Valuation of derivative assets
Lecture 7

Magnus Wiktorsson

September 23, 2019
Contingent Claim (B: Def 7.4 p. 94)

**Definition**

Let \( \{ \mathcal{F}_t^S \}_{t \geq 0} \) be the filtration generated be the asset process \( S \).

A **contingent claim** with maturity \( T \) is any \( \mathcal{F}_T^S \)-measurable r.v. \( X \).

\( X \) is a **simple claim** if \( X = \Phi(S(T)) \), where \( \Phi \) is called a contract function.
Theorem

Assume that we have a market consisting of a risky asset $S$ and a bank-account $B$, where the corresponding $\mathbb{P}$-dynamics are given as

$$dS(t) = S(t)\mu(t, S(t)) \, dt + S(t)\sigma(t, S(t)) \, dW^\mathbb{P}(t),$$

$$S(0) = s_0$$

$$dB(t) = r(t)B(t) \, dt,$$

$$B_0 = 1.$$

Assume that $F(t, s)$ is the value at time $t$ of a simple claim with maturity $T$ of the form $F(T, s) = \Phi(s)$. Then the value $F$ is a solution to the boundary value problem

$$\frac{\partial F(t, s)}{\partial t} = -r(t)s \frac{\partial F(t, s)}{\partial s} - \frac{1}{2} s^2 \sigma(t, s)^2 \frac{\partial^2 F(t, s)}{\partial s^2} + r(t)F(t, s),$$

$$F(T, s) = \Phi(s).$$
Theorem

Assume that $F$ is a solution to the boundary value problem

$$
\frac{\partial F(t, s)}{\partial t} = -r(t)s \frac{\partial F(t, s)}{\partial s} - \frac{1}{2} s^2 \sigma(t, s)^2 \frac{\partial^2 F(t, s)}{\partial s^2} + r(t)F(t, s)
$$

$F(T, s) = \Phi(s)$

and assume that $E \left[ \int_0^T S(u)^2 \sigma(u, S(u))^2 \left( \frac{\partial F(u, s)}{\partial s} \right)^2_{s=S(u)} du \right] < \infty$.

Then $F$ has the representation (RNVF)

$$
F(t, s) = E^Q \left[ e^{-\int_t^T r(u) du} \Phi(S(T)) \mid S_t = s \right],
$$

where $S$ has the $Q$ dynamics

$$
\begin{align*}
    dS(u) &= r(u) S(u) du + S(u) \sigma(u, S(u)) dW^Q(u), \quad t \leq u \leq T, \\
    S(t) &= s
\end{align*}
$$
Hedge/Replicating portfolio

Definition (B: Def 8.1 p. 111)

Let $X$ be a contingent claim with maturity $T$. If there exists a self-financing portfolio $h$ such that

$$\mathbb{P}(V_T^h = X) = 1,$$

then $h$ is a **hedge** or a **replicating portfolio** for the contingent claim $X$. 
Complete market

Definition

If all contingent claims on an arbitrage free market have replicating portfolios then the market is said to be complete otherwise the market is called incomplete.
Replicating portfolio

Let $X$ be a simple claim with contract function $\Phi$ with a value function $F$ that satisfies the Black-Scholes equation. A portfolio $h = (h^S, h^B)$ with value process

$$V^h(t) = h^S(t)S(t) + h^B(t)B(t),$$

where

$$h_t^S = \left. \frac{\partial F(t, s)}{\partial s} \right|_{s=S(t)} \quad \text{(Delta-hedge)}$$

$$h_t^B = \frac{F(t, S(t)) - h^S(t)S(t)}{B(t)},$$

is a replicating portfolio for the claim $X$. 
Black-Scholes model

Assume that we have a market consisting of a risky asset $S$ and a bank-account $B$, where the corresponding $\mathbb{P}$-dynamics are given as

\[
\begin{align*}
    \text{d}S_u &= \mu S_u \text{d}u + \sigma S_u \text{d}W^\mathbb{P}_u, \\
    S_t &= s \\
    \text{d}B_u &= r B_u \text{d}u, \\
    B_0 &= 1.
\end{align*}
\]

We then have that

\[
S_T = s e^{(\mu - \sigma^2/2)(T-t) + \sigma (W(T) - W(t))}, \quad B_T = B_t e^{r(T-t)}.
\]
Black-Scholes formula (B: Prop 7.10 p. 101)

**Theorem**

Assume that $S$ and $B$ have the dynamics on the previous slide. Then the value $\Pi^c_E(t, K, T, s)$ of a European call option with maturity $T$, strike $K$ and $S_t = s$ is given as

\[
\Pi^c_E(t, K, T, s) = sN(d_1(t, s)) - e^{-r(T-t)}K N(d_2(t, s)),
\]

\[
d_1(t, s) = \frac{1}{\sigma \sqrt{T-t}} \left( \ln \left( \frac{s}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \right),
\]

\[
d_2(t, s) = \frac{1}{\sigma \sqrt{T-t}} \left( \ln \left( \frac{s}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t) \right)
= d_1(t, s) - \sigma \sqrt{T-t},
\]

where $N(x)$ is the distribution function of a standard Gaussian random variable.
Replicating portfolio for European call

Let $X$ be a European call option with a value function $F$ that satisfies the Black-Scholes formula on the previous slide.

A portfolio $h = (h^S, h^B)$ with value process

$$V^h(t) = h^S(t)S(t) + h^B(t)B(t),$$

where

$$h^S_t = \frac{\partial F(t, s)}{\partial s} \bigg|_{s=S(t)} = N(d_1(t, S(t))) \text{ (Delta)}$$

$$h^B_t = \frac{F(t, S(t)) - h^S(t)S(t)}{B(t)}$$

$$= -e^{-rT}KN(d_2(t, S(t))),$$

is a replicating portfolio for the European call option.
Put-Call parity

Let $\Pi_{E}^{c}(t, K, T, s)$, $\Pi_{E}^{p}(t, K, T, s)$ and $\Pi_{F}(t, K, T, s)$ respectively be the values of a European call option, European put option and the (payers position) Forward all with maturity $T$ and strike $K$. For any arbitrage free market we have

$$\Pi_{E}^{c}(t, K, T, s) - \Pi_{E}^{p}(t, K, T, s) = \Pi_{F}(t, K, T, s).$$
American options

American options usually are of put or call type.  
**American call** Payoff=$\max(S(\tau) - K, 0)$ where we can choose $\tau$ to be any time between zero and $T$.  
**American put** Payoff=$\max(K - S(\tau), 0)$ where we can choose $\tau$ to be any time between zero and $T$.

The exercise time $\tau$ should be a stopping time, that is given the information available upto time $t$ we must be able to decide if $\tau \leq t$, i.e.  
$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t^S$. 

American options 2

The value function of the option satisfies a Black-Scholes type of PDE in the region where we should not exercise the option:

\[
\begin{align*}
    f'_t + rsf'_s + \frac{\sigma(t, s)^2 s^2}{2} f''_{ss} - rf &= 0, \\
    f(t, s) &= \Phi(s), \quad t < T
\end{align*}
\]

where

\[
\Phi(x) = \begin{cases} 
    \max(s - K, 0) & \text{Call case,} \\
    \max(K - s, 0) & \text{Put case.}
\end{cases}
\]
American call option

If the underlying asset $S$ does not pay dividends and the short rate $r$ is positive, it is not optimal to exercise the call option before $T$. This holds since:

It is obvious that $\Pi_A^c(t, K, T, S(t)) \geq \Pi_E^c(t, K, T, S(t))$

Jensen’s inequality gives

$\Pi_E^c(t, K, T, S(t)) \geq \max(S(t) - \exp(-r(T - t))K, 0)$

For $r > 0$ and $t < T$ we have

$max(S(t) - \exp(-r(T - t))K, 0) > max(S(t) - K, 0)$. Therefore

$\Pi_A^c(t, K, T, S(t)) > max(S(t) - K, 0), \ 0 \leq t < T.$

But this is the pay-off we get if we exercise the American option at time $t$ before $T$ so therefore we should wait until $T$ to exercise.
American Put Option

If the interest rate is non-positive it is not optimal to exercise before $T$. However if the interest rate is positive it is sometimes optimal to exercise before $T$. One can show that there exist a level $L(T-t)$ so that it is optimal to exercise as soon the underlying $S(t)$ goes below $L(T-t)$ for the first time. No closed form is known for $L$. For the Black-Scholes model we have $L(0) = K,$

$$\frac{2Kr}{2r + \sigma^2} \leq L(T - t) \leq K,$$

for $0 \leq t \leq T$, $T > 0$. The function $L$ is decreasing and convex for $r > 0$. Its derivative w.r.t $T$ tends to $-\infty$ as $T - t$ tends to zero.
The value of an American put option
(no dividends no upward jumps, r positive)

\[ \Pi_A^p(t, s, T, K) = \Pi_E^p(t, s, T, K) \]
\[ + \int_t^T rKe^{-r(u-t)}\mathbb{Q}(S(u) \leq L(T - u)|S(t) = s) \, du \]

\( L \) can be found by solving the non-linear integral equation

\[ L(T) = K - \Pi_E^p(0, L(T), T, K) \]
\[ - \int_0^T rKe^{-ru}\mathbb{Q}(S(u) \leq L(T - u)|S(0) = L(T)) \, du \]
American Put Option cont 2

The exercise level $L$ as a function of time to maturity in the Black-Scholes model calculated by numerical methods:
European and American put values in the Black-Scholes model calculated by numerical methods: $K = 100$, $T = 1$, $r = 0.05$ and $\sigma = 0.2$
The perpetual American put in the Black Scholes model

If with have infinite time to maturity we call it a perpetual option. The optimal exercise level is

$$L(\infty) = \frac{2Kr}{2r + \sigma^2}.$$ 

Moreover we have that the value at time zero (or any other time for that matter) is given by

$$\begin{cases} K - S & S \leq L(\infty), \\ \frac{\sigma^2}{2r} L(\infty) \frac{2r}{\sigma^2} + 1 S^{-\frac{2r}{\sigma^2}} & S \geq L(\infty), \end{cases}$$

where $S'$ is the corresponding stock price.