Solution.

1. We can start to look at the interval \((K \leq S_T \leq K_1)\). Here we add \(K\) times a ZCB since the pay-off is \(K\). Moving on to the second interval \((K \leq S_T \leq 2K)\) we see that we can add a European call option with strike \(K\) giving \((S_T - K)^+ = S_T - K + K = S_T\). This does not change the pay-off in the first interval. Moving on to the final interval \((2K \leq S_T)\) we see that if we also subtract one European call options with strike \(2K\) we get \(K + (S_T - K)^+ - (S_T - 2K)^+ = K + S_T - K - (S_T - 2K) = K + S_T - K - S_T + 2K = 2K\). This leaves the pay-off unchanged in the first two intervals. So let \(\Pi_E^T(t, H, T)\) be the price at time \(t\) of a European call with strike \(H\) and maturity \(T\). So the price of the derivative at time \(t\), \(\Pi(t)\), is given by

\[
\Pi(t) = Kp(t, T) + \Pi_E^T(t, K, T) - \Pi_E^T(t, 2K, T).
\]

To see this assume that the price of the derivative and the static replication differs at some time \(s\) say. Sell the most expensive of the two and buy the cheapest put the rest of the money into the bank account. At maturity the pay-off of the derivative and its replication cancels but we still have money in the bank and thus we have constructed an arbitrage opportunity. Therefore the price of the static replication and the derivative must coincide for all \(0 \leq t \leq T\).

**Alternative replication 1**: Using the put call parity on the first European call option we obtain that the price of the contract can alternatively be written as

\[
\Pi(t) = S(t) + \Pi_E^T(t, K, T) - \Pi_E^C(t, 2K, T),
\]

where \(\Pi_E^T(t, H, T)\) is the price at time \(t\) of a European put with strike \(H\) and maturity \(T\).

**Alternative replication 2**: Using the put call parity on both the European call options we obtain that the price of the contract can alternatively be written as

\[
\Pi(t) = 2Kp(t, T) + \Pi_E^T(t, K, T) - \Pi_E^P(t, 2K, T),
\]

where \(\Pi_E^P(t, H, T)\) is the price at time \(t\) of a European put with strike \(H\) and maturity \(T\).

2. We have

\[
M(t) = X(t)Y(t).
\]

Applying Ito’s formula to \(X(t)Y(t)\) we obtain

\[
\begin{align*}
\text{d}X(t)Y(t) &= Y(t)\text{d}X(t) + X(t)\text{d}Y(t) + \text{d}X(t)\text{d}Y(t), \\
&= Y(t)(W_1(t)\text{d}W_1(t) + W_2(t)\text{d}W_2(t)) + X(t)(W_2(t)\text{d}W_1(t) - W_1(t)\text{d}W_2(t)) \\
&\quad + (W_1(t)\text{d}W_1(t) + W_2(t)\text{d}W_2(t))(W_2(t)\text{d}W_1(t) - W_1(t)\text{d}W_2(t)), \\
&= Y(t)(W_1(t)\text{d}W_1(t) + W_2(t)\text{d}W_2(t)) + X(t)(W_2(t)\text{d}W_1(t) - W_1(t)\text{d}W_2(t)) \\
&\quad + (W_1(t)\text{d}W_1(t) - W_1(t)\text{d}W_1(t))\text{d}t, \\
&= Y(t)(W_1(t)\text{d}W_1(t) + W_2(t)\text{d}W_2(t)) + (Y(t)W_2(t) - X(t)W_1(t))\text{d}W_2(t).
\end{align*}
\]

This is a drift free Ito process. So provided that \(\mathbb{E}[|M(t)|] < \infty\) it is a Martingale. We can either use that \(|xy| \leq (x^2 + y^2)/2\) which gives that

\[
\mathbb{E}[|M(t)|] \leq \frac{1}{2}\mathbb{E}[X(t)^2 + Y(t)^2] < \infty
\]
or the Cauchy-Schwartz inequality which gives
\[ E[|M(t)|] \leq \sqrt{E[X(t)^2]E[Y(t)^2]} < \infty \]

In both cases we use that \( X \) and \( Y \) are square integrable Martingales.

3. The dynamics is
\[ dr(t) = a \, dt + b \, dW_t \]
this gives that
\[ \alpha(t) \equiv 0, \quad \beta(t) \equiv a, \quad \gamma(t) \equiv 0, \quad \delta(t) \equiv b^2. \]
This lead to the following ATS equations
\[
\begin{align*}
B'_t(T, T) &= -1, \quad B(T, T) = 0 \\
A'_t(T, T) &= aB(t, T) - \frac{1}{2} b^2 B^2(t, T), \quad A(T, T) = 0
\end{align*}
\]
Solving the first equation by just integrating up we get
\[ B(t, T) = \int_0^t -1 \, ds + C = -t + C, \]
The condition \( B(T, T) = 0 \) gives \( C = T \) and thus
\[ B(t, T) = T - t. \]
Plugging this into the second equition and integrating up gives
\[ A(t, T) = \int_0^t \left( a(T - s) - b^2 \frac{(T - s)^2}{2} \right) \, ds + C'. \]
The condition \( A(T, T) = 0 \) gives
\[ C' = - \int_0^T \left( a(T - s) - b^2 \frac{(T - s)^2}{2} \right) \, ds. \]
and thus
\[
A(t, T) = - \int_t^T \left( a(T - s) - b^2 \frac{(T - s)^2}{2} \right) \, ds,
\]
\[ = - \frac{(T - t)^2}{2} + \frac{b^2 (T - t)^3}{6}. \]
We then finally obtain
\[ p(t, T) = e^{-\frac{(T-t)^2}{2} + \frac{b^2 (T-t)^3}{6} - (T-t)r(t)} \]

4. According to Feynman-Kac's representation theorem the PDE is solved by
\[ f(t, x) = E[(X_T)^r | X_t = x], \]
where \( X \) has the dynamics
\[
\begin{align*}
dX_s &= \mu X_s \, ds + \sigma X_s \, dW_s, \quad t \leq s \leq T
\end{align*}
\]
\[ X_t = x. \]

It is straightforward to see (at least it should be) that \( X \) is a Geometric BM starting at \( x \) at time \( t \), i.e.
\[ X_T = xe^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)}. \]

Looking at \((X_T)^\gamma\) we see that it is given by
\[ (X_T)^\gamma = xe^{\gamma \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \gamma \sigma(W_T - W_t)}. \]

We thus get that
\[
\begin{align*}
f(t, x) &= E[(X_T)^\gamma | X_t = x] = E \left[ x^{\gamma} e^{\gamma \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \gamma \sigma(W_T - W_t)} \right] \\
&= x^{\gamma} e^{\gamma \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + 2\gamma^2(T-t)} = x^{\gamma} e^{\gamma \mu + \gamma(\gamma - 1)\frac{\sigma^2}{2}(T-t)}. 
\end{align*}
\]

We should also check that the obtained solution fulfills the PDE and the boundary condition. We start with the last task \( f(T, x) = xe^\gamma = x^\gamma \) as prescribed. Finally we get that
\[
\frac{\partial f(t, x)}{\partial t} + \mu x \frac{\partial f(t, x)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2} \\
= -(\gamma \mu + \gamma(\gamma - 1)\frac{\sigma^2}{2}) f(t, x) + \gamma x^{-1} \mu x f(t, x) + \gamma(\gamma - 1)x^{-2} \frac{\sigma^2 x^2}{2} f(t, x) \\
= f(t, x) \left( -\gamma \mu - \gamma(\gamma - 1)\frac{\sigma^2}{2} + \gamma \mu + \gamma(\gamma - 1) \frac{\sigma^2}{2} \right) = 0,
\]
which verifies that the obtained solution is correct. \[ \blacksquare \]

5. (a) We should choose the option which has the highest value at time \( T_1 \). Let \( \Pi^F_E(t, H, T) \) be the value at time \( t \) of a European put with strike \( H \) and maturity \( T \). Let \( \Pi^C_E(t, H, T) \) be the value at time \( t \) of a European call with strike \( H \) and maturity \( T \). Let \( \Pi^F(t, H, T) \) be the value at time \( t \) of a forward with strike \( H \) and maturity \( T \). So we look at
\[ \max(\Pi^C_E(T_1, K, T_2), \Pi^P_E(T_1, K, T_2)) = \max(\Pi^C_E(T_1, K, T_2) - \Pi^P_E(T_1, K, T_2), 0) + \Pi^P_E(T_1, K, T_2) \]

Using the Put-call parity (call-put-forward) we obtain
\[
\begin{align*}
\max(\Pi^C_E(T_1, K, T_2) - \Pi^P_E(T_1, K, T_2), 0) + \Pi^P_E(T_1, K, T_2) \\
= \max(\Pi^F(T_1, K, T_2), 0) + \Pi^P_E(T_1, K, T_2) \\
= \max(S(T_1) - Ke^{-(T_2-T_1)}, 0) + \Pi^P_E(T_1, K, T_2).
\end{align*}
\]

This is now the sum of one European call with strike \( Ke^{-(T_2-T_1)} \) and maturity \( T_1 \) and a European put with strike \( K \) and maturity \( T_2 \). So this is the representation using two standard contracts one with maturity \( T_1 \) and one with maturity \( T_2 \). We can also apply the put-call parity to both these options to obtain the solution as the sum of one European put with strike \( Ke^{-(T_2-T_1)} \) and maturity \( T_1 \) and a European call with strike \( K \) and maturity \( T_2 \).

**Alternative derivation:** Say that we have put from the beginning and want to have an option which let us swap to a call if the call is worth more at time \( T_1 \). The cash flow which changes a put to a call is the value of a forward (put-call parity). The call is worth more than put when the forward is worth more than zero. So we should have a put plus the max of zero and a \( T_2 \)-forward at time \( T_1 \). Using the value of the \( T_2 \)-forward at time \( T_1 \) gives that the max of zero and a \( T_2 \)-forward at time \( T_1 \) can be seen as a call option with strike \( Ke^{-(T_2-T_1)} \) and maturity \( T_1 \). So the chooser is equivalent to the sum of one European call with strike \( Ke^{-(T_2-T_1)} \) and maturity \( T_1 \) and a European put with strike \( K \) and maturity \( T_2 \).
We can also start with a call and buy an option which let us change to a put if that is worth more. The required cash flow is minus a forward. The put is worth more than the call if the forward is worth less than zero. So we should have a call plus the max of zero and minus a $T_2$-forward at time $T_1$. Using the value of the forward at time $T_1$ gives that the max of zero and minus a $T_2$-forward at time $T_1$ can be seen as a put option with strike $K e^{-r(T_2-T_1)}$ and maturity $T_1$. So the chooser is also equivalent to the sum of one European put with strike $K e^{-r(T_2-T_1)}$ and maturity $T_1$ and a European call with strike $K$ and maturity $T_2$.

(b) Using the result from and the RNVF the value of the chooser option, $\Pi(t)$ is given as

$$\Pi(t) = e^{-r(T-t)} E^Q[(S(T_1) - Ke^{-r(T_2-T_1)})^+] + e^{-r(T-t)} E^Q[(K - S(T_2))^+] | \mathcal{F}_t]$$

Using that $S$ follows the standard Black Schol’s model we see that

$$S(T) = S(t) e^{(r - \sigma^2/2)(T-t) + \sigma(W_T - W_t)} \overset{d}{=} S(t) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} G},$$

where $G$ is standard Gaussian random variable. We thus obtain that

$$\Pi_C(t, K, T) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(S(t) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} y} - K, 0) \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} \left( y \geq \ln\left(\frac{K}{S(t)} \right) - \frac{(r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \left( S(t) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} y} - K \right) \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= e^{-r(T-t)} \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} (S(t) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} y} - K) \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= e^{-r(T-t)} \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} (S(t) e^{(r - \sigma^2/2)(T-t) + \sigma \sqrt{T-t} y} - K) \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= S(t) \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} e^{-\frac{1}{2}((\sigma^2(T-t)-2\sigma \sqrt{T-t} y) + y^2)} \frac{1}{\sqrt{2\pi}} dy$$

$$- e^{-r(T-t)} K \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= S(t) \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} e^{-\frac{1}{2}(\sigma^2(T-t)-2\sigma \sqrt{T-t} y) + y^2} \frac{1}{\sqrt{2\pi}} dy$$

$$- e^{-r(T-t)} K \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= S(t) \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$- e^{-r(T-t)} K \int_{\ln(K/S(t)) - (r - \sigma^2/2)(T-t)}^{\infty} \frac{e^{-y^2}}{\sqrt{2\pi}} dy$$

$$= S(t) N \left( \frac{\ln(K/S(t)) - (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) - e^{-r(T-t)} K N \left( \frac{\ln(K/S(t)) - (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right),$$

Where $N$ is the distribution function of a standard Gaussian random variable. This is just the Black-Scholes formula which gives that

$$\Pi_C(t, K, T) = S(t) N(d(t, T, K)) - e^{-r(T-t)} K N(d(t, T, K) - \sigma \sqrt{T-t}),$$

where

$$d(t, T, K) = \frac{\ln(S(t)/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.$$
Using put-call parity we obtain
\[
\Pi_p(t, K, T) = e^{-r(T-t)}KN(-d(t, T, K) + \sigma\sqrt{T-t}) - S(t)N(-d(t, T, K)).
\]

Plugging this into the expression for \(\Pi(t)\) we obtain
\[
\Pi(t) = S(t)N(d(t, T_1, K)e^{-r(T_1-T_0)})) - e^{-r(T_1-T_0)}K e^{-r(T_1-T_0)}N(d(t, T_1, K) - \sigma \sqrt{T_1-t})
\]
\[+ e^{-r(T_1-T_0)}KN(-d(t, T_2, K) + \sigma \sqrt{T_2-t}) - S(t)N(-d(t, T_2, K))
\]
\[+ e^{-r(T_1-T_0)}K(N(-d(t, T_2, K) + \sigma \sqrt{T_2-t}) - N(d(t, T_1, K) - \sigma \sqrt{T_1-t}))
\]

6. (a) By using that \(Z(t) = S(t)/p(t, T)\), for \(0 \leq t \leq T\), is a \(\mathbb{Q}^T\) martingale and that volatilities do note change when we change measure, we see that we can calculate the volatility using the diffusion part of the \(\mathbb{Q}\)-dynamics for \(S(t)/p(t, T)\). This gives that
\[
dZ(t) = \left(\frac{\partial S(t)}{\partial S(p(t, T))}\right)S(t)a(t)dW_1^{\mathbb{Q}^T}(t) + \left(\frac{\partial p(t, T)}{\partial p(t, T)}\right)p(t, T)b(t, T)dW_2^{\mathbb{Q}^T}(t)
\]
\[= Z(t)a(t)dW_1^{\mathbb{Q}^T}(t) - Z(t)b(t, T)dW_2^{\mathbb{Q}^T}(t)
\]
\[= Z(t)[a(t) - b(t, T)]\begin{bmatrix} dW_1^{\mathbb{Q}^T}(t) \\ dW_2^{\mathbb{Q}^T}(t) \end{bmatrix}.
\]

(b) Using that \(S(T) = Z(T)\) we can view the contract as written on the process \(Z\) instead of \(S\). So the price of the European call option can thus be calculated as
\[
\Pi(t) = p(t, T)E^\mathbb{Q}^T[\min(Z(T), K)|\mathcal{F}_t].
\]
To calculate this we must find the distribution of \(Z(T)\) under \(\mathbb{Q}^T\). Solving the SDE for \(Z\) under \(\mathbb{Q}^T\) gives that
\[
Z(T) = Z(t)\exp\left(\frac{1}{2}\int_t^T \sigma(s)^2 + b(s, T)^2ds + \int_t^T \sigma(s)dW_1^{\mathbb{Q}^T}(s) - \int_t^T b(s, T)dW_2^{\mathbb{Q}^T}(s)\right).
\]
This now has the same distribution as
\[
Z(t)\exp\left(-\frac{1}{2}\Sigma(t, T)^2(T-t) + \Sigma(t, T)\sqrt{T-t}G\right),
\]
where
\[
\Sigma(t, T) = \sqrt{\frac{\int_t^T \sigma(s)^2 + b(s, T)^2ds}{T-t}},
\]
and where \(G\) is a standard Gaussian random variable.

If we look at the pay-off we find that that we can decompose it as
\[
\min(S(T), K) = S(T) + \min(0, K - S(T)),
\]
\[= S(T) - \max(S(T) - K, 0),
\]
\[= Z(T) - \max(Z(T) - K, 0).
\]

So it is the stock minus a European call with strike \(K\). By almost the same calculations as in the derivation of the standard Black-Scholes formula from problem 5 but with \(\sigma\) replaced by \(\Sigma(t, T)\) and \(e^{-r(T-t)}\) replaced by \(p(t, T)\) we obtain that we should evaluate the following integral to obtain the European call option value
\[
p(t, T)\int_{\min(K/Z(t), T)\Sigma(t, T)^{-1/2}}^{\infty} (Z(t)e^{-\Sigma(t, T)^2(T-t)/2 + \Sigma(t, T)\sqrt{T-t}y - K}) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy,
\]
\[
\begin{align*}
\int_{\ln(S(t)/K)}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy &= \int_{-d}^{\infty} (S(t)e^{-\Sigma(t,T)^2(T-t)/2+\Sigma(t,T)\sqrt{T-t}} - p(t,T) K) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy, \\
\end{align*}
\]

We then finally obtain
\[
\Pi(t) = S(t) - (p(t,T) Z(t) N(d + \Sigma(t,T) \sqrt{T-t}) - p(t,T) K N(d))
\]
where
\[
d = \frac{\ln(S(t)/K) - \Sigma(t,T)^2(T-t)/2}{\Sigma(t,T)\sqrt{T-t}} = \frac{\ln(S(t)/K) - \ln(p(t,T)) - \Sigma(t,T)^2(T-t)/2}{\Sigma(t,T)\sqrt{T-t}},
\]
and where \( N \) is the distribution function of the standard Gaussian distribution.

(c) Using expression (*) we see that the derivative taken w.r.t to \( S \) and \( p \) of \( d \) is multiplied by zero (the integrand is zero at the point \(-d\)). We thus obtain
\[
b_{S}(t) = \frac{\partial}{\partial S} \Pi(t), \\
b_{S}(t) = 1 - N(d + \Sigma(t,T) \sqrt{T-t}),
\]
and
\[
b_{p}(t) = \frac{\partial}{\partial p} \Pi(t), \\
b_{p}(t) = K N(d).
\]

We see that the hedge weight for \( S \) becomes smaller as \( S \) increases and then more and more money is moved into the ZCB.