= S_A(T)I(S_A(T) \geq S_B(T)) + S_B(T)I(S_A(T) < S_B(T))$$

$$= max(S_A(T), S_B(T)).$$

This was exactly what we needed to show since now the values in (a) and (b) coincide for all possible outcomes at maturity and thus they must also coincide for all previous times to avoid arbitrage.

(c) The Black-Scholes like market in this problem is complete so we can hedge all contingent claims. To find the hedge we use the standard $\Delta$-hedge approach.

$$h_A(t) = \frac{1}{B(t)} (P(t, S_A(t), S_B(t)) - h_A(t)S_A(t) - h_B(t)S_B(t))$$

$$h_A(t) = \frac{\partial}{\partial S_A} P(t, S_A(t), S_B(t))$$

$$h_B(t) = \frac{\partial}{\partial S_B} P(t, S_A(t), S_B(t))$$

We start with $h_A(t)$

$$h_A(t) = \frac{\partial}{\partial S_A} (S_A(t)N(d_1) + S_B(t)N(d_2))$$

$$= N(d_1) + S_A(t)n(d_1) \frac{\partial}{\partial S_A} (d_1) + S_B(t)n(d_2) \frac{\partial}{\partial S_A} (d_2)$$

$$= N(d_1) + \frac{1}{\bar{\sigma}\sqrt{T-t}} \left( \frac{S_B(t)}{S_A(t)} n(d_1) - \frac{S_A(t)}{S_B(t)} n(d_2) \right)$$

$$= N(d_1) + \frac{1}{\bar{\sigma}\sqrt{T-t}} \left( n(d_1) - \frac{S_A(t)}{S_B(t)} n(d_2) \right)$$

where $n(x) = (d/dx)N(x) = e^{-x^2/2}/\sqrt{2\pi}.$

Using almost similar calculations we obtain that

$$h_B(t) = N(d_2) + \frac{1}{\bar{\sigma}\sqrt{T-t}} \left( \frac{S_A(t)}{S_B(t)} n(d_1) + n(d_2) \right).$$
Using this we finally obtain

\[ h_B(t) = \frac{1}{B(t)} (\Pi(t, S_A(t), S_B(t)) - h_{S_B}(t)S_A(t) - h_{S_A}(t)S_B(t)) \]

\[ = \frac{1}{B(t)} (\Pi(t, S_A(t), S_B(t)) - S_A(t)N(d_1) - S_B(t)N(d_2)) \]

\[ - \frac{1}{\sigma \sqrt{T-t}} (S_A(t)n(d_1) - S_B(t)n(d_2) - S_A(t)n(d_2) + S_B(t)n(d_1)) \]

\[ = \frac{1}{B(t)} (\Pi(t, S_A(t), S_B(t)) - \Pi(t, S_A(t), S_B(t))) = 0, \]

\[ h_{S_A}(t) = N(d_1) + \frac{1}{\sigma \sqrt{T-t}} \left( n(d_1) - \frac{S_B(t)}{S_A(t)}n(d_2) \right), \]

\[ h_{S_B}(t) = N(d_2) + \frac{1}{\sigma \sqrt{T-t}} \left( -\frac{S_A(t)}{S_B(t)}n(d_1) + n(d_2) \right). \]

(d) From (a) we get that the price of derivative at time zero is

\[ \Pi(0, S_A(0), S_B(0)) = S_A(0)N \left( \frac{\ln \left( \frac{S_A(0)}{S_A(T)} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \right) + S_B(0)N \left( \frac{\ln \left( \frac{S_B(0)}{S_B(T)} \right) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \right). \]

Now using that \( S_A(0) = S_B(0) = S \) and that \( \sigma_A = \sigma_B = \sigma \) we get

\[ \Pi(0, S_A(0), S_B(0)) = 2SN \left( \frac{\sigma \sqrt{T-\rho}}{\sqrt{2}} - \sqrt{T} \right) \]

We can then compare this to buying both stocks which costs 2S. So buying both stocks are always more expensive. Now looking at the potential pay-off we have that the derivative gives the best of the two stocks. We have that the correlation is negative so if one stocks goes up the other goes down compared to the risk free rate. In most cases one of the stocks will be worth considerably more than the other so most of the sum of both stocks will come from the best performing stock. So the derivative will in most cases give almost as high pay off as the sum of the two stocks but with less initial investment. So if our assumption about the correlation is true we should prefer the derivative to the sum of the two stocks.

Sol: 1.6.4 In this problem we first observe that we can view the bet as a derivative written on two assets. The key step will be to find a clever choice of numeraire to avoid calculating two dimensional integrals. According to the general risk neutral valuation formula we have that the price at time \( t \) of a derivative \( X \) with maturity \( T \) is given as

\[ \Pi(t, X) = E^{Q^N} \left[ \frac{N(t)}{N(T)} X \mid \mathcal{F}_t \right], \]

where \( N \) is a numeraire and \( Q^N \) is the corresponding numeraire measure.

(a) We now take a look at the pay off from Anna's point of view (pow)

\[ \Phi(S_A(T), S_B(T)) = \begin{cases} S_A(T) & S_B(T) < S_A(T), \\ 0 & S_B(T) = S_A(T), \\ -S_B(T) & S_A(T) < S_B(T), \end{cases} \]

This can be re-written as

\[ \Phi(S_A(T), S_B(T)) = \begin{cases} S_A(T) & S_B(T)/S_A(T) < 1, \\ 0 & S_B(T)/S_A(T) = 1, \\ S_B(T) & S_A(T)/S_B(T) < 1, \end{cases} \]

and we further see that it can also be written as

\[ \Phi(S_A(T), S_B(T)) = S_A(T)I(S_B(T)/S_A(T) \leq 1) - S_B(T)I(S_A(T)/S_B(T) \leq 1). \]
Plugging this into the general RNVF gives

\[
\Pi(t, X) = \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} \Phi(S_A(T), S_B(T)) \right]_{\mathcal{F}_t}
\]

\[
= \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} S_A(T) I(S_B(T)/S_A(T) \leq 1) \right]_{\mathcal{F}_t}
\]

\[
= \mathbb{E}^{Q^X} \left[ \frac{N(t)}{N(T)} S_B(T) I(S_A(T)/S_B(T) \leq 1) \right]_{\mathcal{F}_t}
\]

So if we choose \(S_A\) as numeraire in the first expectation and \(S_B\) as numeraire in the second expectation we get the following simplification

\[
\Pi(t, X) = S_A(t) \mathbb{E}^{Q_A} \left[ I(S_B(T)/S_A(T) \leq 1) \right]_{\mathcal{F}_t} - S_B(t) \mathbb{E}^{Q_B} \left[ I(S_A(T)/S_B(T) \leq 1) \right]_{\mathcal{F}_t} \quad (\ast)
\]

Now we have under \(S_A\) that \(S_B/S_A\) is the ratio of a traded asset and the numeraire and under \(S_B\) that \(S_A/S_B\) is the ratio of a traded asset and the numeraire. This further simplifies our calculations since both ratios are martingales under their respective numeraire measure, i.e. \(S_A\) and \(S_B\). So when we calculate the dynamics we only need to consider the diffusion parts of the dynamics since we know that the drift parts should be zero. First we consider \(S_B/S_A\) under \(S_A\)

\[
dS_B(u)/S_A(u) = -(S_B(u)/S_A(u)^2)S_A(u) \sigma_B dW_1^{S_A}(u) + (1/S_A(u))S_B(u)(\sigma_B dW_1^{S_A}(u) + \sigma_B \sqrt{1 - \sigma_B^2} dW_2^{S_A}(u))
\]

\[
= (S_B(u)/S_A(u))(\sigma_B - \sigma_A) dW_1^{S_A}(u) + \sqrt{1 - \sigma_B^2} \sigma_B dW_2^{S_A}(u).
\]

We then get that

\[
(S_B(T)/S_A(T)) \overset{d}{=} (S_B(t)/S_A(t)) \exp\left\{ -\frac{(\sigma_B - \sigma_A)^2}{2} + \frac{(1 - \sigma_B^2)}{2} \right\}
\]

\[
= (S_B(t)/S_A(t)) \exp\left\{ -\frac{(\sigma_B - \sigma_A)^2 + (1 - \sigma_B^2)}{2} \right\} \exp\left\{ (\sigma_B - \sigma_A)^2 T/2 + \Sigma^2(T - t)/2 + \Sigma T - \pi G_1 \right\},
\]

where \(\Sigma^2 = (\sigma_B - \sigma_A)^2 + (1 - \sigma_B^2) = \sigma_A^2 - 2\sigma_A \sigma_B + \sigma_B^2\) and where \(G_1\) is a standard Gaussian random variable.

Using the same type of calculations we get that \(S_A/S_B\) under \(S_B\) has the distribution

\[
(S_A(T)/S_B(T)) \overset{d}{=} (S_A(t)/S_B(t)) \exp\left\{ -\frac{(\sigma_A - \sigma_B)^2}{2} + \frac{(1 - \sigma_B^2)}{2} \right\} \exp\left\{ (\sigma_A - \sigma_B)^2 T/2 + \Sigma^2(T - t)/2 + \Sigma T - \pi G_2 \right\},
\]

where \(\Sigma^2 = \sigma_A^2 - 2\sigma_A \sigma_B + \sigma_B^2\) and where \(G_2\) is a standard Gaussian random variable. We can now plug this into the pricing equation (\ast) to obtain

\[
\Pi(t, X) = S_A(t) \int_{-\infty}^{d_1} \mathbb{E}^{Q_A} \left[ I(S_B(T)/S_A(T) \leq 1) \right]_{\mathcal{F}_t} e^{\frac{-x^2}{2}} dx - S_B(t) \int_{-\infty}^{d_2} \mathbb{E}^{Q_B} \left[ I(S_A(T)/S_B(T) \leq 1) \right]_{\mathcal{F}_t} e^{\frac{-x^2}{2}} dx \quad (\ast\ast)
\]

where

\[
d_1 = \frac{\ln(S_A(t)/S_B(t)) + \Sigma^2(T-t)/2}{\Sigma \sqrt{T-t}}, \quad d_2 = \frac{\ln(S_A(t)/S_B(t)) + \Sigma^2(T-t)/2}{\Sigma \sqrt{T-t}}
\]

and where \(N\) is the distribution function of a standard Gaussian random variable. The value from Belles pow is just minus the value from Anna’s pow. Using that \(S_A(0) = S_B(0)\) we obtain that the value at time zero from both Anna’s and Belle’s pow is zero. So the value at time zero for both participants is zero and they both also make a zero investment at time zero so we can see the bet as fair.

(b) See the calculations in (a).
(c) To find a replicating portfolio we want to find a self-financing portfolio consisting of the assets in our market that has the same dynamics as the derivative. Let \( V \) be the value of the self-financing portfolio \( V(t) = a_0(t)B(t) + a_1(t)S_A(t) + a_2(t)S_B(t) \). The self-financing condition gives that the dynamics of \( V \) is

\[
dV(t) = a_0(t)dB(t) + a_1(t)dS_A(t) + a_2(t)dS_B(t)
\]

We now look at the dynamics of \( \Pi \) and it is given by

\[
d\Pi(t, X) = \Pi_t(t, X)dt + \Pi_{S_A}(t, X)dS_A(t) + \Pi_{S_B}(t, X)dS_B(t) + \Pi_{S_A,S_B}(t, X)(d\Pi_A(t))(d\Pi_B(t)) + \Pi_{S_A,S_B}(t, X)(d\Pi_B(t))^2/2, \quad (***)
\]

where

\[
\begin{align*}
\Pi_t(t, X) &= \frac{\partial}{\partial t} \Pi(t, X) \\
\Pi_{S_A}(t, X) &= \frac{\partial}{\partial S_A} \Pi(t, X) \\
\Pi_{S_B}(t, X) &= \frac{\partial}{\partial S_B} \Pi(t, X) \\
\Pi_{S_A,S_B}(t, X) &= \frac{\partial^2}{\partial S_A \partial S_B} \Pi(t, X) \\
\Pi_{S_B,S_B}(t, X) &= \frac{\partial^2}{\partial S_B^2} \Pi(t, X)
\end{align*}
\]

Using that \( \Pi \) satisfies the Black-Scholes equation we obtain that

\[
\Pi_t(t, X)dt + \Pi_{S_A,S_B}(t, X)(d\Pi_A(t))(d\Pi_B(t)) + \Pi_{S_A,S_B}(t, X)(d\Pi_B(t))^2/2 = r(\Pi(t, x) - S_A(t)\Pi_{S_A}(t, X) - S_B(t)\Pi_{S_B}(t, X))dt
\]

Plugging this into (***) gives

\[
d\Pi(t, X) = r(\Pi(t, X) - S_A(t)\Pi_{S_A}(t, X) - S_B(t)\Pi_{S_B}(t, X))dt + \Pi_{S_A}(t, X)dS_A(t) + \Pi_{S_B}(t, X)dS_B(t).
\]

Comparing with the dynamics for \( V \) we obtain

\[
\begin{align*}
a_0(t) &= (\Pi(t, x) - S_A(t)\Pi_{S_A}(t, X) - S_B(t)\Pi_{S_B}(t, X))/B(t), \\
a_1(t) &= \Pi_{S_A}(t, X), \\
a_2(t) &= \Pi_{S_B}(t, X).
\end{align*}
\]

This is the general formula for hedging a derivative written on two assets for the model in this problem. This is in fact true for any complete market consisting of two risky assets and a bank account. This is just the usual delta-hedge in two dimensions and could have been considered as given. The derivation of the formula was done mostly in order for the solution to be complete. We can now calculate the exact portfolio weights using the formula for \( \Pi \). We right up the hedge from Belle’s port which is the portfolio that replicates the payoff from Anna’s port. The hedge from Anna’s port is just the reversed positions. We start with \( \Pi_{S_A}(t, X) \)

\[
\Pi_{S_A}(t, X) = \frac{\partial}{\partial S_A} \Pi(t, X) = \frac{\partial}{\partial S_A} (S_A(t)N(d_1) - S_B(t)N(d_2))
\]

\[
= N(d_1) + S_A(t)n(d_1) \frac{\partial}{\partial S_A} (d_1) - S_B(t)n(d_2) \frac{\partial}{\partial S_A} (d_2)
\]

\[
= N(d_1) + 1/(\Sigma \sqrt{T-t}) \left(S_A(t)n(d_1)/S_A(t) + S_B(t)n(d_2)/S_A(t)\right)
\]

where \( n(x) = (d/dx)N(x) = e^{-x^2/2}/\sqrt{2\pi} \).
We now plug this into the previous equation and using the formula for \( d_1 \) and \( d_2 \) we obtain

\[
\Pi_N(t, X) = N(d_1)
\]

\[
+ \frac{1}{\Sigma \sqrt{T - t}} \exp \left( -\ln \left( \frac{S_B(t)}{S_A(t)} \right) / 2 \right) \left( \exp \left( -\ln \left( \frac{S_A(t)}{S_B(t)} \right) / 2 \right) \right)
\]

\[
+ \frac{S_B(t)}{S_A(t)} \exp \left( -\ln \left( \frac{S_B(t)}{S_A(t)} \right) / 2 \right)
\]

\[
= N(d_1)
\]

\[
+ \frac{2}{\Sigma \sqrt{T - t}} \exp \left( -\ln \left( \frac{S_B(t)}{S_A(t)} \right) / 2 \right) \left( \exp \left( -\ln \left( \frac{S_A(t)}{S_B(t)} \right) / 2 \right) \right)
\]

\[
= N(d_1) + \frac{2}{\Sigma \sqrt{T - t}} n(d_1).
\]

Using almost similar calculations we obtain that

\[
\Pi_N(t, X) = -N(d_2) - \frac{n(d_2)}{\Sigma \sqrt{T - t}} \frac{S_A(t)}{S_B(t)} \frac{n(d_1)}{\Sigma \sqrt{T - t}}
\]

\[
= -N(d_2) - \frac{S_A(t)}{S_B(t)} \frac{2}{\Sigma \sqrt{T - t}} n(d_1).
\]

Using this we finally obtain

\[
a_0(t) = \left( \Pi(t, x) - S_A(t)N(d_1) + S_B(t)N(d_2) - \frac{2}{\Sigma \sqrt{T - t}} \left( S_A(t)n(d_1) - S_B(t) \frac{S_A(t)}{S_B(t)} n(d_1) \right) \right) / B(t)
\]

\[
= (\Pi(t, X) - \Pi(t, X))/B(t) = 0,
\]

\[
a_1(t) = N(d_1) + \frac{2}{\Sigma \sqrt{T - t}} n(d_1),
\]

\[
a_2(t) = -N(d_2) - \frac{S_A(t)}{S_B(t)} \frac{2}{\Sigma \sqrt{T - t}} n(d_1).
\]

The fact that we here always should hold a zero amount in the bank account is due to the specific nature of the contract. We also see that price and the hedge do not depend on the interest rate.

(d) If one should hedge or not has multiple aspects. The first is that among friends it is not considered "commes il faut" (appropriate) to hedge a bet. Moreover it would not be much of a bet if you neither could gain nor lose anything. In addition to this I guess you would not enter into the bet without believing in your own stock. Even if we disregard these aspects there is one problem with the hedge defined in (c). If \( t \) is close to \( T \) and \( S_A(t) \) is close to \( S_B(t) \) the hedge weights will assume very large values. The hedge will be long in \( S_A \) and short in \( S_B \) with really large positions. This problem comes from that the pay-off is discontinous at \( S_A = S_B \). This lead to large risks if there is any trouble with liquidity in either \( S_A \) or \( S_B \). One way out of this is to use a partial static hedge. Belle could for instance buy \( S_A \) and sell \( S_B \) which is a zero intial investment since \( S_A(0) = S_B(0) \). At maturity she will then have \( -S_A I(S_A > S_B) + S_B I(S_B > S_A) + S_A - S_B = S_A I(S_A < S_B) - S_B I(S_B < S_A) \) this position has less risk than the original bet. It is however somewhat strange from Belle’s pow if she really believes in here stock. The bottomline is that you should probably not hedge the bet.

\[\square\]

Sol: 1.7 Volatility

Sol: 1.7.1 (a) Using that

\[e^{-r(t+\tau-\eta)} E^\mathbb{Q}[S(t+\tau)|\mathcal{F}_t] = S(t)\]

we get that

\[F^\mathbb{Q}_t^{t+\tau} = E^\mathbb{Q}[S(t+\tau)|\mathcal{F}_t] = e^{\sigma S(t)}\].
(b) By applying Ito’s formula to \( \ln(S(t)) \) we get that
\[
d \ln(S(t)) = (r - \sigma^2(u)/2)du + \sigma(u)dW(u).
\]
We solve this by direct integration to obtain
\[
\ln(S(t + \tau)) = \ln(S(t)) + r\tau - \int_0^{t+\tau} \sigma^2(u)/2du + \int_0^{t+\tau} \sigma(u)dW(u).
\]
Using that the last term is a martingale we obtain
\[
\mathbb{E}^Q[\ln(S(t + \tau)|\mathcal{F}_t] = \ln(S(t)) + r\tau - \int_0^{t+\tau} \sigma^2(u)/2du.
\]
This together with the result in (a) then gives that
\[
\mathbb{E}^Q\left[ -\frac{2}{\tau} \ln(S(t + \tau/F_t^{t+\tau})|\mathcal{F}_t) \right] = -\frac{2}{\tau} \left( \ln(S(t)) + r\tau - \frac{1}{2} \int_0^{t+\tau} \sigma^2(u)du - \ln(S(t)) - r\tau \right)
\]
\[
= \frac{1}{\tau} \int_0^{t+\tau} \sigma^2(u)du.
\]
So here we see that the VIX-index squared really is an approximation of the average squared future volatility.

(c) To ease the notational complexity we put \( S = S(t + \tau) \) and \( F = F_t^{t+\tau} \). We start by direct calculation of the integrals
\[
\int_0^{F} (K - S)^+ \frac{dK}{K^2} + \int_0^{\infty} (S - K)^+ \frac{dK}{K^2} = \int_0^{F} (K - S) \frac{dK}{K^2} I(S < F) + \int_0^{S} (S - K) \frac{dK}{K^2} I(F < S)
\]
\[
= \int_0^{F} 1 - \frac{S}{K^2} \frac{dK}{K^2} I(S < F) + \int_0^{S} \frac{S}{K^2} - 1 \frac{dK}{K^2} I(F < S)
\]
\[
= \left[ \ln(K) + \frac{S}{K} \right]_S^F I(S < F) + \left[ -\frac{S}{K} - \ln(K) \right]_S^F I(F < S)
\]
\[
= (\ln(F) - \ln(S) + \frac{S}{F} - 1)(I(S < F) + I(F < S))
\]
\[
= \ln(F) - \ln(S) + \frac{S}{F} - 1 = -\ln(S/F) + \frac{S}{F} - 1.
\]
Taking conditional expectation and using the definition of \( F_t^{t+\tau} \) we obtain
\[
\mathbb{E}^Q\left[ -\ln(S(t + \tau)/F_t^{t+\tau}) + \frac{S(t + \tau)/F_t^{t+\tau} - 1}{F_t^{t+\tau}} | \mathcal{F}_t \right] = \mathbb{E}^Q\left[ -\ln(S(t + \tau)/F_t^{t+\tau}) + \frac{F_t^{t+\tau} - 1}{F_t^{t+\tau}} | \mathcal{F}_t \right],
\]
which shows that
\[
\mathbb{E}^Q\left[ \int_0^{F_t^{t+\tau}} (K - S(t + \tau))^+ \frac{dK}{K^2} + \int_0^{\infty} (S(t + \tau) - K)^+ \frac{dK}{K^2} | \mathcal{F}_t \right] = \mathbb{E}^Q\left[ -\ln \left( \frac{S(t + \tau)}{F_t^{t+\tau}} \right) | \mathcal{F}_t \right].
\]

Sol: 1.7.2 By applying Ito’s formula to \( \ln(S(u)) \) we get that
\[
d \ln(S(u)) = (r - V(u)/2)du + \sqrt{V(u)}dW_1(u) + \sqrt{1 - \rho^2}dW_2(u).
\]
We solve this by direct integration to obtain
\[
\ln(S(t + \tau)) = \ln(S(t)) + r\tau - \int_0^{t+\tau} V(u)/2du + \int_0^{t+\tau} \sqrt{V(u)}dW_1(u) + \sqrt{1 - \rho^2}dW_2(u).
\]
Using that the last term is a martingale we obtain
\[
\mathbb{E}^Q[\ln(S(t + \tau)|\mathcal{F}_t] = \ln(S(t)) + r\tau - \mathbb{E}^Q\left[ \int_0^{t+\tau} V(u)/2du|\mathcal{F}_t \right].
\]

This together with the that $F_t^{t+	au} = e^{\tau}S(t)$ then gives that

$$\mathbb{E}^Q \left[ -\frac{2}{\tau} \ln \left( S(t + \tau) / F_t^{t + \tau} \right) \mid F_t \right] = -\frac{2}{\tau} \left( \ln(S(t)) + r\tau - \mathbb{E}^Q \left[ \frac{1}{2} \int_t^{t+	au} V(u)du \mid F_t \right] - \ln(S(t)) - r\tau \right)$$

$$= \frac{1}{\tau} \mathbb{E}^Q \left[ \int_t^{t+	au} V(u)du \mid F_t \right].$$

Now to calculate this we need to use the dynamics of $V$. Using direct integration we see that we can represent $V(u), u > t$ as

$$V(u) = V(t) + \int_t^u \kappa (\theta - V(s))ds + \int_t^u \sqrt{V(s)}\sigma dW(s).$$

Taking conditional expectation on both sides using that the last term is a martingale we obtain

$$\mathbb{E}^Q[V(u) \mid F_t] = \mathbb{E}^Q[V(t) + \int_t^u \kappa (\theta - V(s))ds \mid F_t] = V(t) + \kappa \theta (u - t) - \kappa \mathbb{E}^Q\left[ \int_t^u \kappa dW(s) \mid F_t \right].$$

Now put $m(u) = \mathbb{E}^Q[V(u) \mid F_t]$ then get

$$m(u) = V(t) + \int_t^u \kappa (\theta - m(s))ds.$$

Taking derivatives w.r.t $u$ of both sides we obtain the ODE

$$\dot{m}(u) = \kappa (\theta - m(u))$$

$$m(t) = V(t).$$

Making the change of variables $\tilde{m}(u) = \vartheta - m(u)$ we obtain a standard linear ODE

$$\dot{\tilde{m}}(u) = -\kappa \tilde{m}(u)$$

$$\tilde{m}(t) = \vartheta - V(t).$$

We solve this straight away yielding

$$\tilde{m}(u) = (\vartheta - V(t))e^{-\kappa (u-t)}$$

and then we get that

$$m(u) = \vartheta - (\vartheta - V(t))e^{-\kappa (u-t)}.$$

We can now finally calculate

$$\frac{1}{\tau} \mathbb{E}^Q \left[ \int_t^{t+	au} V(u)du \mid F_t \right] = \frac{1}{\tau} \int_t^{t+	au} m(u)du$$

$$= \frac{1}{\tau} \int_t^{t+	au} \vartheta - (\vartheta - V(t))e^{-\kappa (u-t)}du$$

$$= \vartheta - \frac{\vartheta - V(t)}{\kappa \tau} (1 - e^{-\kappa \tau})$$

$$= \frac{V(t) (1 - e^{-\kappa \tau})}{\kappa \tau} + \frac{\vartheta e^{-\kappa \tau} - 1 + \kappa \tau}{\kappa \tau}.$$

Note that this expression tends to $V(t)$ as $\tau$ tends to zero and it tends to $\vartheta$ as $\tau$ tends to infinity. This makes sense since $V(t)$ is the current squared volatility and $\vartheta$ is the long term mean of squared volatility.

**Sol: 1.8 Simple interest rate contracts and Martingale models for the short rate**

**Sol: 1.8.1 The price of the ZCB, $p(t, T)$, is given by**

$$p(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s)ds} \right].$$

The short rate $r(s)$ for $s > t$ is given by

$$r(s) = r + \int_t^s \Theta(u)du + \int_t^s \sigma dW(u) = r - c(e^{-s-t} - e^{-t}) + \frac{\sigma^2}{2}(s^2 - t^2) + \sigma \int_t^s dW(u).$$
The integral of the short rate \( r(t) \) between \( t \) and \( T \) is then given by

\[
- \int_t^T r(s) ds = - \int_t^T \left( r - c(e^{-t} - e^{-s}) + \frac{\sigma^2}{2}(s - t) + \sigma \int_t^s dW(u) \right) ds
\]

\[
= -r(T - t) - c(e^{-T} - e^{-t}) - ce^{-t}(T - t) - \frac{\sigma^2}{6}(T^3 - t^3) + \frac{\sigma^2}{2}(T - t) - \sigma \int_t^T \int_t^s dW(u) ds.
\]

The expectation of the stochastic term is

\[
E \left[ \int_t^T \int_t^s dW(u) ds \right] = 0,
\]

and the variance is

\[
\text{Var} \left( - \int_t^T \int_t^s dW(u) ds \right) = E \left( - \int_t^T \int_t^s dW(u) ds \right)^2
\]

\[
= E \left( \int_t^T \int_t^s dW(u) \right)^2 = E \left( \int_t^T (T - u) dW(u) \right)^2
\]

\[
= \int_t^T (T - u)^2 du = \left[ -\frac{(T - u)^3}{3} \right]_t^T = \frac{(T - t)^3}{3}.
\]

We see that the stochastic term has a normal distribution, and therefore

\[
-\sigma \int_t^T \int_t^s dW(u) ds \sim N \left( 0, \frac{\sigma^2}{3}(T - t)^3 \right).
\]

Moreover we have that

\[
E \left[ e^{-\sigma \int_t^T \int_t^s dW(u) ds} \right] = e^{\frac{\sigma^2}{6}(T - t)^3}.
\]

This gives that \( p(t, T) \) is given by

\[
p(t, T) = \exp \left\{ -r(T - t) - c(e^{-T} - e^{-t}) - ce^{-t}(T - t) - \frac{\sigma^2}{6}(T^3 - t^3) + \frac{\sigma^2}{2}(T - t) + \frac{\sigma^2}{6}(T - t)^3 \right\}.
\]

Sol: 1.8.2 Let \( X(T_0) \) denote the value of the floating rate bond at time \( T_0 \). We start by noting that the value of the floating rate bond is just all expected cash flows discounted back to time \( T_0 \). This gives that

\[
X(T_0) = E^Q \left[ \frac{B(T_0)}{B(T_0)} A + \sum_{i=1}^n \frac{B(T_0)}{B(T_i)} A(T_i - T_{i-1})L_{T_{i-1} \leq [T_{i-1}, T_i]} | F_{T_0} \right]
\]

\[
= E^Q \left[ \frac{B(T_0)}{B(T_0)} A| F_{T_0} \right] + \sum_{i=1}^n E^Q \left[ \frac{B(T_0)}{B(T_i)} A(T_i - T_{i-1})L_{T_{i-1} \leq [T_{i-1}, T_i]} | F_{T_0} \right]
\]

\[
= E^Q \left[ \frac{B(T_0)}{B(T_0)} A| F_{T_0} \right] + \sum_{i=1}^n E^Q \left[ \frac{B(T_0)}{B(T_i)} A \left( \frac{1}{p(T_i - 1, T_i)} - 1 \right) | F_{T_0} \right]
\]

Now we change numeraires so that each cash flow is evaluated under the corresponding forward measure, that is a cash flow at time \( T_i \) is evaluated using the measure \( Q^{T_i} \) with numeraire \( p(t, T_i) \). This then gives

\[
X(T_0) = p(T_0, T_0)E^{Q_{T_0}} \left[ A| F_{T_0} \right] + \sum_{i=1}^n p(T_0, T_i)E^{Q_{T_i}} \left[ A \left( \frac{1}{p(T_i - 1, T_i)} - 1 \right) | F_{T_0} \right].
\]
Sol: 1.9  The HJM framework

Sol: 1.9.1  (a) Using that

\[ f(t, u) = f(0, u) + \int_0^t \sigma^2(u - s) ds + \int_0^t \sigma dW_t^Q \]

and that

\[ p(t, T) = e^{-\int_t^T f(s, u) du} \]

we get that

\[ p(t, T) = e^{-\int_t^T f^*(0, u) du + \frac{\sigma^2}{2} \int_t^T (u - t)^2 du - \sigma \int_t^T \sigma W_t^Q du} \]

\[ = e^{-\int_t^T f^*(0, u) du + \frac{\sigma^2}{2} \int_t^T (u - t)^2 du - \sigma \int_t^T \sigma W_t^Q du} \]

Note that \( p^*(0, T) = p^*(0, t) = \exp(-\int_t^T f^*(0, u) du) \) and that \( r(t) = f(t, t) = f^*(0, t) + t^2 \sigma^2 / 2 + \sigma W_t^Q \), so we can further simplify the expression as

\[ p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left( (T - t) f^*(0, t) - \frac{\sigma^2}{2} (T - t)^2 - (T - t) r(t) \right), \]

which is the Ho-Lee model’s ZCB-price when calibrated to the initial forward curve. This form is however, not so well suited for further calculations.

(b) Using the definition of \( X_t \) and \( L_t\) we see that

\[ X_t = 1 + (S - T) \frac{p(t, T) - p(t, S)}{p(t, S)} = \frac{p(t, T)}{p(t, S)} \]

If we now use the result from (a) we get that

\[ X_t = e^{-\int_t^T f^*(0, u) du + \frac{\sigma^2}{2} \int_t^T (u - t)^2 du + \sigma \int_t^T \sigma W_t^Q du + \int_t^T f^*(0, u) du + \frac{\sigma^2}{2} \int_t^T (u - t)^2 du - \sigma \int_t^T \sigma W_t^Q du} \]

\[ = e^{\sigma \int_t^T f^*(0, u) du + \frac{\sigma^2}{2} \int_t^T (u - t)^2 du + \sigma \int_t^T \sigma W_t^Q du + \int_t^T f^*(0, u) du - \sigma \int_t^T \sigma W_t^Q du} \]

To find the dynamics under \( Q^S \) we note that \( X_t \) is the ratio of the traded asset \( p(t, T) \) and the numeraire \( p(t, S) \) therefore we must have that \( X_t \) is a martingale under \( Q^S \). Therefore we only need to calculate the diffusion part of the dynamics since we know that the drift part must be zero, doing this we obtain that \( X_t \) has the following dynamics under \( Q^S \)

\[ dX_t = \sigma(S - T) X_t dW_t^{Q^S} \]

where \( W_t^{Q^S} \) is a standard BM under \( Q^S \). This gives that

\[ X_T = X_0 e^{-\sigma(S - T)^2 / 2 + \sigma^2(S - T)^2} \]

\[ \sigma\int_0^T f^*(0, u) du + \frac{\sigma^2}{2} \int_0^T (u - t)^2 du} \]

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(c) We start by observing that we can express the pay-off in terms of \( X_t \) instead of \( L_t \) and \( S \) giving that

\[
(S - T) \max(L_T[T, S] - K, 0) = (S - T) \max \left( \frac{X_T - 1}{S - T} - K, 0 \right) = \max(X_T - (1 + (S - T)K), 0).
\]

This can now be seen as a standard European call option on \( X_T \) with strike level \((1 + (S - T)K)\), except that we will not get paid at until time \( S \) as opposed to time \( T \) in the standard case. Using the RNVF for the numeraire measure \( \mathbb{Q}^3 \) we get that the price of the Caplet at time \( t \) \( \Pi_t \) for \( 0 \leq t \leq T \) is given by

\[
\Pi_t = p(t, S) \mathbb{E}^{\mathbb{Q}^3} \left[ \max(X_T - (1 + (S - T)K), 0) \bigg| \mathcal{F}_t \right].
\]

Using the result from (b) we get that

\[
\Pi_t = p(t, S)X_t \mathbb{N}(d_1) - p(t, S)(1 + (S - T)K) \mathbb{N}(d_2) = p(t, T) \mathbb{N}(d_1) - p(t, S)(1 + (S - T)K) \mathbb{N}(d_2),
\]

where \( d_1 = (\log (X_t/(1 + (S - T)K)) + (S - T)^2 \sigma^2(T - t)/2)/(S - T)\sigma\sqrt{T - t} \) and \( d_2 = d_1 - (S - T)\sigma\sqrt{T - t} \).

\[\blacksquare\]

Sol: 1.9.2 (a) We should calculate the fair value of \( Y = (T_2 - T_1)L_{T_2}[T_2, T_3] \) at time \( T_1 \). Using the formula

\[
1 + (S_2 - S_1)L_{[S_1, S_2]} = p(t, S_1)/p(t, S_2)
\]

where we put \( S_1 = T_2, S_2 = T_3 \) and \( t = T_2 \) we can express \( Y \) in terms of ZCB values as

\[
Y = 1 + (T_2 - T_1)L_{[T_2, T_3]} = \frac{p(T_2, T_2)}{p(T_2, T_3)}.
\]

To find the fair value at time \( T_1 \) we use the change of numeraire trick. So we get that

\[
\Pi^Y(T_1) = \mathbb{E}^{\mathbb{Q}^3} \left[ \frac{N(T_1)}{N(T_2)} Y \bigg| \mathcal{F}_{T_1} \right] = \mathbb{E}^{\mathbb{Q}^3} \left[ \frac{N(T_1) p(T_2, T_2)}{N(T_2) p(T_2, T_3)} \bigg| \mathcal{F}_{T_1} \right].
\]

A reasonable choice of numeraire is to use \( P(t, T_3) \), which gives that we use the numeraire measure \( \mathbb{Q}^{T_3} \). This also is the measure under which the dynamics for the forward rate is given in the problem. We thus obtain

\[
\Pi^Y(T_1) = \mathbb{E}^{\mathbb{Q}^{T_3}} \left[ \frac{p(T_1, T_3)}{p(T_2, T_3)} \bigg| \mathcal{F}_{T_1} \right] = p(T_1, T_3) \mathbb{E}^{\mathbb{Q}^{T_3}} \left[ \frac{1}{p(T_2, T_3)^2} \bigg| \mathcal{F}_{T_1} \right].
\]

We now re-express this using forward rates

\[
\Pi^Y(T_1) = e^{-\int_{T_1}^{T_3} f(T, a)da} \mathbb{E}^{\mathbb{Q}^{T_3}} \left[ e^{\int_{T_1}^{T_3} f(T, a)da} \bigg| \mathcal{F}_{T_1} \right] = e^{-\int_{T_1}^{T_3} f(T, a)da + 2 \int_{T_2}^{T_3} f(T, a)da} \mathbb{E}^{\mathbb{Q}^{T_3}} \left[ e^{\int_{T_1}^{T_3} f(T, a)da} \bigg| \mathcal{F}_{T_1} \right] = e^{\int_{T_1}^{T_3} f(T, a)da - 2 \int_{T_2}^{T_3} f(T, a)da} \mathbb{E}^{\mathbb{Q}^{T_3}} \left[ e^{\int_{T_1}^{T_3} f(T, a)da} \bigg| \mathcal{F}_{T_1} \right].
\]

So far the calculations hold for all models. We now plug in the dynamics given in the problem and use that
\[ T_3 - T_2 = T_2 - T_1. \] We then get that

\[
\Pi^Y(T_1) = e^{\int_{T_1}^{T_2} \sigma(t) \, dt} \left[ e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - T_1) \, dt} \right] \left[ e^{\int_{T_1}^{T_2} \sigma(t)(u - T_1) \, du} \right] \left[ e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - u) \, du} \right] \frac{dW(t)}{dW(T_1)} \
\]

\[
= e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - T_1) \, dt} e^{\int_{T_1}^{T_2} \sigma(t)(u - T_1) \, du} e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - u) \, du} \frac{dW(T_2)}{dW(T_1)} \
\]

\[
= e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - T_1) \, dt} e^{\int_{T_1}^{T_2} \sigma(t)(u - T_1) \, du} e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - u) \, du} \frac{dW(T_2)}{dW(T_1)} \
\]

So we get that

\[
\Pi^Y(T_1) = e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - T_1) \, dt} e^{\int_{T_1}^{T_2} \sigma(t)(u - T_1) \, du} e^{\int_{T_1}^{T_2} \sigma(t)(T_2 - u) \, du} \frac{dW(T_2)}{dW(T_1)}. 
\]

(b) Plugging that that \( b = -0.01, \sigma(t) = \tilde{\sigma} = 0.1 \) and that \( T_2 - T_1 = 0.25 \) we get that

\[
\Pi^Y(T_1) = e^{-0.01(1 - e^{-0.05})^2 + 0.25^2/0.01} \approx 0.9997.
\]

So the value is slightly less than one.

(c) Since the value of \( Y \) is slightly less than one the bank will on average pay out less than the contract \( X \) which has value one. This mean that the bank will on average make a small profit on the error. The gain is due to that the term structure for the forward rate is decreasing. Testing the stability of the result by slightly changing the model parameters reveal that the result is not very stable. If we instead had \( b = 0.01 \) then the value of \( Y \) would be \( \approx 1.0006 \) and the bank would on average lose instead. Keeping \( b \) at -0.01 and increasing \( \tilde{\sigma} \) to 0.2 would give \( \approx 1.0001 \) so the bank would also here lose. We thus see that it is quite a delicate matter if the bank would win or lose on the error.

**Sol: 1.10 Dividend paying stocks**

Sol: 1.10.1 (a) Since \( S \) is a GBM, straight forward calculations give

\[
S(u) = S(t) \exp((\alpha - \sigma^2/2)(u-t) + \sigma(W(u) - W(t))) \overset{d}{=} S(t) \exp((\alpha - \sigma^2/2)(u-t) + \sigma\sqrt{u-t}G), \, G \in \mathcal{N}(0,1), \, u > t.
\]

Using this we obtain

\[
\mathbb{E}^Q \left[ S(T) \frac{B(t)}{B(T)} \right] = \int_0^T \mathbb{E}^Q \left[ S(t) \exp((\alpha - \sigma^2/2)(T-t) + \sigma\sqrt{T-t}G) \right] \frac{dW(t)}{dW(T)} \
\]

\[
= \int_0^T \mathbb{E}^Q \left[ S(t) \exp((\alpha - \sigma^2/2)(T-t)) \right] \frac{dW(t)}{dW(T)} \
\]

\[
= \int_0^T \mathbb{E}^Q \left[ qS(t) \exp((\alpha - \sigma^2/2)(u-t)) \right] du \
\]

\[
= S(t) \exp((\alpha - \sigma^2/2)(T-t)) \int_0^T qS(t) \exp((\alpha - \sigma^2/2)(u-t)) \, du \
\]

This expression should equal \( S(t) \) for all \( 0 < t < T \). This is possible only if

\[
\frac{q}{\alpha - r} = -1,
\]

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which gives that \( \alpha = r - q \).

**Alternative solution:** We can using the Ito formula express \( S(T)B(t)/B(T) \) as

\[
S(T) \frac{B(t)}{B(T)} = S(t) + \int_t^T \frac{B(t)}{B(u)} d \left( \frac{S(u)}{B(u)} \right)
\]

\[
= S(t) + \int_t^T \frac{B(t)}{B(u)} dS(u) + S(u)d \left( \frac{B(t)}{B(u)} \right)
\]

\[
= S(t) + \int_t^T \frac{B(t)}{B(u)} 2S(u)du + \frac{B(t)}{B(u)} \sigma S(u)dW^Q(u) - \frac{rS(t)}{B(u)} du
\]

\[
= S(t) + \int_t^T \frac{B(t)}{B(u)} (x - r)S(u)du + \frac{B(t)}{B(u)} \sigma S(u)dW^Q(u).
\]

This gives that

\[
S(T) \frac{B(t)}{B(T)} + \int_t^T \frac{B(t)}{B(u)} dD(u)
\]

\[
= S(t) + \int_t^T \frac{B(t)}{B(u)} (x - r)S(u)du + \frac{B(t)}{B(u)} \sigma S(u)dW^Q(u) + \int_t^T \frac{B(t)}{B(u)} qS(u)du
\]

\[
= S(t) + \int_t^T \frac{B(t)}{B(u)} (x - r + q)S(u)du + \frac{B(t)}{B(u)} \sigma S(u)dW^Q(u).
\]

We when immediately see that the integral will be a good candidate for being a martingale if \( x - r + q = 0 \), which is equivalent to \( x = r - q \). Plugging this \( x \) into the SDE for \( S \) and observing that \( S \) is a GBM we straightforward see that

\[
\int_t^T \frac{B(t)}{B(u)} \sigma S(u)dW^Q(u)
\]

is martingale (using the Ito isometry, since GBMs have finite moments of all orders) so that

\[
E^Q \left[ S(T) \frac{B(t)}{B(T)} + \int_t^T q \frac{B(t)}{B(u)} S(u)du | \mathcal{F}_t \right] = E^Q \left[ S(t) + \int_t^T \frac{B(t)}{B(u)} \sigma S(u)dW^Q(u) | \mathcal{F}_t \right] = S(t).
\]

(b) According to the fundamental theorems of asset pricing we have: A model is free of arbitrage if and only if there exists at least one probability measure \( Q \) such that any discounted traded asset is a martingale under \( Q \). Moreover: If a model is free of arbitrage then it is complete if and only if \( Q \) is unique. Since we have a unique solution in a) the market is free of arbitrage and complete.

(c) Using the result in a) we get

\[
\Pi(t) = E^Q \left[ S(T) \frac{B(t)}{B(T)} | \mathcal{F}_t \right]
\]

\[
= S(t) \exp((r - q - r)(T - t))
\]

\[
= S(t) \exp(-q(T - t))
\]

(d) Following the hint we get

\[
a(t) = \frac{\partial}{\partial S(t)} \Pi = \exp(-q(T - t))
\]

\[
b(t) = \frac{\Pi(t) - a(t)S(t)}{B(t)} = \frac{S(t) \exp(-q(T - t)) - S(t) \exp(-q(T - t))}{B(t)} = 0.
\]

This now gives us the portfolio \( (a(t), b(t)) = (\exp(-q(T - t)), 0) \) with value process

\[
V(t) = a(t)S(t) + b(t)B(t) = S(t) \exp(-q(T - t)).
\]
We can now check the self-financing condition
\[ dV(t) = a(t)dS(t) + b(t)dD(t) + \sigma(t)dB(t). \]

We start with the LHS
\[ dV(t) = \exp(-q(T - t))dS(t) + q\exp(-q(T - t))S(t)dt. \]

We then calculate the RHS
\[
a(t)dS(t) + b(t)dD(t) + b(t)dB(t) = \exp(-q(T - t))dS(t) + \exp(-q(T - t))q\exp(-qT)dt + 0 = \exp(-q(T - t))dS(t) + \exp(-q(T - t))q\exp(-qT)dt.
\]

Thus we have LHS = RHS which was to be shown.

(e) So we should here find the value \( \Pi_E \) of a standard European call on stocks paying continuous dividends. According to the RNVF we have
\[
\Pi_E(t) = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \max(S(T) - K, 0) | \mathcal{F}_t \right],
\]

where
\[
G \in \mathcal{N}(0, 1), \quad T > t
\]

\[
= \int_{-\infty}^{\infty} \frac{B(t)}{B(T)} \max(S(t) \exp(-q(T - t)) \exp\left(-\frac{1}{2}\left(\sigma^2(t - t) - 2\sigma\sqrt{T - tx} + x^2\right)\right)}{\sqrt{2\pi}} I(x > -d)dx
\]

\[= \int_{-\infty}^{\infty} \frac{K \exp(-r(T - t)) \exp(-x^2/2)}{\sqrt{2\pi}} I(x > -d)dx
\]

\[
= \int_{-d - \sqrt{T - t}}^{-d} S(t) \exp(-q(T - t)) \exp(-x^2/2) dx
\]

\[= S(t) \exp(-q(T - t)) \left(1 - N(-d - \sqrt{T - t}) - K \exp(-r(T - t)) \left(1 - N(-d)\right)\right)
\]

\[= S(t) \exp(-q(T - t)) N(d + \sigma\sqrt{T - t}) - K \exp(-r(T - t)) N(d),
\]

where
\[
d = \frac{\ln \left(\frac{S(t)}{K}\right) + (r - q - \frac{1}{2} \sigma^2)(T - t)}{\sigma\sqrt{T - t}}
\]

and where \( N \) is the cumulative distribution function of the standard Gaussian distribution. Comparing this with the standard BS formula we see that putting \( q = 0 \) the formula reduces to the standard BS formula. Moreover we see that by replacing \( S(t) \) by \( S(t) \exp(-q(T - t)) \) in the original BS formula (this observation is in fact true for all simple claims, but it is in general not true for path dependent options), we obtain the formula we calculated above and since the original BS formula is monotonically increasing in \( S \) we get that the European call options always are cheaper with dividends than without.

(f) So we want to find a self-financing portfolio \((a(t), b(t))\) with value process \( V(t) = a(t)S(t) + b(t)B(t) \) which replicates the call option. Using the result from e) that by replacing \( S(t) \) by \( S(t) \exp(-q(T - t)) \) in the original BS formula we obtain the formula we calculated above, the hedge calculation in the standard BS case and the chain rule we immediately obtain
\[
a(t) = \frac{\partial}{\partial S(t)} \Pi_E(t) = \exp(-q(T - t)) N(d + \sigma\sqrt{T - t})
\]

\[
b(t) = \frac{\Pi_E(t) - a(t)S(t)}{B(t)} = -K \exp(-rT) N(d).
\]
We can use the Itô formula to express $S(T)B(t)/B(T)$ as

$$ S(T)\frac{B(t)}{B(T)} = S(t) + \int_t^T \frac{B(t)}{B(u)} dS(u) + \frac{B(t)}{B(t)} d\left(\frac{B(t)}{B(u)}\right) $$

$$ = S(t) + \int_t^T \frac{B(t)}{B(u)} dS(u) + \frac{B(t)}{B(u)} \left(\frac{\sigma S(u) dW^Q(u) - \delta S(u-) dN(u) - rS(u) \frac{B(t)}{B(u)} du}{\frac{B(t)}{B(u)}}\right) $$

This gives that

$$ S(T)\frac{B(t)}{B(T)} + \int_t^T \frac{B(t)}{B(u)} d\mathcal{D}(u) $$

$$ = S(t) + \int_t^T \frac{B(t)}{B(u)} (x - r)S(u) du + \frac{B(t)}{B(u)} \sigma S(u) dW^Q(u) - \frac{B(t)}{B(u)} \delta S(u-) dN(u) + \int_t^T \frac{B(t)}{B(u)} \delta S(u-) dN(u) $$

$$ = S(t) + \int_t^T \frac{B(t)}{B(u)} (x - r)S(u) du + \frac{B(t)}{B(u)} \sigma S(u) dW^Q(u). $$

We can immediately see that the integral will be a good candidate for being a martingale if $x = r$, which is equivalent to $x = r$. Since $S$ is positive and only jumps downwards, looking at another process $Y$ which behaves like $S$ but without the jumps, we find that $Y$ is always larger than $S$. Now $Y$ is a standard GBM with all moments finite so then $S$ must also have all moments finite. Using this we straight forward see that

$$ \int_t^T \frac{B(t)}{B(u)} \sigma S(u) dW^Q(u) $$

is martingale (using the Itô isometry) so that

$$ \mathbb{E}^Q \left[ S(T)\frac{B(t)}{B(T)} + \int_t^T \frac{B(t)}{B(u)} S(u-) dN(u) \mid \mathcal{F}_t \right] = S(t). $$

According to the fundamental theorems of asset pricing we have: A model is free of arbitrage if and only if there exists at least one probability measure $\mathbb{Q}$ such that any discounted traded asset is a martingale under $\mathbb{Q}$. Moreover: If a model is free of arbitrage then it is complete if and only if $\mathbb{Q}$ is unique. Since we have a unique solution in a) the market is free of arbitrage and complete.

To get out a formula for the stock process we apply the Ito formula to $\ln(S(u))$. This gives us

$$ d\ln(S(u)) = r - \sigma^2/2 du + \sigma dW^Q(u) + (\ln(S(u-))(1 - \delta) - \ln(S(u-))) dN(u) $$

$$ = r - \sigma^2/2 du + \sigma dW^Q(u) + \ln(1 - \delta) dN(u). $$

Integrating this from $t$ to $T$ we get

$$ \ln(S(T)) = \ln(S(u)) + \int_t^T d\ln(S(u)) $$

$$ = \ln(S(u)) + (r - \sigma^2/2)(T - t) + \sigma(W^Q(T) - W^Q(t)) + \ln(1 - \delta)(N(T) - N(t)). $$

Taking the exponential of both sides we obtain

$$ S(T) = S(t) \exp(\ln(1 - \delta)(N(T) - N(t)) \exp((r - \sigma^2/2)(T - t) + \sigma(W^Q(T) - W^Q(t)). $$

Using this result we get

$$ \Pi(t) = \mathbb{E}^Q \left[ S(T)\frac{B(t)}{B(T)} \mid \mathcal{F}_t \right] $$

$$ = S(t) \exp(\ln(1 - \delta)(N(T) - N(t)) \exp((r - \sigma^2/2)(T - t)) $$

$$ = S(t) \exp(\ln(1 - \delta)(N(T) - N(t)). $$
(d) Following the hint we get
\[
\begin{align*}
a(t) &= \frac{\partial}{\partial S(t)} P = \exp(\ln(1 - \delta)(N(T) - N(t))) \\
b(t) &= \frac{P(t) - a(t)S(t)}{B(t)} = \frac{S(t)(\exp(\ln(1 - \delta)(N(T) - N(t))) - \exp(\ln(1 - \delta)(N(T) - N(t))))}{B(t)} = 0.
\end{align*}
\]
This now gives us the portfolio \((a(t), b(t)) = (\exp(\ln(1 - \delta)(N(T) - N(t))), 0)\) with value process
\[
V(t) = a(t)S(t) + b(t)B(t) = S(t) \exp(\ln(1 - \delta)(N(T) - N(t))).
\]
We can now check the self-financing condition
\[
dV(t) = a(t-)dS(t) + a(t-)dD(t) + b(t)dB(t).
\]
We start with the LHS
\[
dV(t) = a(t-)dS(t) + S(t-)da(t) + dS(t)da(t) \\
a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t) \\
a(t-)\delta S(t-)dN(t) + a(t-)S(t-)\exp(\ln(1 - \delta)) - 1dN(t) \\
a(t-)S(t-)\delta(\exp(\ln(1 - \delta)) - 1)dN(t) \\
a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t) \\
a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t) \\
a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t) \\
= a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t).
\]
We then calculate the RHS
\[
a(t-)dS(t) + a(t-)dD(t) + b(t)dB(t) \\
a(t-)dS(t) + a(t-)\delta S(t-)dN(t) + 0 \\
a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t) + a(t-)S(t-)(-\delta + \delta)dN(t) \\
a(t-)rS(t)dt + a(t-)\sigma S(t)dW(t).
\]
Thus we have LHS=RHS which was to be shown.
(e) So we should here find the value \(\Pi_{S}^{E}\) of a standard European call on stock paying discrete dividends. According to the RNVF we have
\[
\Pi_{S}^{E}(t) = \mathbb{E}^{Q}\left[ \frac{B(t)}{B(T)} \max(S(T) - K, 0)|\mathcal{F}_{t} \right] \\
= \mathbb{E}^{Q}\left[ \frac{B(t)}{B(T)} \max(S(t) \exp(\ln(1 - \delta)(N(T) - N(t))) \exp((r - \sigma^2/2)(T - t) + \sigma\sqrt{T-t}G) - K, 0)|\mathcal{F}_{t} \right] \\
, \ G \in N(0, 1), \ T > t
\]
We here immediately see that this would be equivalent to calculating the standard BS formula with \(S(t)\) replaced by \(S(t) \exp(\ln(1 - \delta)(N(T) - N(t)))\) (this observation is in fact true for all simple claims, but it is in general not true for path dependent options). We thus obtain
\[
\Pi_{S}^{E}(t) = S(t) \exp(\ln(1 - \delta)(N(T) - N(t))) \Phi(d + \sigma\sqrt{T-t}) - K \exp(-r(T-t))\Phi(d),
\]
where
\[
d = \frac{\ln \left( \frac{S(t) \exp(\ln(1 - \delta)(N(T) - N(t)))}{K} \right) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}},
\]
33
and where $\Phi$ is the cumulative distribution function of the standard Gaussian distribution. Comparing this with the standard BS formula we see that putting $q = 0$ the formula reduces to the standard BS formula. Moreover as noted above we see that by replacing $S(t)$ by $S(t) \exp(-q(T - t))$ in the original BS formula we obtain the formula we calculated above and since the original BS formula is monotonically increasing in $S$ we get that the European call options always are cheaper with dividends than without.

(f) So we want to find a self-financing portfolio $(a(t), b(t))$ with value process $V(t) = a(t)S(t) + b(t)B(t)$ which replicates the call option. Using the result from e) that by replacing $S(t)$ by $S(t) \exp(ln(1 - \delta)(N(T) - N(t)))$ in the original BS formula we obtain the formula we calculated above, the hedge calculation in the standard BS case and the chain rule we immediately obtain

\begin{align*}
a(t) & = \frac{\partial}{\partial S(t)} \Pi_c(t) = \exp(ln(1 - \delta)(N(T) - N(t))) \Phi(d + \sigma \sqrt{T - t}) \\
b(t) & = \frac{\Pi_c(t) - a(t)S(t)}{B(t)} = -K \exp(-rT) \Phi(d).
\end{align*}