1. Your bank offers you a derivative. The derivative has maturity $T$ and the following pay-off function:

$$
\begin{cases}
K & S(T) \leq K, \\
S(T) & K \leq S(T) \leq 2K, \\
2K & 2K \leq S(T),
\end{cases}
$$

where $K > 0$. Show how you can replicate this pay-off by using standard contracts. You are allowed to use the stock $S$, European put and call options, and a zero-coupon bond each with maturity $T$. Express the fair value for $0 < t < T$ of the derivative using a linear combination of the fair values of standard contracts.

2. Let the square integrable Martingales $X$ and $Y$, for $t \geq 0$, be given by the following SDE:s,

$$
\begin{align*}
\text{d}X(t) &= W_1(t)\text{d}W_1(t) + W_2(t)\text{d}W_2(t), \quad X(0) = 0, \\
\text{d}Y(t) &= W_2(t)\text{d}W_1(t) - W_1(t)\text{d}W_2(t), \quad Y(0) = 0,
\end{align*}
$$

where $W_1$ and $W_2$ are two independent standard Brownian motions. Show that the process $M(t) = X(t)Y(t)$ is a Martingale for $t \geq 0$.

3. For an affine term structure model we have the ZCB-value $p(t, T)$ is of the form $p(t, T) = \exp(A(t, T) - B(t, T)r(t))$, where $A, B$ are deterministic functions which do not depend on $r$. This is true for all short rate models with $\mathbb{Q}$-dynamics of the form

$$
\text{d}r(t) = \alpha(t)r(t)\text{d}t + \beta(t)\text{d}W(t),
$$

The functions $A$ and $B$ satisfy the following system of ordinary differential equations (ODE:s)

$$
\begin{align*}
B'_1(t, T) &= -\alpha(t)B(t, T) + \frac{1}{2}\gamma(t)B^2(t, T) - 1, \quad B(T, T) = 0, \\
A'_1(t, T) &= \beta(t)B(t, T) - \frac{1}{2}\delta(t)B^2(t, T), \quad A(T, T) = 0
\end{align*}
$$

Solve these equations to find the ZCB-value for the short rate model with following $\mathbb{Q}$-dynamics

$$
\text{d}r(t) = adt + bdW(t),
$$

where $a, b$ are deterministic constants and $W$ a standard $\mathbb{Q}$-Brownian motion.

Please turn over!
4. Solve the PDE

\[
\frac{\partial f(t,x)}{\partial t} + \mu x \frac{\partial f(t,x)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f(t,x)}{\partial x^2} f(T,x) = 0,
\]

where \( \gamma \) is a real-valued constant, for \( t \in [0,T], \ x > 0 \) using the Feynman-Kac representation.

5. An investor, we can call her Deidre, knows that a certain company’s stock will either decrease or increase quite a lot during the next quarter, depending on the failure or success of their latest project. One way to speculate in either a decrease or an increase is to buy both a European put and a European call with the current stock price as strike. Deidre thinks that this alternative is too expensive. She has heard of a contract called a chooser option, where you can before some later time point \( T_1 \), choose if the contract should be a European put or call option both with maturity \( T_2 \) and strike \( K \) where \( T_1 < T_2 \). Deidre believes that this contract will fit her purpose better. So your task is now to find the fair value of the contract. We assume that you are allowed to up to and including time \( T_1 \) to decide if the option should be a put or a call. When valuating the contract assume that we choose the best alternative at time \( T_1 \), i.e. at time \( T_1 \) you should pick the alternative which has the highest value of the European put and call option. Assume that you are in the standard Black-Scholes market.

(a) Show that this contract can be seen as a sum of two standard contracts one with maturity \( T_1 \) and one with maturity \( T_2 \). Hint: Put-Call parity. (8)

(b) Us the result from (a) to find the fair value of the contract with strike price \( K \) at time \( t \) where \( t < T_1 \). (12)

6. In a realistic situation the short interest rate \( r \) is not a deterministic constant. What one wants to do is to use observed prices of Zero Coupon bonds (ZCB) as a discounting factor when valuating derivatives. The way to accomplish this is to express the dynamics of the underlying stock \( S \) under the forward measure \( Q^T \), i.e. the martingale measure which has the ZCB as numeraire. Under the measure \( Q^T \) we have that the discounted stock process \( Z(t) = S(t)/p(t,T) \) should be a martingale, where \( p(t,T) \) is the value at time \( t \) of a ZCB with maturity \( T \). Note that by definition we have \( Z(T) = S(T) \) since \( p(T,T) = 1 \), so any simple contract on \( S \) with maturity \( T \) can be seen as a simple contract on \( Z(T) \).

(a) Now assume that under the usual martingale measure \( \mathbb{Q} \), we have the following model for the ZCB and the stock,

\[
\begin{align*}
\mathrm{d}S(t) & = r(t)S(t)\mathrm{d}t + a(t)S(t)\mathrm{d}W_{1}^Q(t), \\
\mathrm{d}p(t,T) & = r(t)p(t,T)\mathrm{d}t + p(t,T)b(t,T)\mathrm{d}W_{2}^Q(t), \quad 0 \leq t \leq T,
\end{align*}
\]

where \( W_{1}^Q \) and \( W_{2}^Q \) are two independent standard \( \mathbb{Q} \)-Brownian motions and where \( a \) is a positive deterministic function and \( b \) is a deterministic function such that \( b(t,T) > 0 \) for \( t < T \) and \( b(T,T) \equiv 0 \) for all \( T \). Calculate the the \( Q^T \) dynamics of \( Z(t) = S(t)/p(t,T) \). Hint: Recall that the volatilities do not change when we change the measure. (5)

(b) Use the results from (a) to find the fair value of a derivative with maturity \( T \) and pay-off

\[
\Phi(S(T)) = \min(S(T), K),
\]

for this model, for \( t \) such that \( 0 < t < T \). (10)

(c) Find the replicating portfolio for the contract trading in \( S \) and \( p \). (5)

Good luck!