Beyond Black-Scholes

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FMSN25/MAFM24 Valuation of Derivative Assets

October 4, 2019
Stylized facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
OMXS30-index
OMXS30-index (Daily log-returns)

\[ r_t = \log(S_t) - \log(S_{t-\delta}) \]
Are daily log-returns Gaussian?
What Do Real Option Prices Look like?

Optionprices 20110927 10:10:51

Time to maturity
Strike

Option prices are plotted against time to maturity and strike price in a 3D graph. The x-axis represents the time to maturity, ranging from 0 to 1200. The y-axis represents the strike price, ranging from 0 to 200. The z-axis represents the option price, ranging from 0 to 200. The graph shows the relationship between time to maturity, strike price, and option price.
What Do Real Option Prices Look like?

Optionprices 20110927 12:55:00

Time to maturity

Strike

0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0

0 200 400 600 800 1000 1200
What Do Real Option Prices Look like?

Optionprices 20110928  9:40:00

Time to maturity

Strike

0
600
800
1000
1200
1400

0
50
100
150
200

0
0.2
0.4
0.6
0.8

0.6
0.4
0.2

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What Do Real Option Prices Look like?
What Do Real Option Prices Look like?
Implied volatility
If the Black-Scholes model were true all we need to know is the volatility to price options.
Implied volatility 20110927 10:10 OMXS30
If the Black-Scholes model was true the implied volatility would be constant!
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If the Black-Scholes model was true the implied volatility would be constant!
How bad is the Black-Scholes fit?

Only 6.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 5.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 8.2% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 7.3% of the model prices are within the ASK-BID bounds!
What can we do about this?

We need more advanced models!!

- Stochastic volatility
- Stock models with jumps (Exponential Lévy processes)
- Stock models with jumps and stochastic volatility
- Local volatility models
- Markov switched models
How can we model volatility?

Continuous time stochastic volatility Heston

\[ \begin{align*}
    dV_t &= \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW^{(1)}_t \\
    dS_t &= \mu S_t dt + S_t \sqrt{V_t} \left( \rho dW^{(1)}_t + \sqrt{1 - \rho^2} dW^{(2)}_t \right)
\end{align*} \]
Heston model is not complete

Equation for Q-dynamics:

\[ r = \mu - g_1(t) \sqrt{V(t)} \rho - g_2(t) \sqrt{V(t)} \sqrt{1 - \rho^2} \]

\[ \beta = \kappa(\theta - V(t)) - g_1(t) \sqrt{V(t)} \beta \]

How should volatility risk be priced?
No general criteria available since volatility is not explicitly traded.
What about VIX?
Possible $\mathbb{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathbb{Q} \theta^\mathbb{Q}}{\mu - r} \rho, \quad b = \frac{\kappa^\mathbb{Q} - \kappa}{\mu - r} \beta,$$

which gives the $\mathbb{Q}$-dyn

$$dS_0(t) = rS_0(t)dt,$$

$$dS_1(t) = S_1(t)r dt + S_1(t)\sqrt{V(t)}(\rho dW_1^\mathbb{Q}(t) + \sqrt{1 - \rho^2}dW_2^\mathbb{Q}(t)),$$

$$dV(t) = \kappa^\mathbb{Q}(\theta^\mathbb{Q} - V(t))dt + \sigma_V \sqrt{V(t)}dW_1^\mathbb{Q}(t),$$
OMXS30 Heston-volatility - Estimated from option prices
Lévy processes

A process $X$ with following properties is called a Lévy process

- $X_0 = 0$
- Independent increments $X_{t+s} - X_t$ independent of $X_t$ for all $s > 0$ all $t > 0$
- Stationary increments $X_{s+t} - X_t \overset{d}{=} X_s$ for all $s > 0$ all $t > 0$
Examples of Lévy processes

- Wiener process
- Poisson
- Compound Poisson
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift [Merton, 1976]
- Gamma process
- Normal Inverse Gaussian (NIG) process [Barndorff-Nielsen, 1997]
- Variance Gamma (VG) process [Madan and Seneta, 1990]
- Carr Geman Madan Yor (CGMY) process [Carr et al., 2002]
- Finite Moment Log Stable (FMLS) process (crash model) [Carr and Wu, 2003]
General Lévy processes

A general Lévy process can be written as

\[ X(t) = \mu t + \sigma W(t) + Z(t) \]

Linear drift \( \mu t \),
Brownian motion with variance \( \sigma^2 \): \( \sigma W(t) \).
Pure jump process \( Z(t) \)
Lévy-Khintchine representation

The characteristic function of any one-dimensional Lévy process can be written as

$$\phi(y, t) = \mathbb{E}[\exp(ivX(t))] = \exp(tK(y)),$$

where

$$K(y) = i\mu y + (iy)^2 \sigma^2 / 2 + K_z(y)$$

with

$$K_z(y) = i\gamma y + \int_\mathbb{R} (e^{iyx} - 1 - iyxI(|x| < 1))\nu(dx),$$

$\nu$ is called the Lévy measure.
Lévy measures

Interpretation
The number

$$\int_a^b \nu(dx),$$

equals the average number of jumps with sizes between a and b per time unit.

General restriction on $\nu$

$$\int_{\mathbb{R}} \min(x^2, 1) \nu(dx) < \infty$$

This is equivalent to that all Lévy processes have finite quadratic variation.
Expectation and variance

Expectation

\[ \mathbb{E}[X(t)] = tK'(0)/i = t\left(\mu + \gamma + \int_{|x|>1} x\nu(dx)\right) \]

Variance

\[ \text{Var}[X(t)] = -tK''(0) = t\left(\sigma^2 + \int_{\mathbb{R}} x^2\nu(dx)\right) \]

But note that neither the variance nor the expectation needs to be finite!

Moment relations

The expectation \( \mathbb{E}[|g(X(t))|] \) is finite for all \( t > 0 \) if

\[ \int_{|x|>1} |g(x)|\nu(dx) < \infty, \]

provided that \( |g(x + y)| \leq c|g(x)g(y)| \) for some \( c > 0 \) \( \forall x, y \in \mathbb{R} \).
Exponentially affine stock price models under $Q$

A stock price model is called exponentially affine if [Duffie et al., 2000]

$$\mathbb{E}[e^{iy \ln(S(T))} \mid S(t)] = \exp(iy \ln(S(t)) + iy(T-t) + A(t, T, iy) + B(t, T, iy)V(t)),$$

where $A$ and $B$ does not depend on $S$ (or $V$). Note that $B$ is related to stochastic volatility and is set to zero for models with out stochastic volatility. Almost all recent stock price models fall into this class.

**Examples:** Black-Scholes, Heston, Bates, Merton, VG, CGMY, NIG and NIG-CIR etc ...

**Not in the class:** Constant elasticity of Variance (CEV), Stochastic alpha-beta-rho (SABR) and Local volatility models.
Condition for the discounted price process to be a $\mathbb{Q}$-martingale

The discounted price process is a martingale if

$$\mathbb{E}^\mathbb{Q}[e^{-r(T-t)}S(T)|\mathcal{F}_t] = S(t).$$

This is true if $A(t, T, 1) = 0$ and $B(t, T, 1) = 0$. 
The Merton model [Merton, 1976]

\[
dS_t = rS_t dt + \sigma S_t dW_t + S_{t-}(e^{J_t} - 1) dN_t - S_t \lambda (e^{\mu J + \sigma J^2 / 2} - 1) dt
\]

where \( J_t \in N(\mu J, \sigma^2 J) \), \( N \) is a Poisson process with intensity \( \lambda \).

\[
\mathbb{E}[e^{iy \ln(S(T))}|S(t)] = \exp(iy \ln(S(t)) + iy r (T - t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T - t)((-\sigma^2 / 2)iy + (iy)^2 \sigma^2 / 2 + \lambda \left( (e^{iy \mu J + (iy)^2 \sigma^2 / 2} - 1) - iy(e^{\mu J + \sigma^2 / 2} - 1) \right)
\]

Note that \( S_{t-} = \lim_{s \uparrow t} S_s \).
How bad is the Merton fit?

Only 8.4% of the model prices are within the ASK-BID bounds!
The Heston model [Heston, 1993]

\[ \begin{align*}
    dV_t &= \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t}dW_t^{(1)} \\
    dS_t &= \mu S_t dt + S_t \sqrt{V_t}(\rho dW_t^{(1)} + \sqrt{1 - \rho^2}dW_t^{(2)})
\end{align*} \]

\[ \mathbb{E}[e^{iy\ln(S(T))}|S(t)] = \exp(iy\ln(S(t)) + iy(\frac{1}{2}T-t) + A(t,T,iy) + B(t,T,iy)V(t)) \]

\[ A(t,T,iy) = \frac{\kappa\theta}{\sigma_v^2} \left( (\kappa - \rho \sigma_v iy - d)(T-t) \\
    \quad - 2 \log\left( (\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1) \right) \right), \]

\[ B(t,T,iy) = (1 - e^{-d(T-t)}) \frac{(iy)^2 - iy}{(\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1)}, \]

\[ d = \sqrt{(\rho \sigma_v iy - \kappa)^2 + \sigma_v^2 (iy + y^2)}. \]
How good is the Heston fit?

About 72% of the model prices are within the ASK-BID bounds!
The Bates model $\sim$ Heston+Merton [Bates, 1996]

\[ \text{d}S_t = rS_t \text{d}t + \sqrt{V_t} S_t \text{d}W_t + S_t (e^{J_t} - 1) \text{d}N_t - S_t \lambda (e^{\mu_j + \sigma_j^2/2} - 1) \text{d}t \]

where $J_t \in \text{Norm}(\mu_J, \sigma_J^2)$, $N$ is Poisson a process with intensity $\lambda$ and $V$ is as in Heston.

\[
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy r (T - t) + A(t, T, iy) \\
+ B(t, T, iy) V(t))
\]

\[
A(t, T, iy) = A_{Merton}(t, T, iy)|_{\sigma=0} + A_{Heston}(t, T, iy)
\]

\[
B(t, Y, iy) = B_{Merton}(t, T, iy) + B_{Heston}(t, T, iy) \\
= B_{Heston}(t, T, iy)
\]
How good is the Bates fit?

About 80% of the model prices are within the ASK-BID bounds!
How good is the Bates fit?

About 55\% of the model prices are within the ASK-BID bounds!
The Normal Inverse Gaussian (NIG) model [Barndorff-Nielsen, 1997]

\[ S_t = S_0 \exp(rt + X(t)), \]

where \( X(t) \) is NIG Lévy process.

\[ \mathbb{E}[e^{iy \ln(S(T))}|S(t)] = \exp(iy \ln(S(t)) + iyr(T - t) + A(t, T, iy)) \]

\[ A(t, T, iy) = (T - t) \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2} \right) \]

\[ -iy(T - t) \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2} \right), \]

where \( \alpha > |\beta + 1|, \delta > 0. \)
The NIGCIR model [Carr et al., 2003b]

This is a stochastic volatility (stochastic time change) model with jumps

\[ S_t = S_0 \exp(\rt + X(I_t)) \]
\[ I_t = \int_0^t V_s ds \]

where \( X \) is a NIG Lévy process, and \( V \) is as in Heston.

\[ A(t, T, iy) = A_{ICIR}(t, T, A_{NIG}(0, 1, iy)), \]
\[ B(t, T, iy) = B_{ICIR}(t, T, A_{NIG}(0, 1, iy)), \]

where \( \mathbb{E}[\exp(z \int_t^T V_s ds)|\mathcal{F}_t] = \exp(A_{ICIR}(t, T, z) + B_{ICIR}(t, T, z)V(t)).\) with
\[ A_{ICIR}(t, T, z) = A_{Heston}(t, T, iy)|_{(iy+y^2)=-2z,\rho=0}, \]
\[ B_{ICIR}(t, T, z) = B_{Heston}(t, T, iy)|_{(iy+y^2)=-2z,\rho=0}. \]
Fourier methods for pricing exponentially affine models

Let $\tilde{s}_T = \ln(S(T))$, $\tilde{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

\[
\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{s}_T} - e^{\tilde{k}}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z + 1)}, \text{ if } \text{Re } z > 0.
\]

- The Fourier transform for the pay-off a European put.

\[
\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{k}} - e^{\tilde{s}_T}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z + 1)}, \text{ if } \text{Re } z < -1.
\]

- The Fourier transform for $-\min(S(T), K)$.

\[
- \int_{\mathbb{R}} e^{z\tilde{k}} \min(e^{\tilde{k}}, e^{\tilde{s}_T}) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z + 1)}, \text{ if } -1 < \text{Re } z < 0.
\]
Fourier methods for pricing exponentially affine models

Let $\tilde{s}_T = \ln(S(T))$, $\tilde{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{s}_T} - e^{\tilde{k}}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } \text{Re}z > 0.$$  

- The Fourier transform for the pay-off a European put.

$$\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{k}} - e^{\tilde{s}_T}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } \text{Re}z < -1.$$  

- The Fourier transform for $-\min(S(T), K)$.

$$-\int_{\mathbb{R}} e^{z\tilde{k}} \min(e^{\tilde{k}}, e^{\tilde{s}_T}) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } -1 < \text{Re}z < 0.$$
Fourier methods for pricing exponentially affine models

Let $\bar{s}_T = \ln(S(T))$, $\bar{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \Re z > 0.$$ 

- The Fourier transform for the pay-off a European put.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } \Re z < -1.$$ 

- The Fourier transform for $-\min(S(T), K)$.

$$- \int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}, \text{ if } -1 < \Re z < 0.$$
Inverse Fourier transform

Now we have that

$$\max(S_T - K, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\overline{k}} \frac{e^{(z+1)\overline{s}_T}}{z(z + 1)} \left|_{z = \bar{z} + iw} \right. \quad \text{dw}, \quad \bar{z} > 0$$

Thus we can write

$$\Pi(t) = \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \max(S_T - K, 0) | S_t \right]$$

$$= \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\overline{k}} \frac{e^{(z+1)\overline{s}_T}}{z(z + 1)} \left|_{z = \bar{z} + iw} \right. \right] \text{dw} | S_t$$

$$= e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\overline{k}} \mathbb{E}^Q \left[ e^{(z+1)\overline{s}_T} | S_t \right] \left|_{z = \bar{z} + iw} \right. \text{dw}$$
Use that $S_T$ comes from an exponentially affine model

Then

\[
\mathbb{E}^Q \left[ e^{(z+1)\bar{S}_T} \middle| S_t \right] = e^{r(t,T)(t-T)(z+1)+(z+1)\ln(S_t)+A(t,T,z+1)+B(t,T,z+1)V_t} = g(t, T, z + 1)
\]

So that

\[
\Pi(t) = e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{g(t, T, z + 1)}{z(z + 1)} \bigg|_{z=\bar{z}+iw} \, dw
\]

\[
= e^{-r(t,T)(T-t)} \frac{1}{\pi} \int_0^{\infty} \text{Re} \left( e^{-z\bar{k}} \frac{g(t, T, z + 1)}{z(z + 1)} \bigg|_{z=\bar{z}+iw} \right) \, dw
\]
Calculation of the inverse Fourier transform

We can use the fast Fourier transform.

We can use quadrature methods.

\[ \Pi(t) \approx \sum_{j=1}^{N} w_j^{(N)} e^{\gamma_j^{(N)}} e^{-\gamma(t,T)(T-t)} \frac{1}{\pi} \text{Re} \left( e^{-\bar{z}k} \frac{g(t, T, z + 1)}{z(z + 1)} \bigg|_{z = \bar{z} + i\gamma_j^{(N)}} \right), \]

where \( \gamma_j^{(N)}, w_j^{(N)} \) are weights coming from the Gauss-Laguerre quadrature method.
References


Ito’s formula for Lévy processes

\[
df(X(t)) = f'(X(t)) \mu dt + \frac{f''(X(t)) \sigma^2}{2} dt + \sigma f'(X(t)) dW(t) \\
+ f'(X(t-)) dZ(t) \\
+ f(X(t-) + \Delta Z(t)) - f(X(t-)) - f'(X(t-)) \Delta Z(t),
\]

where \( \Delta Z(t) \) is the jump in \( Z \).
Origin of NIG

The original NIG distribution depend on four parameters \((\alpha, \beta, \delta, \mu)\) and it is related to two independent Brownian motions \(W_1\) and \(W_2\). Let \(W_1\) be a Brownian motion starting at \(\mu\) with drift \(\beta\) and let \(W_2\) be a Brownian motion starting at 0 with drift \(\sqrt{\alpha^2 - \beta^2}\). Let \(\tau_\delta = \inf\{s > 0 : W_2(s) > \delta\}\). Now \(X = W_1(\tau_\delta)\) has a NIG distribution with parameters \((\alpha, \beta, \delta, \mu)\) and

\[
\mathbb{E}[e^{iyX}] = \exp(iy\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2}))
\]

In order to get the right model for stocks we should choose

\[
\mu = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2})
\]