Valuation of derivative assets
Lecture 10

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Monte Carlo methods

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Introduction

In valuation we want to calculate

\[ \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \Phi(S(T)) \middle| \mathcal{F}_t \right] . \]

This can in practise be very hard to do!

- Fourier methods (tomorrow)
- PDE-methods
- Tree-methods (first week)
- Monte Carlo methods (today, simulation)
MC-methods

Two main theorems from probability:

**Law of large numbers (LLN):**

If $X$, $\{X_i\}_{i=1,2,\ldots}$ iid with $\mathbb{E}[|g(X)|] < \infty$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g(X_i) = \mathbb{E}[g(X)]$$

**Central limit theorem (CLT):**

If $X$, $\{X_i\}_{i=1,2,\ldots}$ iid with $V(g(X)) < \infty$ then

$$\frac{\left( \frac{1}{n} \sum_{i=1}^{n} g(X_i) - \mathbb{E}[g(X)] \right)}{\sqrt{V \left( \frac{1}{n} \sum_{i=1}^{n} g(X_i) \right)}} \xrightarrow{d} N(0, 1),$$
LLN
Crude MC-estimate

We estimate the expectation with the mean

\[
\mathbb{E}[g(X)]_{CM}^{CM} = \frac{1}{n} \sum_{i=1}^{n} g(X_i).
\]

How accurate is this?

\[
V(\mathbb{E}[g(X)]_{CM}) = \frac{V(g(X))}{n}.
\]
Example

Calculate

\[ \Pi(t) = e^{-r(T-t)} \mathbb{E}^Q[\Phi(S(T))|\mathcal{F}_t] \]

in the standard BS model. Simulate

\[ S^{(i)}(T) = S(t) \exp((r - \sigma^2/2)(T-t) + \sigma G_i \sqrt{T-t}), \]

where \( G_i \) iid with \( G_i \in N(0, 1) \).

Estimate \( \Pi(t) \) with

\[ \hat{\Pi}(t) = e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^{n} \Phi(S(t) \exp((r - \sigma^2/2)(T-t) + \sigma G_i \sqrt{T-t})). \]
Confidence interval for crude MC-estimate

\[ I_{\mathbb{E}[g(X)]} \approx \mathbb{E}[\hat{g}(X)]_{CM}^n \pm \lambda_{\alpha/2} \sqrt{V[\hat{g}(X)]/\sqrt{n}} \]

where \( N(\lambda_{\alpha/2}) = 1 - \alpha/2 \), \( \mathbb{E}[\hat{g}(X)]_{CM}^n = \frac{1}{n} \sum_{i=1}^{n} g(X_i) \) and

\[ V[\hat{g}(X)] = \frac{1}{n-1} \sum_{i=1}^{n} \left( g(X_i) - \mathbb{E}[\hat{g}(X)]_{CM}^n \right)^2. \]

Drawback: If we want to double the precision we need to quadruple \( n \).

Are there better ways?
Variance reduction techniques

- Antithetic variables
- Control variates
- Importance sampling
Anti-thetic variables (symmetric case)

If $X$ has a symmetric distribution then $g(X) \overset{d}{=} g(-X)$ for all $g$ so we have $\mathbb{E}[g(X)] = \mathbb{E}[g(-X)]$ provided that $\mathbb{E}[|g(X)|] < \infty$.

So we form the new estimate

$$\widehat{\mathbb{E}}[g(X)]_{n}^{AV} = \frac{1}{n} \sum_{i=1}^{n} \frac{g(X_i) + g(-X_i)}{2}$$

Variance:

$$V(\widehat{\mathbb{E}}[g(X)]_{n}^{AV}) = (1 + \rho)V(\widehat{\mathbb{E}}[g(X)]_{2n}^{CM}),$$

where $\rho = C(g(X), g(-X))/V(g(X))$. 

Anti-thetic variables (symmetric case) cont

So if \( \rho < 0 \) we do better than crude MC with \( 2n \). If \( g \) is monotone then \( \rho < 0 \) always.

**BEST CASE:**
If \( g \) is an odd function (+ a constant) then \( V(\hat{E}[g(X)]_{AV}^n) = 0 \).

**WORST CASE:**
If \( g \) is an even function then \( V(\hat{E}[g(X)]_{AV}^n) = V(\hat{E}[g(X)]_{CM}^n) \).
Example

Estimate $\mathbb{E}[e^{W_t}]$ where $W$ is standard BM.
Anti-thetic variables (general case)

Draw $X_i$ with inverse method $X_i = F^{-1}(U_i)$ and $\tilde{X}_i = F^{-1}(1 - U_i)$ where $U_i \in U(0, 1)$. Form

$$\hat{\mathbb{E}}[g(X)]_{n}^{AV} = \frac{1}{n} \sum_{i=1}^{n} \frac{g(X_i) + g(\tilde{X}_i)}{2}.$$ 

In some cases the inverse method is not possible (or very hard)!
Control variates

Find a random variable $Y$ with \underline{known expectation} which is correlated to $X$. Form

$$
\hat{E}[g(X)]_{CV}^{n} = \frac{1}{n} \sum_{i=1}^{n} g(X_i) - b \frac{1}{n} \sum_{i=1}^{n} (Y_i - E[Y]).
$$

Why does this work? How do we choose $b$?
Optimal choice of $b$

The best choice for $b$ is

$$b = \frac{C(g(X), Y)}{V(Y)}.$$ 

This gives

$$V(\overline{\mathbb{E}[g(X)]}_{n}^{CV}) = V(CrudeMC_{n}) - \frac{C(g(X), Y)^{2}}{nV(Y)}.$$ 

Problem: We generally do not know $C(g(X), Y)$ or perhaps not even $V(Y)$. Estimate $b$ (using a CM-estimate)!

$$\hat{b} = \frac{\sum_{i=1}^{n} g(X_i)(Y_i - \mathbb{E}[Y])}{\sum_{i=1}^{n}(Y_i - \mathbb{E}[Y])^{2}}.$$
Example European call

Use \( Y = S(T)e^{-r(T-t)} \) as control variate. \( \mathbb{E}^Q[Y] = ? \)

Simulate (in the BS-case) the stock as

\[
S^{(i)}(T) = S(t) \exp((r - \sigma^2/2)(T - t) + \sigma G_i \sqrt{T - t}),
\]

where \( G_i \in N(0, 1) \).

\[
\hat{\Pi}(t) = e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^{n} (S^{i}(T) - K)^+ - b \frac{1}{n} \sum_{i=1}^{n} (S^{i}(T)e^{-r(T-t)} - S(t))
\]

Estimate \( b \) as before!

Interesting fact: As \( (T-t) \) approaches zero the optimal \( b \) will approach \( \Delta \) of the option! Why?
Example Control Variate

Estimation of European call option price
\[ r = 0.02, \sigma = 0.5, T = 1, S(0) = 100, K = 1 \]
using 1000 samples
Importance sampling

We want to estimate $\mathbb{E}^Q[g(X)]$. If the function $g(X)$ is only non-zero with a low probability Crude MC might lead to bad estimates.

Idea: Use measure change to fix this. Assume that we have another measure $Q'$ with $Q' \sim Q$ with LR $L = dQ/dQ'$.

Now we have

$$\mathbb{E}^Q[g(X)] = \mathbb{E}^{Q'}[g(X)L].$$
Importance sampling (cont)

Choose \( Q' \) such the probability of \( Q'(g(X) > 0) \) is reasonably high.
Simulate \( X_i \) under \( Q' \).
Estimate \( \mathbb{E}^Q[g(X)] \) as

\[
\mathbb{E}^Q[g(X)]^{IS} = \frac{1}{n} \sum_{i=1}^{n} g(X_i) L_i.
\]
Example: Valuation of out of the money call option

If the moneyness $\frac{\mathbb{E}_Q[S(T)|\mathcal{F}_t]}{K} = \frac{S(t)e^{r(T-t)}}{K}$ is low the function we take expectation over is zero for most probable outcomes.

Define $\mathbb{Q}'$ such that $S$ has dynamics

$$dS(t) = aS(t)dt + \sigma S(t)dW^{\mathbb{Q}'}(t)$$
Example (cont)

Rule of thumb: Choose $a$ such that

$$E^Q'[S(T)|F_t]/K = e^{a(T-t)}S(t)/K = 1.$$ 

This gives $a = \ln(K/S(t))/(T - t)$. We then get that the Girsanov kernel $g$ in the measure change should solve: $a - \sigma g = r$, which gives that $g = (a - r)/\sigma$. We use this to get $L$,

$$L = e^{-g^2(T-t)/2-g(W^Q'(T) - W^Q'(t))}.$$
Example (cont)

Estimation of European call option price

\[ r = 0.02, \sigma = 0.5, T = 1, S(0) = 1, K = 5e^{rT}, M = e^{rT}S(0)/K = 0.2 \]

\[ \mathbb{Q}(S(T) > K) = 2.6132 \cdot 10^{-4}, \quad \mathbb{Q}'(S(T) > K) = 0.40129 \quad \text{1000 samples} \]

Importance sampling vs Crude MC
Example (cont)

Estimation of European call option price

\[ r = 0.02, \sigma = 0.5, T = 1, S(0) = 1, K = 2e^{rT}, M = e^{rT}S(0)/K = 0.5 \]

\[ Q(S(T) > K) = 0.0509, Q'(S(T) > K) = 0.40129 \]

1000 samples
Example (cont)

Estimation of European call option price

\[ r = 0.02, \sigma = 0.5, T = 1, S(0) = 1, K = 1.25e^{rT}, M = e^{rT}S(0)/K = 0.8 \]

\[ Q(S(T) > K) = 0.2431, \; Q'(S(T) > K) = 0.40129 \]

1000 samples
Example (cont) Really extreme case

Estimation of European call option price

\[ r = 0.02, \sigma = 0.5, T = 1, S(0) = 1, K = 100e^{rT}, M = e^{rT}S(0)/K = 0.01 \]

\[ Q(S(T) > K) = 1.5347 \cdot 10^{-21}, \quad Q'(S(T) > K) = 0.40129 \]

1000 samples
Simulation of SDE:s Euler method

Suppose we have the SDE:

\[ dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \]
\[ X(0) = x \]

To simulate \( X(t) \) from 0 to \( T \) form \( \hat{X}_0 = x \) for \( k = 1, 2, \cdots, n \)

\[ \hat{X}_k = \hat{X}_{k-1} + \mu((k - 1)h, \hat{X}_{k-1})h + \sigma((k - 1)h, \hat{X}_{k-1})\sqrt{h}G_k \]

where \( G_k \in N(0, 1) \), \( h=T/n \) is called stepsize.

Expected error:

\[ \mathbb{E}|X(T) - \hat{X}_n| \leq C\sqrt{h}. \]
Simulation of SDE:s Milstein method

To simulate $X(t)$ from 0 to $T$ form $\hat{X}_0 = x$ for $k = 1, 2, \cdots, n$

$$\hat{X}_k = \hat{X}_{k-1} + \mu((k-1)h, \hat{X}_{k-1})h + \sigma((k-1)h, \hat{X}_{k-1})\sqrt{h}G_k$$

$$+ \sigma((k-1)h, \hat{X}_{k-1})\sigma'((k-1)h, \hat{X}_{k-1})\frac{h}{2}(G_k^2 - 1)$$

where $G_k \in N(0, 1)$, $h = T/n$.

Expected error:

$$\mathbb{E}|X(T) - \hat{X}_n| \leq C'h.$$
Simulation of geometric BM

\[ x = 10, \mu = 0.2, \sigma = 0.3, T = 10 \]

\[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \]
Example Heston model (Computer exercise)

\[
\begin{align*}
    dS(t) &= rS(t)dt + \sqrt{V(t)}S(t) \left( \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right) \\
    dV(t) &= \kappa(\theta - V(t))dt + \beta \sqrt{V(t)}dW_1(t)
\end{align*}
\]

Simulate \( S \) (on log scale) with Euler and \( V \) with Milstein.

\[
\begin{align*}
    \hat{S}_k &= \hat{S}_{k-1} \exp((r - \frac{\hat{V}_{k-1}}{2})h + \sqrt{\hat{V}_{k-1}} \sqrt{h}(\rho G_{1,k} + \sqrt{1 - \rho^2} G_{2,k})) \\
    \hat{V}_k &= \hat{V}_{k-1} + \kappa(\theta - \hat{V}_{k-1})h + \sqrt{\hat{V}_{k-1}} \beta \sqrt{h} G_{1,k} + \frac{\beta^2 h}{4} (G_{1,k}^2 - 1),
\end{align*}
\]

where \( G_{1,k} \) for each \( k \) is independent of \( G_{2,k} \) and \( G_{1,k}, G_{2,k} \in N(0,1) \) are iid.
Example Heston model (alternative simulation)

\[
\begin{align*}
    d &= \frac{4\theta \kappa}{\beta^2} \\
    \lambda &= \frac{4\kappa e^{-h\kappa}}{\beta^2(1 - e^{-h\kappa})} \\
    C &= \frac{\beta^2(1 - e^{-h\kappa})}{4\kappa} \\
    \hat{V}_k &= C \times \text{ncx2rnd}(d, \hat{V}_{k-1}\lambda) \text{(MATLAB's non-central-}\chi^2\text{ random number)} \\
    \hat{S}_k &= \hat{S}_{k-1} \exp \left( h \left( \left( r - \frac{\rho \kappa \theta}{\beta} \right) + \frac{\hat{V}_k + \hat{V}_{k-1}}{2} \left( \frac{\kappa \rho}{\beta} - \frac{1}{2} \right) ight) \right) \\
    &\quad + \frac{\rho}{\beta}(\hat{V}_k - \hat{V}_{k-1}) + \sqrt{h \frac{\hat{V}_k + \hat{V}_{k-1}}{2} (1 - \rho^2)G_k}
\end{align*}
\]

where \( \{G_k\}_{k=1}^n \) are iid standard Gaussian r.v.