Valuation of derivative assets
Lecture 15

Magnus Wiktorsson

October 17, 2019
Affine term structure (B Prop 24.2 p. 379)

If
\[
p(t, T) = \exp(A(t, T) - B(t, T)r(t)) = F(t, r(t), T),
\]
where \( A, B \) are deterministic functions which do not depend on \( r \). We say that the ZCB price have an **affine term structure**

This is true for all short rate models of the form
\[
dr(t) = \alpha(t)r(t) + \beta(t)\,dt + \sqrt{\gamma(t)r(t) + \delta(t)}\,dW_t.
\]

The functions \( A \) and \( B \) satisfy the following system of ordinary differential equations (ODE:s)

\[
B_t'(t, T) = -\alpha(t)B(t, T) + \frac{1}{2}\gamma(t)B^2(t, T) - 1, \quad B(T, T) = 0
\]
\[
A_t'(t, T) = \beta(t)B(t, T) - \frac{1}{2}\delta(t)B^2(t, T), \quad A(T, T) = 0
\]
Solution of ATS ODE:s

If \( \gamma \equiv 0 \) then there is an immediate solution to the equations which is given by:

\[
B(t,T) = \int_t^T e^{\int_t^u \alpha(v) \, dv} \, du
\]

\[
A(t,T) = -\int_t^T \beta(s) B(s,T) \, ds + \int_t^T \frac{\delta(s)}{2} B(s,T)^2 \, ds
\]

If \( \alpha, \beta \) and \( \delta \) are complicated then the integrals may have to be calculated numerically.
Example:

**Hull-White (extended Vašiček)**

\[ dr(t) = (\Theta(t) - ar(t)) \, dt + \sigma(t) \, dW_t, \quad (\Theta(t), \, a, \, \sigma(t) > 0). \]

This gives

\[ \alpha(t) \equiv -a, \quad \beta(t) \equiv \Theta(t), \quad \gamma(t) \equiv 0, \quad \delta(t) \equiv \sigma(t)^2. \]

and thus

\[
B(t,T) = \int_t^T e^{\int_t^u -a \, dv} \, du = \int_t^T e^{-a(u-t)} \, du = \left[ -\frac{e^{-a(u-t)}}{a} \right]_t^T
\]

\[= \frac{1 - e^{-a(T-t)}}{a} \]

\[A(t,T) = -\int_t^T \Theta(s) B(s,T) \, ds + \int_t^T \frac{\sigma(s)^2}{2} B(s,T)^2 \, ds\]

\[= -\int_t^T \Theta(s) \frac{1 - e^{-a(T-s)}}{a} \, ds + \int_t^T \frac{\sigma(s)^2}{2} \left( \frac{1 - e^{-a(T-s)}}{a} \right)^2 \, ds\]
Preparation for Heath-Jarrow-Morton (HJM) framework

A ZCB with maturity $T$ is a traded asset and should therefore have $\mathbb{Q}$-dynamics of the form

$$dp(t, T) = r(t)p(t, T) \, dt + p(t, T)v(t, T) \, dW_t$$

where $v(t, T)$ is some $\mathcal{F}_t$-adapted function (possibly multi-dim). Assume that we have a $\mathbb{Q}$-model for the forward rate $f(t, u)$ for every $u > 0$,

$$df(t, u) = \alpha(t, u) \, dt + \sigma(t, u) \, dW_t,$$

where $\alpha$ (1-dim) and $\sigma$ (possibly multi-dim) are $\mathcal{F}_t$-adapted functions. We then have that

$$p(t, T) = e^{-\int_t^T f(t, u) \, du}.$$

We will now look for conditions on $\alpha$ and $\sigma$ which makes these two models for $p(t, T)$ to be consistent.
Drift condition for the forward rate

We must have

\[ \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u)^* \, du \]

for the forward dynamics to be consistent with the ZCB dynamics.

This is sometimes also called the HJM drift condition.
Forward rates for ATS models

For ATS models we have

\[ p(t, T) = \exp(A(t, T) - B(t, T)r(t)). \]

Now we have that

\[ f(t, T) = -\frac{\partial}{\partial T} \ln(p(t, T)) = -A_T'(t, T) + B_T'(t, T)r(t). \]

This gives that

\[ df(t, T) = \alpha(t, T) \, dt + B_T'(t, T) \sqrt{\gamma(t)r(t) + \delta(t)} \, dW_t \]

where easy but somewhat lengthy calculations using the ATS ODE:s give

\[ \alpha(t, T) = B(t, T)B_T'(t, T)(\gamma(t)r(t) + \delta(t)) \]
The HJM framework

Suppose that

\[ df(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dW(t) \]

\[ f(0, T) = f^*(0, T) \]

under \( Q \), where \( W \) is a d-dim BM and \( \alpha \) (1-dim) and \( \sigma \) (d-dim) are adapted. To avoid arbitrage we should have

\[ \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u)^* \, du. \]

This is called the HJM drift condition.
The good thing about HJM models is that we immediately fit the observed initial term structure for ZCB:s.

Moreover the d-dim BM makes it possible also to capture the forward curve dynamics.

In the model we only need to specify the volatility structure.

One problem is that most non-deterministic volatility functions lead to non-Markovian forward rates.
The HJM framework and corresponding short rate dynamics

So we have

\[ r(t) = f(t, t) \]

and thus

\[
\begin{align*}
\text{dr}(t) & = df(t, t) = \frac{\partial}{\partial T} f(t, T) \big|_{T=t} \, dt + d_t f(t, T) \big|_{T=t} \\
& = \frac{\partial}{\partial T} f(t, T) \big|_{T=t} \, dt + \alpha(t, t) \, dt + \sigma(t, t) \, dW(t) \\
& = \frac{\partial}{\partial T} f(t, T) \big|_{T=t} \, dt + \sigma(t, t) \, dW(t),
\end{align*}
\]

since

\[ \alpha(t, t) = \sigma(t, t) \int_t^t \sigma(t, u)^* \, du = 0. \]
Example:
The simplest possible HJM-model is the one where $\sigma(t, T) \equiv \bar{\sigma}$ where $\bar{\sigma}$ is a deterministic constant. This gives

$$df(t, T) = \bar{\sigma} \int_t^T \bar{\sigma} \, du + \bar{\sigma} \, dW(t) = \bar{\sigma}^2 (T - t) \, dt + \bar{\sigma} \, dW(t),$$

and thus

$$f(t, T) = f^*(0, T) + \int_0^t \bar{\sigma}^2 (T - s) \, ds + \int_0^t \bar{\sigma} \, dW(s)$$

$$= f^*(0, T) + \bar{\sigma}^2 (tT - t^2 / 2) + \bar{\sigma} W(t).$$

This gives the short rate

$$r(t) = f(t, t) = f^*(0, t) + \bar{\sigma}^2 t^2 / 2 + \bar{\sigma} W(t),$$

which gives

$$dr(t) = \frac{\partial}{\partial t} f^*(0, t) + \bar{\sigma}^2 t \, dt + \bar{\sigma} \, dW(t), \text{ (Calibrated Ho-Lee model).}$$
The Ho-Lee model calibrated to initial ZCB prices

Remember that

\[ p(t, T) = e^{-\int_t^T \Theta(s)(T-s) \, ds + \frac{1}{2} \sigma^2 (T-t)^3 - (T-t)r(t)} \]  

We have for calibrated model that

\[ \Theta(t) = \frac{\partial}{\partial t} f^*(0, t) + \sigma^2 t. \]

Plugging in this expression for \( \Theta \) into Eq. (*) gives

\[
\begin{align*}
p(t, T) &= \exp \left( - \int_t^T \left( \frac{\partial}{\partial s} f^*(0, s) + \sigma^2 s \right)(T-s) \, ds \
&\quad + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t) \right) \\
&= \exp \left( - \left[ (f^*(0, s) + \sigma^2 \frac{s^2}{2})(T-s) \right]_{t}^{T} - \int_{t}^{T} f^*(0, s) + \sigma^2 \frac{s^2}{2} \, ds \
&\quad + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t) \right) \\
&= \exp \left( f^*(0, t)(T-t) - \int_{0}^{T} f^*(0, s) \, ds - \int_{0}^{t} f^*(0, s) \, ds \
&\quad + \sigma^2 \frac{t^2}{2} (T-t) - \sigma^2 \frac{T^3-t^3}{6} + \frac{1}{2} \sigma^2 \frac{(T-t)^3}{3} - (T-t)r(t) \right)
\end{align*}
\]
Calibration to initial ZCB prices for Ho-Lee model 2

Simplying the above expression we obtain

\[ p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left( (T - t)f^*(0, t) - \frac{1}{2}\sigma^2 t(T - t)^2 - (T - t)r(t) \right) . \]

This leads to a forward rate as

\[ f(t, T) = -\frac{\partial}{\partial T} \ln(p(t, T)) = f^*(0, T) - f^*(0, t) + \sigma^2 t(T - t) + r(t), \]

which gives

\[
\begin{align*}
\mathrm{d}f(t, T) &= \left( -\frac{\partial}{\partial t} f^*(0, t) + \sigma^2 (T - t - t) \right) \mathrm{d}t + \mathrm{d}r(t) \\
&= \left( -\frac{\partial}{\partial t} f^*(0, t) + \sigma^2 (T - 2t) \right) \mathrm{d}t + \left( \frac{\partial}{\partial t} f^*(0, t) + \sigma^2 t \right) \mathrm{d}t + \sigma \mathrm{d}W(t) \\
&= \sigma^2 (T - t) \mathrm{d}t + \sigma \mathrm{d}W(t)
\end{align*}
\]
LIBOR dynamics in the Gaussian HJM framework

Assume that $\sigma(t, s)$ is deterministic

$$df(t, s) = \alpha(t, s)dt + \sigma(t, s)dW(t)^{Q}$$

and recall that

$$X(t) = L_t[T_1, T_2] = \frac{1}{T_2 - T_1} \left( \frac{p(t, T_1)}{p(t, T_2)} - 1 \right)$$

$$= \frac{1}{T_2 - T_1} \left( e^{\int_{T_1}^{T_2} f(t, s)ds} - 1 \right).$$

This gives the $Q^{T_2}$-dynamics for $t \leq u \leq T_1$

$$dX(u) = \frac{1}{T_2 - T_1} e^{\int_{T_1}^{T_2} f(u, s)ds} \left( \int_{T_1}^{T_2} \sigma(u, s)ds \right) dW^{Q^{T_2}}(u)$$

$$= \left( X(u) + \frac{1}{T_2 - T_1} \right) v(u, T_1, T_2) dW^{Q^{T_2}}(u).$$

This gives that

$$X(T_1) = \left( X(t) + \frac{1}{T_2 - T_1} \right) e^{-\frac{1}{2} \int_{T_1}^{T_2} |v(u, T_1, T_2)|^2du + \int_{T_1}^{T_2} v(u, T_1, T_2)dW^{Q^{T_2}}(u)} - \frac{1}{T_2 - T_1}. $$
Corresponding Caplet in the Gaussian HJM framework

Pay-off time at $T_2$: 
$$(1 + (T_2 - T_1)X(T_1)) - (1 + (T_2 - T_1)K))^+ = (T_2 - T_1)(X(T_1) - K)^+.$$ 

Using the dynamics on previous page we get:

$$\Pi^{Caplet}(t, [T_1, T_2], K) = p(t, T_2) \left( (1 + (T_2 - T_1)X(T_1))N(d_1) - (1 + (T_2 - T_1)K)N(d_2) \right),$$

where

$$d_1 = \ln \left( \frac{1+(T_2-T_1)X(T_1)}{1+(T_2-T_1)K} \right) + \frac{1}{2} \bar{\sigma}^2(T_1 - t) \frac{1}{\bar{\sigma} \sqrt{T_1 - t}},$$

$$d_2 = \ln \left( \frac{1+(T_2-T_1)X(T_1)}{1+(T_2-T_1)K} \right) - \frac{1}{2} \bar{\sigma}^2(T_1 - t) \frac{1}{\bar{\sigma} \sqrt{T_1 - t}},$$

$$\bar{\sigma}^2 = \frac{\int_t^{T_1} |v(u, T_1, T_2)|^2 du}{T_1 - t} = \frac{\int_t^{T_1} \left| \int_T^{T_2} \sigma(u, s) ds \right|^2 du}{T_1 - t}.$$