Solution.

1. We can start to look at the interval \((K \leq S_T \leq K + \Delta)\). Here we substract a call option with strike \(K\) since the pay-off is \(K - S_T\). Moving on to the second interval \((K + \Delta \leq S_T \leq K + 3\Delta)\) we see that we can add two European call options with strike \(K + \Delta\) giving \(-(S_T - K)^+ + 2(S_T - (K + \Delta)) = K - S_T - 2K - 3\Delta = S_T - K - 2\Delta\). This does not change the pay-off in the first interval. Moving on to the final interval \((K + 3\Delta \leq S_T)\) we see that if we also subtract one European call option with strike \(K + 3\Delta\) we get \(-(S_T - K)^+ + 2(S_T - (K + \Delta)) + (S_T - (K + 3\Delta))^+ = (K - S_T) + 2(S_T - K - \Delta) - (S_T - (K + 3\Delta)) = K + S_T - K - 2\Delta - S_T + K + 3\Delta = \Delta\). This leaves the pay-off unchanged in the first two intervals. So let \(\Pi(t)\) be the price at time \(t\) of a European call with strike \(H\) and maturity \(T\). So the price of the static replication at time \(t\), \(\Pi(t)\), is given by

\[\Pi(t) = -\Pi_E^C(t, K, T) + 2\Pi_E^C(t, K + \Delta, T) - \Pi_E^C(t, K + 3\Delta, T)\]

To see this assume that the price of the derivative and the static replication differs at some time \(s\) say. Sell the most expensive of the two and buy the cheapest put the rest of the money into the bank account. At maturity the pay-off of the derivative and its replication cancels but we still have money in the bank and thus we have constructed an arbitrage opportunity. Therefore the price of the static replication and the derivative must coincide for all \(0 \leq t \leq T\).

**Alternative replication:** Using the put call parity on all European call option we obtain that the price of the contract can alternatively be written as

\[\Pi(t) = \Delta p(t, T) - \Pi_E^P(t, K, T) + 2\Pi_E^P(t, K + \Delta, T) - \Pi_E^P(t, K + 3\Delta, T),\]

where \(\Pi_E^P(t, H, T)\) is the price at time \(t\) of a European put with strike \(H\) and maturity \(T\).

2. We have

\[M(t) = e^{2t} \left( \cos^2(W(t)) - \frac{1}{2} \right) = \frac{e^{2t}}{2} \cos(2W(t)).\]

Applying Ito’s formula to \(M(t) = \frac{e^{2t}}{2} \cos(2W(t))\) we obtain

\[
dM(t) = e^{2t} \cos(2W(t)) \, dt - e^{2t} \sin(2W(t)) \, dW(t) - e^{2t} \cos(2W(t)) \, dt,
\]

\[
= -e^{2t} \sin(2W(t)) \, dW(t).
\]

This is a drift free Ito process. So provided that

\[
E \left[ \left( \int_0^t e^{2s} \sin(2W(s)) \, dW(s) \right)^2 \right] < \infty
\]

for \(t \geq 0\) it is a Martingale. Now using the Ito isometry we get

\[
E \left[ \left( \int_0^t e^{2s} \sin(2W(s)) \, dW(s) \right)^2 \right] = E \left[ \int_0^t e^{4s} \sin^2(2W(s)) \, ds \right]
\]

\[
\leq \int_0^t e^{4s} \, ds = \frac{e^{4t} - 1}{4} < \infty.
\]
3. We have that
\[ p(u, S) = \mathbb{E}^Q \left[ e^{-\int_0^T r(t) \, dt} \big| \mathcal{F}_u \right]. \]

The \( Q \)-dynamics for \( r \) is
\[ dr(y) = a e^{-y} + \sigma^2 y \, dy + \sigma \, dW_y, \]
this gives that
\[ p(u, S) = \mathbb{E}^Q \left[ e^{-\int_0^T r(t) \, dt} \big| \mathcal{F}_u \right], \]
\[ = \mathbb{E}^Q \left[ e^{-\int_0^T r(u) + \int_u^T dr(y) \big| \mathcal{F}_u \right], \]
\[ = e^{-(S-u)r(u)} \mathbb{E}^Q \left[ e^{-\int_0^T r(t) \, dt + \sigma^2 t \, dt + \int_0^T \sigma \, dW_t \big| \mathcal{F}_u \right], \]
\[ = e^{-(S-u)r(u)} \mathbb{E}^Q \left[ e^{-\int_0^T r(t) \, dt + \sigma^2 t \, dt + \int_0^T \sigma \, dW_t \big| \mathcal{F}_u \right], \]
\[ = e^{-(S-u)r(u)} \mathbb{E}^Q \left[ e^{-\int_0^T r(t) \, dt + \sigma^2 t \, dt + \int_0^T \sigma \, dW_t \big| \mathcal{F}_u \right], \]
\[ = e^{-(S-u)r(u) - (S-u)^2 + \frac{\sigma^2}{2} (S-u)^2} \mathbb{E}^Q \left[ e^{-\int_0^T r(t) \, dt + \sigma^2 t \, dt + \int_0^T \sigma \, dW_t \big| \mathcal{F}_u \right], \]
\[ = e^{-(S-u)r(u) - (S-u)^2 + \frac{\sigma^2}{2} (S-u)^2 - \frac{1}{2} (S-u)^2 \sigma^2}, \]
\[ = e^{-(S-u)r(u) - (S-u)^2 + \frac{\sigma^2}{2} (S-u)^2 - \frac{1}{2} (S-u)^2 \sigma^2}. \]

\[ \square \]

4. According to Feynman-Kac’s representation theorem the PDE is solved by
\[ f(t, x) = \mathbb{E}[e^{-r(T-t)}(X_T - K)^2 | X_t = x], \]
where \( X \) has the dynamics
\[ dX_t = rX_t \, dt + \sigma X_t \, dW_t, \quad t \leq s \leq T, \]
\[ X_t = x. \]

It is straightforward to see (at least it should be) that \( X \) is is a Geometric BM starting at \( x \) at time \( t \), i.e.
\[ X_T = xe^{r(T-t) + \sigma(W_T - W_t)}. \]

Looking at \((X_T - K)^2\) we see that it is given by
\[ (X_T - K)^2 = x^2 e^{2r(T-t) + 2\sigma(W_T - W_t)} - 2Kxe^{r(T-t) + \sigma(W_T - W_t)} + K^2. \]

We thus get that
\[ f(t, x) = e^{-r(T-t)} \mathbb{E} \left[ x^2 e^{2r(T-t) + 2\sigma(W_T - W_t)} - 2Kxe^{r(T-t) + \sigma(W_T - W_t)} + K^2 \right], \]
\[ = e^{-r(T-t)} \left( x^2 e^{2r(T-t) + 2\sigma(W_T - W_t)} - 2Kxe^{r(T-t) + \sigma(W_T - W_t)} + K^2 \right), \]
\[ = x^2 e^{(r + \sigma^2)(T-t)} - 2Kx + K^2 e^{-r(T-t)}. \]

We should also check that the obtained solution fulfills the PDE and the boundary condition. We start with the last task \( f(T, x) = x^2 e^{0} - 2xK + K^2 e^{0} = (x - K)^2 \) as prescribed. Finally we get that
\[ \frac{\partial f(t, x)}{\partial t} + rx \frac{\partial f(t, x)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2}. \]

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5. (a) We start by examining the pay-off:
\[
\text{max}(aS_t(T) - bS_u(T), 0) = aS_t(T)I(aS_t(T) \geq bS_u(T)) - aS_t(T)I(aS_t(T) \geq bS_u(T))
\]
which verifies that the obtained solution is correct.

After this rewriting we are ready to attack the problem with the RNVF. Let \( \Pi(t, S_t(t), S_u(t)) \) be the fair value of the contract at time \( t \). Thus we have
\[
\Pi(t, S_t(t), S_u(t)) = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \max(aS_t(T) - bS_u(T), 0) | F_t \right]
\]
\[
= \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} aS_t(T)I \left( \frac{S_u(T)}{S_t(T)} \leq \frac{a}{b} \right) | F_t \right]
- \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} bS_u(T)I \left( \frac{S_t(T)}{S_u(T)} \geq \frac{b}{a} \right) | F_t \right].
\]

Now we apply the change of numeraire technique to simply the calculations:
\[
\Pi(t, S_t(t), S_u(t)) = \mathbb{E}^{Q^S_1} \left[ \frac{S_t(t)}{S_1(T)} aS_t(T)I \left( \frac{S_u(T)}{S_t(T)} \leq \frac{a}{b} \right) | F_t \right]
- \mathbb{E}^{Q^S_1} \left[ \frac{S_u(t)}{S_2(T)} bS_u(T)I \left( \frac{S_t(T)}{S_u(T)} \geq \frac{b}{a} \right) | F_t \right]
\]
\[
= aS_t(t) \mathbb{E}^{Q^S_1} \left[ I \left( \frac{S_u(T)}{S_t(T)} \leq \frac{a}{b} \right) | F_t \right] - bS_u(t) \mathbb{E}^{Q^S_1} \left[ I \left( \frac{S_t(T)}{S_u(T)} \geq \frac{b}{a} \right) | F_t \right].
\]

We can now use that \( S_u(t)/S_t(t) \) is a \( Q^S_1 \) martingale and that \( S_1(t)/S_2(t) \) is a \( Q^S_2 \) martingale, since they are both ratios of traded assets and numeraires. We then get the following \( Q^S_1 \)-dynamics for \( S_u(t)/S_t(t) \) (using Ito and the MG-property)
\[
d\frac{S_u(t)}{S_1(t)} = \frac{S_u(t)}{S_1(t)} \left( (\sigma_{21} - \sigma_{11}) dW^Q_1(u) + (\sigma_{22} - \sigma_{12}) dW^Q_2(u) \right)
\]
\[
\frac{d}{d} \frac{S_u(t)}{S_1(t)} \sqrt{(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2} dW^Q_1(u)
\]
\[
= \frac{S_u(t)}{S_1(t)} \tilde{\sigma} dW^Q_1(u),
\]
where \( \tilde{\sigma} = \sqrt{(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2} \) and \( W^Q_1 \) is a standard \( Q^S_1 \) BM. So then we get that
\[
\frac{S_u(T)}{S_1(T)} \frac{d}{d} \frac{S_u(t)}{S_1(t)} e^{-\frac{\tilde{\sigma}^2}{2}(T-t) + \tilde{\sigma} \sqrt{T-t} G},
\]
where \( G \) is standard Gaussian random variable.

Using the same type of arguments we get the following distribution for \( S_t(T)/S_u(T) \) under \( Q^S_2 \).
\[
\frac{S_t(T)}{S_u(T)} \frac{d}{d} \frac{S_t(t)}{S_u(t)} e^{-\frac{\tilde{\sigma}^2}{2}(T-t) + \tilde{\sigma} \sqrt{T-t} G},
\]

which verifies that the obtained solution is correct.
where \( G \) is standard Gaussian random variable and where
\[
\tilde{\sigma} = \sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{12} - \sigma_{22})^2} = \sqrt{(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2}
\]
as before. Putting this together we obtain
\[
\Pi(t, S_1(t), S_2(t)) = \begin{cases} 
    aS_1(t)Q^{s_1}\left( \frac{S_2(t)}{S_1(t)} - \frac{\tilde{\sigma}}{\tilde{\sigma}}(T-t) + \sigma\sqrt{T-t}G \leq \frac{a}{b}\mathcal{F}_t \right) \\
    -bS_2(t)Q^{s_1}\left( \frac{S_1(t)}{S_2(t)} e^{-\frac{\tilde{\sigma}}{\tilde{\sigma}}(T-t) + \sigma\sqrt{T-t}G} \geq \frac{b}{a}\mathcal{F}_t \right)
\end{cases}
\]
\[
= \begin{cases} 
    aS_1(t)Q^{s_1}\left( \frac{\ln \left( \frac{aS_1(t)}{bS_2(t)} \right) + \frac{\tilde{\sigma}}{\tilde{\sigma}}(T-t)}{\tilde{\sigma}\sqrt{T-t}} \right) - bS_2(t)Q^{s_1}\left( \frac{\ln \left( \frac{aS_1(t)}{bS_2(t)} \right) - \frac{\tilde{\sigma}}{\tilde{\sigma}}(T-t)}{\tilde{\sigma}\sqrt{T-t}} \right)
\end{cases}
\]
where
\[
d = \frac{\ln \left( \frac{S_1(t)}{bS_2(t)} \right) - \frac{\tilde{\sigma}}{\tilde{\sigma}}(T-t)}{\tilde{\sigma}\sqrt{T-t}}.
\]

Alternative solution:
We start by examining the pay-off:
\[
\max(aS_1(T) - bS_2(T), 0) = aS_1(T)I(aS_1(T) \geq bS_2(T)) - bS_2(T)I(aS_1(T) \geq bS_2(T))
\]
\[
= aS_1(T)I \left( \frac{S_1(T)}{S_2(T)} \geq \frac{b}{a} \right) - bS_2(T)I \left( \frac{S_1(T)}{S_2(T)} \geq \frac{b}{a} \right).
\]
Now we apply the change of numeraire technique to simply the calculations:
\[
\Pi(t, S_1(t), S_2(t)) = E^{Q^s}\left[ \frac{S_2(t)}{S_1(t)} (aS_1(T) - bS_2(T)) I \left( \frac{S_1(T)}{S_2(T)} \geq \frac{b}{a} \right) \mathcal{F}_t \right]
\]
\[
= S_2(t)E^{Q^s}\left[ \left( \frac{aS_1(T)}{S_2(T)} - b \right) I \left( \frac{S_1(T)}{S_2(T)} \geq \frac{b}{a} \right) \mathcal{F}_t \right]
\]
We can now use that \( S_1(u)/S_2(u) \) is a \( Q^s \)-martingale, since it is the ratio of traded asset and a numeraire. We then get the following \( Q^s \)-dynamics for \( S_1(u)/S_2(u) \) (using Ito and the MG-property)
\[
d\left( \frac{S_1(u)}{S_2(u)} \right) = \left( \sigma_{11} - \sigma_{22} \right) dW_1^{Q^s}(u) + \left( \sigma_{12} - \sigma_{22} \right) dW_2^{Q^s}(u)
\]
\[
\frac{d}{d} \left( \frac{S_1(u)}{S_2(u)} \right) = \sqrt{(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2} dW^{Q^s}(u)
\]
\[
= \frac{S_2(u)}{S_1(u)} \tilde{\sigma} dW^{Q^s}(u),
\]
where \( \tilde{\sigma} = \sqrt{(\sigma_{11} - \sigma_{12})^2 + (\sigma_{12} - \sigma_{22})^2} \) and \( W^{Q^s} \) is a standard \( Q^s \)-BM. So then we get that
\[
\frac{S_1(T)}{S_2(T)} \overset{d}{=} \frac{S_1(t)}{S_2(t)} e^{-\frac{\tilde{\sigma}}{\tilde{\sigma}}(T-t) + \sigma\sqrt{T-t}G},
\]
where $G$ is standard Gaussian random variable. We now plug this in

$$
\Pi(t, S_1(t), S_2(t)) = S_2(t)E^{Q_2}\left[\left(\frac{dS_1(T)}{S_2(T)} \right) I\left(\frac{S_1(T)}{S_2(T)} \geq \frac{b}{a} \right) |\mathcal{F}_t\right]
$$

$$
= S_2(t)E^{Q_2}\left[\left(\frac{S_1(t)}{S_2(t)} \right) e^{-\frac{\sigma^2}{2}(T-t) + \tilde{\sigma} \sqrt{T-t}G} - b \right) I\left(\frac{S_1(t)}{S_2(t)} \geq \frac{b}{a} \right) |\mathcal{F}_t\right]
$$

$$
= E^{Q_2}\left[\left(aS_1(t) e^{-\frac{\sigma^2}{2}(T-t) + \tilde{\sigma} \sqrt{T-t}G} - bs_2(t)\right) I\left(G \geq \frac{\ln \left(\frac{bS_1(t)}{aS_2(t)}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma \sqrt{T-t}}\right) |\mathcal{F}_t\right]
$$

$$
= \int_{-\infty}^{\infty} \left(aS_1(t) e^{-\frac{\sigma^2}{2}(T-t) + \tilde{\sigma} \sqrt{T-t}y} - bs_2(t)\right) e^{-\frac{y^2}{2}}\frac{1}{\sqrt{2\pi}} dy \quad (*)
$$

$$
\int_{-\infty}^{\infty} \left(aS_1(t) e^{-\frac{1}{2}(\tilde{\sigma}^2(t-t) - 2\tilde{\sigma} \sqrt{T-t} + y^2)}\right) \frac{1}{\sqrt{2\pi}} dy - \int_{-\infty}^{\infty} bs_2(t) e^{-\frac{y^2}{2}}\frac{1}{\sqrt{2\pi}} dy
$$

$$
= aS_1(t)(1 - N(-d - \tilde{\sigma} \sqrt{T-t})) - bs_2(t)(1 - N(-d))
$$

$$
= aS_1(t)N(d + \tilde{\sigma} \sqrt{T-t}) - bs_2(t)N(d),
$$

where

$$
d = \frac{\ln \left(\frac{aS_1(t)}{bS_2(t)}\right) - \frac{\sigma^2}{2}(T-t)}{\tilde{\sigma} \sqrt{T-t}}.
$$

(b) The Black-Scholes like market in this problem is complete so we can hedge all contingent claims. To find the hedge we use the standard $\Delta$-hedge approach.

$$
b_B(t) = \frac{1}{B(t)}(\Pi(t, S_1(t), S_2(t)) - h_{S_1}(t)S_1(t) - h_{S_2}(t)S_2(t))
$$

$$
h_{S_1}(t) = \frac{\partial}{\partial S_1}\Pi(t, S_1(t), S_2(t))
$$

$$
h_{S_2}(t) = \frac{\partial}{\partial S_2}\Pi(t, S_1(t), S_2(t))
$$

We start with $h_{S_1}(t)$ using (*) we can see that the derivatives coming from d vanishes

$$
h_{S_1}(t) = aN(d + \tilde{\sigma} \sqrt{T-t}).
$$

We then move on to $h_{S_2}$ again (*) we can see that the derivatives coming from d vanishes,

$$
h_{S_2}(t) = -bN(d).
$$

Using this we finally obtain

$$
b_B(t) = \frac{1}{B(t)}(\Pi(t, S_1(t), S_2(t)) - h_{S_1}(t)S_1(t) - h_{S_2}(t)S_2(t))
$$

$$
= 0,
$$

$$
h_{S_1}(t) = aN(d + \tilde{\sigma} \sqrt{T-t}),
$$

$$
h_{S_2}(t) = -bN(d).
$$
6. (a) By using that \( Z(t) = \frac{p_z(0)}{p_1(0)} = e^{-\int_{T_1}^{T_2} f(u, T) dT} \) we get that

\[
Z(T_1) = Z(t) \frac{Z(T_1)}{Z(t)} = Z(t) e^{-\int_{T_1}^{T_2} f(T_1, T) dT + \int_{T_1}^{T_2} f(u, T) dT}
= Z(t) e^{-\int_{T_1}^{T_2} f(T_1, T) - f(u, T) dT}
= Z(t) e^{-\int_{T_1}^{T_2} f(T_1, T) dT}
\]

Now plugging in the \( \mathbb{Q}^{T_1} \)-dynamics we get

\[
Z(T_1) = Z(t) e^{-\int_{T_1}^{T_2} f(T_1, T) \frac{1}{2} \sigma^2 \left( e^{2 \sigma \sigma(u, T) du} \right)^2 ds + \sigma(T) dW^{\mathbb{Q}^{T_1}}(t) dT}
= Z(t) e^{-\int_{T_1}^{T_2} f(T_1, T) \frac{1}{2} \sigma^2 \left( e^{2 \sigma \sigma(T) dT} \right)^2 ds + \int_{T_1}^{T_2} \sigma(T) dT dW^{\mathbb{Q}^{T_1}}(t)}
\]

\[
\frac{d}{dt} Z(t) e^{-\frac{1}{2} \Sigma^2 + \Sigma G}
\]

where \( \Sigma^2 = \int_{T_1}^{T_2} \left| \int_{T_1}^{T_2} \sigma(T, T) dT \right|^2 ds \) and where \( G \) is standard Gaussian random variable. Using this we obtain

\[
II(t) = p_1(t) \mathbb{E}^{\mathbb{Q}^{T_1}} \left[ \max(Z(T_1) - K, 0) \mid \mathcal{F}_t \right]
= p_1(t) \mathbb{E}^{\mathbb{Q}^{T_1}} \left[ \max(Z(t)e^{-\frac{1}{2} \Sigma^2 + \Sigma G} - K, 0) \mid \mathcal{F}_t \right]
= p_1(t) \mathbb{E}^{\mathbb{Q}^{T_1}} \left[ (Z(t)e^{-\frac{1}{2} \Sigma^2 + \Sigma G} - K) I \left( G \geq \frac{\ln \left( \frac{K}{Z(t)} \right) + \frac{1}{2} \Sigma^2}{\Sigma} \right) \right] \mid \mathcal{F}_t \right]
\]

\[
= p_1(t) \int_{-d}^{\infty} (Z(t)e^{-\frac{1}{2} \Sigma^2 + \Sigma G} - K) e^{\frac{-y^2}{2}} dy
= p_2(t) \int_{-d}^{\infty} e^{-\frac{1}{2} \left( (\Sigma^2 - 2\Sigma y + y^2) \right)} - p_1(t) K \int_{-d}^{\infty} e^{\frac{-y^2}{2}} dy
= p_2(t) N(d + \Sigma) - p_1(t) K N(d),
\]

where

\[
d = \frac{\ln \left( \frac{Z(t)}{K} \right) - \frac{1}{2} \Sigma^2}{\Sigma}.
\]

(b) Using expression (**) we see that the derivative taken w.r.t to \( p_1 \) and \( p_2 \) of \( d \) is multiplied by zero (the integrand is zero at the point -d). We thus obtain

\[
b_{p_1}(t) = \frac{\partial}{\partial p_1} II(t) = -K N(d),
\]

and

\[
b_{p_2}(t) = \frac{\partial}{\partial p_2} II(t) = N(d + \Sigma).
\]