Valuation of derivative assets
Lecture 9

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Girsanov transformation

Theorem (Å: Thm 9.8 p. 225)

Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\{W^\mathbb{P}_t\}_{t \geq 0}$ is a $(d\text{-dim})$ Brownian motion w.r.t. $\mathcal{F}_t$. Let $g_t$ be a $(d\text{-dim})$ process adapted to $\mathcal{F}_t$ for $t \in [0,T]$ which satisfies

$$E^\mathbb{P}\left[\exp\left(\frac{1}{2} \int_0^T |g_t|^2 \, dt\right)\right] < \infty, \quad \text{(Novikov condition)}.$$

Define the process $L_t$ by

$$L_t = \exp\left(- \int_0^t g_s^* \, dW^\mathbb{P}_s - \frac{1}{2} \int_0^t |g_s|^2 \, ds\right), \quad 0 \leq t \leq T.$$

Define a new probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\mathbb{Q}(A) = E^\mathbb{P}[1_A L_T]$ for $A \in \mathcal{F}_T$.

Then $W^\mathbb{Q}_t = W^\mathbb{P}_t + \int_0^t g_s \, ds$ is a standard $(d\text{-dim})$ $\mathbb{Q}$-BM on $[0,T]$. 
The new dynamics after change of measure

Suppose that the market \((N+1 \text{ assets})\) have the \(\mathbb{P}\)-dynamics

\[
\begin{align*}
    dB_t &= r(t)B_t \, dt, \\
    B_0 &= 1, \\
    dS_t &= \text{diag}(S_t)\mu(t, S_t) \, dt + \text{diag}(S_t)\sigma(t, S_t) \, dW_t^\mathbb{P}, \\
    S_0 &= s.
\end{align*}
\]

Using the Girsanov kernel \(g_t\) we get the \(\mathbb{Q}\)-dynamics

\[
\begin{align*}
    dB_t &= r(t)B_t \, dt, \\
    B_0 &= 1, \\
    dS_t &= \text{diag}(S_t)(\mu(t, S_t) - \sigma(t, S_t)g_t) \, dt + \text{diag}(S_t)\sigma(t, S_t) \, dW_t^\mathbb{Q}, \\
    S_0 &= s.
\end{align*}
\]
The likelihood ratio process $L$

Applying the Ito formula to

$$L_t = \exp \left( - \int_0^t g_s^* \, dW_s^\mathbb{P} - \frac{1}{2} \int_0^t |g_s|^2 \, ds \right)$$

we get that

$$dL_t = \left( -\frac{1}{2}|g_t|^2 + \frac{1}{2}|g_t|^2 \right) L_t \, dt - L_t g_t^* \, dW_t^\mathbb{P}$$

$$= -L_t g_t^* \, dW_t^\mathbb{P}.$$ 

So knowing the dynamics of $L$ we can read off the Girsanov kernel $g$. (This will be used on slide 9.)
**Definition (Numeraire)**

A numeraire is the basic unit of currency on the market. Any strictly positive asset of the form

\[ N(t) = N(0) + \int_0^t \sum_{i=0}^n \alpha_i(t) \, dS_i(u), \]

can be used as a numeraire.

That means that \( N \) is a strictly positive self-financing portfolio on the market \( S_0, S_1, \ldots, S_n \).

Numeraires are used as discounting factors.
The numeraire measure $\mathbb{Q}^N$

First note that $\mathbb{Q} = \mathbb{Q}^0$ is the numeraire measure for the numeraire $S_0 = B$ (bank account).

What happens if we want to use $S_1$ as the numeraire instead? What is the corresponding numeraire-measure $\mathbb{Q}^1$?

Note that the values of all contingent claims should remain unchanged!
The numeraire measure $\mathbb{Q}^1$

We should have that $\mathbb{Q}^1 \sim \mathbb{Q}^0$ (and thus also $\mathbb{Q}^1 \sim \mathbb{P}$). Let

$$L_T = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}$$

on $\mathcal{F}_T$. We must then have that

$$\Pi(0; X) = S_0(0)\mathbb{E}^{\mathbb{Q}^0}\left[\frac{X}{S_0(T)}|\mathcal{F}_0\right]$$

$$= S_1(0)\mathbb{E}^{\mathbb{Q}^1}\left[\frac{X}{S_1(T)}|\mathcal{F}_0\right]$$

$$= S_1(0)\mathbb{E}^{\mathbb{Q}^0}\left[\frac{XL_T}{S_1(T)}|\mathcal{F}_0\right]$$

for all $\mathcal{F}_T$-claims $X$ with $\mathbb{E}^{\mathbb{Q}^0}[||X||] < \infty$. 
This then gives that

\[ \frac{S_0(0)}{S_0(T)} = \frac{L_T S_1(0)}{S_1(T)} \]

and thus

\[ L_T = \frac{S_1(T) S_0(0)}{S_0(T) S_1(0)}. \]

and since \( S_1(t)/S_0(t) \) is a \( Q^0 \)-martingale we get that

\[ L_t = \mathbb{E}^{Q^0}[L_T | \mathcal{F}_t] = \frac{S_1(t) S_0(0)}{S_0(t) S_1(0)}. \]
The numeraire measure $\mathbb{Q}^1$ cont 2

Under $\mathbb{Q}^0$ we have (with $S(t) = [S_1(t), \ldots, S_n(t)]^*$)

$$dS_0(t) = r(t)S_0(t) \, dt$$

$$dS(t) = \text{diag}(S(t))1_n r(t) \, dt + \text{diag}(S(t))\sigma(t, S(t)) \, dW^{\mathbb{Q}^0}(t)$$

This gives that

$$dL_t = d \left( \frac{S_1(t)}{S_0(t)} \right) \frac{S_0(0)}{S_1(0)}$$

$$= r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \, dt + \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \sigma_1.(t, S_t) \, dW^{\mathbb{Q}^0}(t)$$

$$- r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \, dt$$

$$= L_t \sigma_1.(t, S(t)) \, dW^{\mathbb{Q}^0}(t).$$

So the Girsanov kernel $g(t) = -\sigma_1^*(t, S(t))$ takes us from $\mathbb{Q}^0$ to $\mathbb{Q}^1$. 
The new dynamics under $Q^1$ (and arbitrary $Q^k \ 1 \leq k \leq n$)

Using the Girsanov kernel $g_1(t) = -\sigma_1^*(t, S(t))$ we get

\[
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt \\
    dS(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma_1^*(t, S(t))) \, dt \\
            &\quad + \text{diag}(S(t))\sigma(t, S(t)) \, dW^{Q^1}(t)
\end{align*}
\]

With the same type of argument we get for $Q^k$ that $g_k(t) = -\sigma_k^*(t, S(t))$ and thus the $Q^k$ dynamics

\[
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt \\
    dS(t) &= \text{diag}(S(t))(1_n r(t) + \sigma(t, S(t))\sigma_k^*(t, S(t))) \, dt \\
            &\quad + \text{diag}(S(t))\sigma(t, S(t)) \, dW^{Q^k}(t)
\end{align*}
\]
The forward measure $Q_T$

If the short rate $r(t)$ is stochastic then $S_0(t)/S_0(T) = e^{-\int_t^T r(s) \, ds}$ is a random variable. This may cause some complications for valuation of derivatives. Suppose we can use a bond that pays out one unit of currency at maturity $T$ as a numeraire instead. This derivative is called a zero coupon bond (ZCB). The value at time $t$ here denoted $p(t, T)$ is given by:

$$p(t, T) = E^Q\left[\frac{S_0(t)}{S_0(T)}1|\mathcal{F}_t\right] = E^Q[e^{-\int_t^T r(s) \, ds}|\mathcal{F}_t]$$

Note $p(T, T) = 1$ since $E^Q[e^{-\int_T^T r(s) \, ds}|\mathcal{F}_T] = E^Q[1|\mathcal{F}_T] = 1$. 
The forward measure cont

Suppose we have a Black-Scholes type of model for $S_1$ but with $r(\cdot)$ stochastic. So assume $\mathbb{Q}^0$-dynamics:

$$dS_0(t) = r(t)S_0(t)\,dt,$$

$$dS_1(t) = S_1(t)r(t)\,dt + S_1(t)\sigma_1.\,dW^{\mathbb{Q}^0}(t),$$

where $W^{\mathbb{Q}^0}$ is $d$-dim $\mathbb{Q}^0$-BM and $\sigma_1.$ deterministic $d$-dim row-vector. Further assume that $p(t, T)$ has $\mathbb{Q}^0$-dynamics

$$dp(t, T) = p(t, T)r(t)\,dt + p(t, T)v(t, T)\,dW^{\mathbb{Q}^0}(t),$$

where $v(t, T)$ deterministic is a $d$-dim row-vector-valued function. This gives with the same arguments as above that the corresponding Girsanov kernel $g_T(t)$ is $-v^*(t, T).$
The forward measure cont 2

We thus get the $\mathbb{Q}^T$ dynamics

\begin{align*}
\text{d}S_0(t) &= r(t)S_0(t) \text{d}t, \\
\text{d}S_1(t) &= S_1(t)(r(t) + \sigma_1 v^*(t, T)) \text{d}t + S_1(t)\sigma_1 \, \text{d}W^{\mathbb{Q}^T}(t), \\
\text{d}p(t, T) &= p(t, T)(r(t) + v(t, T)v^*(t, T)) \text{d}t + p(t, T)v(t, T) \, \text{d}W^{\mathbb{Q}^T}(t)
\end{align*}

Let $X(t) = S_1(t)/p(t, T)$, then $X(T) = S_1(T)/p(T, T) = S_1(T)$. This is now a $\mathbb{Q}^T$-martingale with dynamics

\begin{align*}
\text{d}X(t) &= X(t)(\sigma_1 - v(t, T)) \, \text{d}W^{\mathbb{Q}^T}(t) \overset{d}{=} X(t)\tilde{\sigma}(t) \, \text{d}\tilde{W}^{\mathbb{Q}^T}(t),
\end{align*}

where $\tilde{\sigma}(t) = |\sigma_1 - v(t, T)|$ and $\tilde{W}^{\mathbb{Q}^T}(t)$ is a 1-dim $\mathbb{Q}^T$-BM.

To price derivatives with maturity $T$ we can view them as written on $X(T)$ rather than $S_1(T)$. So

\begin{align*}
\mathbb{E}^{\mathbb{Q}} \left[ \frac{S_0(t)}{S_0(T)} \Phi(S_1(T)) | F_t \right] &= \frac{p(t, T)}{p(T, T)} \mathbb{E}^{\mathbb{Q}^T} [\Phi(S_1(T)) | F_t] = p(t, T) \mathbb{E}^{\mathbb{Q}^T} [\Phi(X(T)) | F_t].
\end{align*}
Pricing of European call under stochastic interest rate

Assume that we have the dynamics on the previous slide. We then have that

\[ X(T) = X(t)e^{\int_t^T -\tilde{\sigma}^2(u) \, du + \int_t^T \tilde{\sigma}(u) \, d\tilde{W}^Q(u)} \overset{d}{=} X(t)e^{-\frac{\Sigma_{t,T}^2}{2}} + \Sigma_{t,T} G, \]

where \( G \in N(0,1) \) and \( \Sigma_{t,T}^2 = \int_t^T \tilde{\sigma}^2(u) \, du = \int_t^T |\sigma_1 - v(u,T)|^2 \, du \).

With almost the same calculation (put \( r = 0 \) and replace \( \sigma \sqrt{T - t} \) by \( \Sigma_{t,T} \)) as in the derivation of the Black-Scholes formula we get

\[ p(t,T) \mathbb{E}^{Q^T} [(X(T) - K)^+ | X(t)] = p(t,T)(X(t)N(d_1) - KN(d_2)) = S(t)N(d_1) - p(t,T)KN(d_2), \]

where

\[
\begin{align*}
  d_1 &= \frac{\ln(S(t)/(Kp(t,T))) + \Sigma_{t,T}^2/2}{\Sigma_{t,T}}, \\
  d_2 &= \frac{\ln(S(t)/(Kp(t,T))) - \Sigma_{t,T}^2/2}{\Sigma_{t,T}}.
\end{align*}
\]
Preparation for the computer exercise (Heston model)

If we look at real stock prices we see that the volatility is not constant.

**Heston model, $\mathbb{P}$-dynamics:**

\[
\begin{align*}
\text{d}S_0(t) &= rS_0(t) \text{d}t, \\
\text{d}S_1(t) &= S_1(t)\mu \text{d}t + S_1(t)\sqrt{V(t)}(\rho \text{d}W_1^\mathbb{P}(t) + \sqrt{1 - \rho^2} \text{d}W_2^\mathbb{P}(t)), \\
\text{d}V(t) &= \kappa(\theta - V(t)) \text{d}t + \beta \sqrt{V(t)} \text{d}W_1^\mathbb{P}(t)
\end{align*}
\]

What about $\mathbb{Q}$-dyn?

\[
\begin{align*}
\mu - g_1(t)\rho \sqrt{V(t)} - g_2(t)\sqrt{1 - \rho^2} \sqrt{V(t)} &= r \\
\kappa(\theta - V(t)) - g_1(t)\beta \sqrt{V(t)} &= ?
\end{align*}
\]

The problem is that volatility is not a traded asset! So we have no unique solution and thus the market is incomplete.
Possible $\mathbb{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathbb{Q} \theta^\mathbb{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathbb{Q} - \kappa}{\mu - r} \frac{\rho}{\beta},$$

which gives the $\mathbb{Q}$-dyn

$$dS_0(t) = rS_0(t) \, dt,$$
$$dS_1(t) = S_1(t) r \, dt + S_1(t) \sqrt{V(t)} \left( \rho \, dW_1^\mathbb{Q}(t) + \sqrt{1 - \rho^2} \, dW_2^\mathbb{Q}(t) \right),$$
$$dV(t) = \kappa^\mathbb{Q} (\theta^\mathbb{Q} - V(t)) \, dt + \beta \sqrt{V(t)} \, dW_1^\mathbb{Q}(t)$$
Solution for the Heston model?

We have that

\[
S(T) = S(t)e^{\int_t^T (r - \frac{V_u}{2})\,du + \int_t^T \sqrt{V(u)}(\rho \,dW_1^P(u) + \sqrt{1-\rho^2}\,dW_2^P(u))}
\]

The problem is that there is no closed form solution for \( V \).

Valuation is usually done by:

1. Fourier methods (Friday 13-15 in MH309A)
2. Monte Carlo methods (Thursday)
3. Numerical PDE methods (Outside the scope of this course)
Simulation of the Heston model

\[ S(0) = 100, \ \mu = 0.04, \ \nu(0) = 0.3, \ \kappa = 3, \ \theta = 0.3, \ \beta = 0.7, \ \rho = -0.6 \]