Valuation of derivative assets
Lecture 4

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A **Filtration** on \((\Omega, \mathcal{F}, \mathbb{P})\) is an increasing family of \(\sigma\)-algebras, \(\{\mathcal{F}_t\}_{t \geq 0}\), such that

i) \(\mathcal{F}_t \subset \mathcal{F}\) for \(t \geq 0\)

ii) \(\mathcal{F}_s \subset \mathcal{F}_t\) for \(0 \leq s \leq t\)

So i) means that \(A \in \mathcal{F}_t \Rightarrow A \in \mathcal{F}\), but if \(A \in \mathcal{F}\) then in some cases we can have \(A \notin \mathcal{F}_t\).
Natural filtration

Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\). The **natural filtration** of \( \{X_t\}_{t \geq 0}, \{\mathcal{F}^X_t\}_{t \geq 0} \), is the filtration generated by the process \( X \) i.e.

\[
\mathcal{F}^X_t = \sigma(X_s, 0 \leq s \leq t)
\]

This can be interpreted as the information we can obtain by observing the trajectory of \( X \) from 0 to \( t \). You can e.g. think of \( X \) as the observed prices of some financial asset.
Adapted process

Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( \{\mathcal{F}_t\}_{t \geq 0} \) be a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). The process \( X \) is \textit{adapted} to \( \{\mathcal{F}_t\}_{t \geq 0} \) if for all \( t \geq 0 \) and \( B \in \mathcal{B}(\mathbb{R}) \),

\[
\{ \omega \in \Omega : X_t(\omega) \in B \} \in \mathcal{F}_t.
\]

Note that a process is always adapted to its natural filtration.
Conditional expectation

**Theorem (Kolmogorov (1933))**

Let $X$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[|X|] < \infty$. Further let $\mathcal{G} \subset \mathcal{F}$ (i.e. $\mathcal{G}$ is a sub-$\sigma$-algebra of $\mathcal{F}$) then there exists a random variable $Y$ such that

i) $Y$ is a random variable on $(\Omega, \mathcal{G}, \mathbb{P})$

ii) $E[|Y|] < \infty$

iii) For every $G \in \mathcal{G}$ we have

$$\int_G Y \, d\mathbb{P} := E[Y 1_G] = E[X 1_G] := \int_G X \, d\mathbb{P}.$$

We will from now on denote $Y$ with $E[X | \mathcal{G}]$.

If $\tilde{Y}$ is another random variable satisfying i)–iii) then $\mathbb{P}(Y = \tilde{Y}) = 1$. We usually express this as: $\tilde{Y}$ is a version of $E[X | \mathcal{G}]$. 
Independence

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two $\sigma$-algebras on $\Omega$ where $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ with probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. If for all $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be independent.
Independence

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A random variable $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be independent of $\mathcal{G} \subset \mathcal{F}$ if $\mathcal{F}^X = \sigma(X)$ is independent of $\mathcal{G}$. 
Independence

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two $\sigma$-algebras on $\Omega$ where $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ with probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. If for all $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be independent.

A random variable $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be independent of $\mathcal{G} \subset \mathcal{F}$ if $\mathcal{F}^X = \sigma(X)$ is independent of $\mathcal{G}$.

Two random variables $X$ and $Y$ on $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be independent if $\mathcal{F}^X = \sigma(X)$ is independent of $\mathcal{F}^Y = \sigma(Y)$. 
Properties of conditional expectation

Assume that $X$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[|X|] < \infty$ and that $\mathcal{G} \subset \mathcal{F}$

i) If $\sigma(X) = \mathcal{F}^X \subset \mathcal{G}$ then $E[X|\mathcal{G}] = X$

ii) If $X$ is independent of $\mathcal{G}$ then $E[X|\mathcal{G}] = E[X]$

iii) (Tower property) If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ then $E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]$ and $E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[X|\mathcal{F}_1]$

iv) (Taking out what is known) Let $Z$ be a random variable such that $\sigma(Z) \subset \mathcal{G}$. If $E[|ZX|] < \infty$ then $E[ZX|\mathcal{G}] = ZE[X|\mathcal{G}]$

v) (Jensen) If $f$ is a convex or a concave function such that $E[|f(X)|] < \infty$ then

$$E[f(X)|\mathcal{G}] \begin{cases} 
\geq f(E[X|\mathcal{G}]) & f \text{ convex} \\
\leq f(E[X|\mathcal{G}]) & f \text{ concave}
\end{cases}$$
Martingales

Let \( \{X_t\}_{t \geq 0} \) be a real valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \{\mathcal{F}_t\}_{t \geq 0} \) a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). The process \( X \) is a continuous time Martingale w.r.t. \( \{\mathcal{F}_t\}_{t \geq 0} \) if

i) \( X \) is adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \)

ii) \( \mathbb{E}[|X_t|] < \infty, \ t \geq 0 \)

iii) For all \( s \leq t \) \( \mathbb{E}[X_t|\mathcal{F}_s] = X_s \)

Ex 1: Brownian motion \( W(t) \) is a Martingale w.r.t. its natural filtration \( \{\mathcal{F}_t^W\}_{t \geq 0} \).

Ex 2: The process \( X(t) = (W(t)^2 - t) \) is a Martingale w.r.t. the filtration \( \{\mathcal{F}_t^W\}_{t \geq 0} \) and also w.r.t. its natural filtration \( \{\mathcal{F}_t^X\}_{t \geq 0} \).
Ito-integral revisited

Suppose \( \{W_t\}_{t \geq 0} \) is a Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( W \) is Martingale w.r.t \( \{\mathcal{F}_t\}_{t \geq 0} \). Further suppose that \( \{f(t)\}_{t \geq 0} \) is adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \) and that

\[
\int_0^T E[f(s)^2] \, ds < \infty,
\]

\( \sigma(X_0) \subset \mathcal{F}_0 \) with \( E[X_0^2] < \infty \) then the process

\[
\{X_t\}_{0 \leq t \leq T} = \left\{ X_0 + \int_0^t f(s) \, dW_s \right\}_{0 \leq t \leq T}
\]

is Martingale w.r.t. \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \).
Stochastic differential equations (SDE:s)

\[
dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t
\]
\[
X_0 = x
\]

This should be interpreted as the stochastic integral equation

\[
X_t = x + \int_0^t \mu(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s.
\]
Ito-formula for SDE:s

If $g \in C^{1,2}([0, T], \mathbb{R})$ then for $0 \leq t \leq T$

$$g(t, X_t) = g(0, X_0)$$
$$+ \int_0^t g'_1(s, X_s) + \mu(s, X_s)g'_2(s, X_s) + \frac{\sigma(s, X_s)^2}{2}g''_{22}(s, X_s) \, ds$$
$$+ \int_0^t \sigma(s, X_s)g'_2(s, X_s) \, dW_s,$$

where $g'_1$ and $g'_2$ mean derivative w.r.t. first and second argument respectively and $g''_{22}$ means second derivative w.r.t. second argument.
Example: Ito-formula for SDE:s

\[ dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = x, \text{ where } x > 0. \]

Do Ito for \( g(t, X_t) = \ln(X_t) \).
Example: Ito-formula for SDE:s

\[ \text{d}X_t = \mu X_t \text{d}t + \sigma X_t \text{d}W_t, \quad X_0 = x, \text{ where } x > 0. \]

Do Ito for \( g(t, X_t) = \ln(X_t) \).

\[
\ln(X_t) = \ln(x) + \int_0^t \frac{\mu X_s}{X_s} - \frac{\sigma^2 X_s^2}{2X_s^2} \text{d}s \\
+ \int_0^t \frac{\sigma X_s}{X_s} \text{d}W_s \\
= \ln(x) + \int_0^t \mu - \frac{\sigma^2}{2} \text{d}s + \int_0^t \sigma \text{d}W_s \\
= \ln(x) + \left( \mu - \frac{\sigma^2}{2} \right)t + \sigma W_t
\]
Multi dimensional Stochastic differential equations (SDE:s)

\[ \begin{align*}
\mathrm{d}X_t &= \mu(t, X_t) \, \mathrm{d}t + \sigma(t, X_t) \, \mathrm{d}W_t, \\
X_0 &= x,
\end{align*} \]

where \( \mu \) is a \( \mathbb{R}^d \) vector, \( \sigma \) is a \( \mathbb{R}^{d \times m} \) matrix and \( W \) is an \( \mathbb{R}^m \)-dimensional Brownian motion (the components in the \( \mathbb{R}^m \) vector \( W \) are independent standard Brownian motions). This should be interpreted as the \( \mathbb{R}^m \)-dimensional stochastic integral equation

\[ X_t = x + \int_0^t \mu(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}W_s, \]

where we do the integrals componentwise.
Multi dimensional Itô-formula for SDE:s

If \( g : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) and \( g \in C^{1,2}([0, T], \mathbb{R}^d) \) then for \( 0 \leq t \leq T \)

\[
g(t, X_t) = g(0, X_0) + \int_0^t \left\{ g'(s, X_s) + \sum_{k=1}^{d} \mu(s, X_s)_k g'_{x_k}(s, X_s) \\
+ \frac{1}{2} \sum_{k=1}^{d} \sum_{l=1}^{d} (\sigma(s, X_s)\sigma(s, X_s)^*)_kl g''_{x_kx_l}(s, X_s) \right\} ds \\
+ \int_0^t \sum_{k=1}^{d} \sum_{j=1}^{m} g'_{x_k}(s, X_s)\sigma(s, X_s)_{kj} (dW_s)_j
\]
Multi dimensional Ito-formula for SDE:s (cont)

Using notation from multi-variate calculus and linear algebra we can write the formula in a more compact way:

\[
g(t, X_t) = g(0, X_0) + \int_0^t g'_s(s, X_s) + (\nabla_x g)(s, X_s)\mu(s, X_s)\, ds
\]
\[
+ \int_0^t \frac{1}{2} \text{tr}((\nabla^2_x g)(s, X_s)\sigma(s, X_s)\sigma^*(s, X_s))\, ds
\]
\[
+ \int_0^t (\nabla_x g)(s, X_s)\sigma(s, X_s)\, dW_s
\]

Another possibility is the use the differential form

\[
\text{d}g(t, X_t) = g'_t(t, X_t)\, dt + (\nabla_x g)(t, X_t)\, dX_t
\]
\[
+ \frac{1}{2} \text{tr}((\nabla^2_x g)(t, X_t)\, dX_t\, dX_t^*)
\]

(Compare with a second order Taylor expansion!)