Solution.

1. We have

\[ X(t) = \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_0^t e^{-\alpha s} \, dW_s \right). \]

Pick two arbitrary time points \( u \) and \( t \) satisfying \( 0 < u < t \). By direct calculation of the conditional expectation we obtain

\[
E[X(t)|\mathcal{F}_t] = E \left[ \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_0^t e^{-\alpha s} \, dW_s \right) \bigg| \mathcal{F}_t \right],
\]

\[
= E \left[ \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_u^t e^{-\alpha s} \, dW_s + \int_u^t e^{-\alpha s} \, dW_s \right) \bigg| \mathcal{F}_t \right],
\]

\[
= \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_0^{u} e^{-\alpha s} \, dW_s + \frac{1}{2} \int_u^t e^{-2ct} \, ds \right),
\]

\[
= \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} - e^{-2cu} + e^{-2ct} \right) + \int_0^u e^{-\alpha s} \, dW_s \right),
\]

\[
= \exp \left( -\frac{1}{4c} \left( 1 - e^{-2cu} \right) + \int_0^u e^{-\alpha s} \, dW_s \right),
\]

\[
= X(u).
\]

Now since \( X \) is positive we have that \( E[|X(t)|] = E[X(t)] \). We thus have

\[
E[X(t)] = E \left[ \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_0^t e^{-\alpha s} \, dW_s \right) \right],
\]

\[
= \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \frac{1}{2} \int_0^t e^{-2ct} \, ds \right),
\]

\[
= \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} + e^{-2ct} - 1 \right) \right) = 1 < \infty.
\]

Any Ito integral with an integrand adapted to the filtration generated by the Brownian motion is adapted to the filtration generated by the Brownian motion. Now \( X(t) \) is a continuous function of such a process therefore it is also adapted. So we have shown that \( X(t) \) is a martingale for \( t \geq 0 \).

An alternative solution:

We have

\[ X(t) = \exp \left( -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_0^t e^{-\alpha s} \, dW_s \right). \]

To simplify the calculations we define

\[ Z(t) = -\frac{1}{4c} \left( 1 - e^{-2ct} \right) + \int_0^t e^{-\alpha s} \, dW_s. \]

We then have that

\[ X(t) = \exp(Z(t)). \]
Applying Ito’s formula to \( \exp(Z(t)) \) we immediately get that
\[
dX(t) = X(t) \, dZ(t) + X(t)(dZ(t))^2 / 2. \quad (*)
\]
where we have that
\[
dZ(t) = - e^{-2ct} \, dt + e^{-ct} \, dW_t.
\]
Plugging this into (*) we obtain
\[
dX(t) = \left( - \frac{1}{2} e^{-2ct} X(t) + \frac{1}{2} e^{-2ct} X(t) \right) \, dt - e^{-ct} X(t) \, dW_t,
\]
\[
= X(t) e^{-ct} \, dW_t.
\]
So \( X \) is a MG for \( t \geq 0 \) if (using the Ito isometry)
\[
E \left[ \left( \int_0^t X(s) e^{-cs} \, ds \right)^2 \right] = E \left[ \int_0^t X(t)^2 e^{-2cs} \, ds \right] < \infty.
\]
Plugging in the definition of \( X \) we get that
\[
E \left[ \int_0^t X(s)^2 e^{-2cs} \, ds \right] = \int_0^t e^{-2cs} \, ds \cdot \left[ \exp \left( \frac{1}{2c} (1 - e^{-2cs}) \right) \right]_0^t,
\]
\[
= \int_0^t e^{-2cs} \, ds \cdot \left[ \exp \left( \frac{1}{2c} (1 - e^{-2cs}) \right) \right]_0^t,
\]
\[
= \exp \left( \frac{1}{2c} (1 - e^{-2ct}) \right) - 1 < \infty.
\]
That \( X \) is adapted follows from the same argument as above. So we have shown that \( X(t) \) is a martingale for \( t \geq 0 \).

**Trick solution - not originally intended by the exam constructor:**
The form of \( X \) is the same form as a likelihood ratio process where the Girsanov kernel is given by \( g(s) = -e^{-cs} \), i.e.
\[
X(t) = \exp \left( - \frac{1}{2} \int_0^t g(s)^2 \, ds - \int_0^t g(s) \, dW_s \right).
\]
Now this is a martingale according to the Girsanov theorem if \( g \) is adapted and satisfies the Novikov condition, i.e. if
\[
E \left[ \exp \left( \frac{1}{2} \int_0^t g(s)^2 \, ds \right) \right] < \infty,
\]
which is trivially true since \( g \) is deterministic and integrable.

2. This contracts is a bet that the stock lies between \( K - \Delta \) and \( K + \Delta \) at maturity. Since the pay-off is always negative, buying this contract means that you receive the contract and cash. So it is like borrowing money from the bank but you only have to pay something back if the stock ends up outside the interval \([K - \Delta, K + \Delta]\).
There are several different ways of accomplishing this pay-off. But if we start systematically with the interval $S(T) \leq K - 2\Delta$ we see that we can match this pay-off selling a ZCB with face-value $\Delta$ and maturity $T$. Moving on the second interval calculating the difference in pay-off we find that this is $S(T) - K + \Delta - (-\Delta) = S(T) - (K - 2\Delta)$ we can match this by buying a European call option with strike $K - 2\Delta$ and maturity $T$ (not changing the pay-off in the previous interval). Moving on the third interval we get $0 - (S(T) - K + \Delta) = -(S(T) - (K - \Delta))$. So we match this by selling a European call option with strike $K - \Delta$ and maturity $T$ (not changing the pay-off in the previous intervals). Moving on to the fourth interval we get $(K + \Delta - S(T)) - 0 = -(S(T) - (K + \Delta))$. So we match this by a European call option with strike $K + \Delta$ and maturity $T$ (not changing the pay-off in the previous intervals). We thus obtain the solution

$$\Pi_X(t) = -\Delta P(t, T) + \Pi^E_S(t, K - 2\Delta, T) - \Pi^E_S(t, K - \Delta, T) - \Pi^E_S(t, K + \Delta, T) + \Pi^E_S(t, K + 2\Delta, T).$$

Alternative solutions: By using the put-call parity on one (four cases), two (six cases), three (four cases) or four (one case) of the European call options we can get out several different equivalent solutions. None of these solutions are simpler or more convenient than the one already stated. In a practical situation the solution which uses the options which have best liquidity is preferred.

3. We have the following model for the short rate

$$dr(u) = \theta(u) \, du + \sigma \, dW(u).$$

We obtain by integrating up the dynamics that

$$r(u) = r(t) + \int_t^u \theta(s) \, ds + \int_t^u \sigma \, dW(s).$$

By the RNVF we have that

$$p(t, T) = E^Q \left[ \exp \left( -\int_t^T r(u) \, du \right) \mid F_T \right],$$

$$= E^Q \left[ \exp \left( -(T - t) r(t) - \int_t^T \int_t^u \theta(s) \, ds \, du - \int_t^T \int_t^u \sigma \, dW(s) \, du \right) \mid F_T \right].$$

The key element is then to change the order of integration so that we first integrate over $u$ and then $s$. We integrate over the triangle $t < s < u, t < u < T$. So changing the order we obtain $s < u < T, t < s < T$ which gives

$$p(t, T) = E^Q \left[ \exp \left( -(T - t) r(t) - \int_t^T \int_t^u \theta(s) \, ds \, du - \int_t^T \int_t^u \sigma \, dW(s) \, du \right) \mid F_T \right],$$

$$= E^Q \left[ \exp \left( -(T - t) r(t) - \int_t^T (T - s) \theta(s) \, ds - \int_t^T (T - s) \sigma \, dW(s) \right) \mid F_T \right].$$
\[
\text{exp of Gaussian} = \exp \left( -(T - t) r(t) - \int_t^T (T - s) \vartheta(s) \, ds + \frac{1}{2} \mathbb{E}^Q \left[ \left( \int_t^T (T - s) \sigma \, dW(s) \right)^2 \bigg| \mathcal{F}_t \right] \right),
\]

\[
\text{Ito Isometry} = \exp \left( -(T - t) r(t) - \int_t^T (T - s) \vartheta(s) \, ds + \frac{1}{2} \int_t^T (T - s)^2 \sigma^2 \, ds \right),
\]

\[
= \exp \left( -(T - t) r(t) - \int_t^T (T - s) \vartheta(s) \, ds + \frac{1}{6} (T - t)^3 \sigma^2 \right).
\]

Comparing with the formula
\[
p(t, T) = \exp(A(t, T) - B(t, T) r(t)),
\]

we immediately obtain
\[
A(t, T) = -\int_t^T (T - s) \vartheta(s) \, ds + \frac{1}{6} (T - t)^3 \sigma^2,
\]
\[
B(t, T) = T - t.
\]

4. Using Feynman-Kac’s representation formula we obtain
\[
f(t, x) = \mathbb{E}[e^{X(T)} | X_t = x],
\]
where \( X \) has the following dynamics for \( t \leq s \leq T \)
\[
dX_s = \mu \, ds + \sigma \, dW_s, \quad X_t = x.
\]
So we have
\[
X(T) = x + \mu(T - t) + \sigma(W(T) - W(t)) = x + \mu(T - t) + \sigma \sqrt{T - t} G, \quad \text{where } G \in \mathbb{N}(0, 1).
\]
Using this we obtain
\[
f(t, x) = \mathbb{E}[e^{x + \mu(T - t) + \sigma \sqrt{T - t} G}] = e^{ax + a\mu(T - t) + \frac{a^2} {2} \sigma^2 (T-t)}.
\]

We start by checking the boundary condition. Now we have
\[
\lim_{t \uparrow T} = e^{ax + a\mu(T - t) + \frac{a^2} {2} \sigma^2 (T-t)} = e^{ax}.
\]
So the boundary condition is satisfied.

To check that the solution satisfies the PDE we start by calculating the partial derivatives:

\[
\frac{\partial}{\partial t} f(t, x) = -a \mu f(t, x) - \frac{a^2 \sigma^2}{2} f(t, x),
\]
\[
\mu \frac{\partial}{\partial x} f(t, x) = a \mu f(t, x),
\]
\[
\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} f(t, x) = \frac{a^2 \sigma^2}{2} f(t, x).
\]
5. (a) So the required solution at time \( t \) solves the PDE. Thus we have that

\[
\frac{\partial}{\partial t} f(t, x) + \mu \frac{\partial}{\partial x} f(t, x) + \left( \frac{\sigma^2}{2} \right) \frac{\partial^2}{\partial x^2} f(t, x),
\]

\[
= f(t, x) \left( -a \mu - \frac{a^2 \sigma^2}{2} + a \mu + \frac{a^2 \sigma^2}{2} \right),
\]

\[
= 0.
\]

Thus we have that

\[
f(t, x) = e^{\alpha x + \alpha d(T-t) + \frac{\sigma^2}{2}(T-t)},
\]

solves the PDE.

(b) We should now calculate the value at some arbitrary time \( 0 < t < T_1 \). Using the tower property of conditional expectation and that \( e^{-r(T_1-t)} = e^{-r(T_1-T)} e^{-r(T-T)} \) we obtain

\[
\Pi_{FSO}(T_1) = \mathbb{E}^Q[e^{-r(T_1-T)} \max(S(T_2) - cS(T_1), 0) | \mathcal{F}_{T_1}],
\]

\[
= S(T_1) e^{-r(T_1-T_1)} \int_{-\infty}^{\infty} \max(e^{(r-\sigma^2/2)(T_2-T_1) + \sigma \sqrt{T_2-T_1} y} - c, 0) \frac{e^{-y^2}}{\sqrt{2\pi}} dy,
\]

\[
= S(T_1) e^{-r(T_1-T_1)} \int_{-d}^{\infty} \left( e^{(r-\sigma^2/2)(T_2-T_1) + \sigma \sqrt{T_2-T_1} y} - c \right) \frac{e^{-y^2}}{\sqrt{2\pi}} dy,
\]

\[
= S(T_1) \int_{-d}^{\infty} e^{-\frac{1}{2}(\sigma^2(T_2-T_1)+2\sigma \sqrt{T_2-T_1} y+\gamma^2)} \frac{1}{\sqrt{2\pi}} dy - cS(T_1) e^{-r(T_2-T_1)} \int_{-d}^{\infty} e^{-y^2} \frac{1}{\sqrt{2\pi}} dy,
\]

\[
= S(T_1) \left( 1 - N(-d - \sigma \sqrt{T_2-T_1}) \right) - ce^{-r(T_2-T_1)} \left( 1 - N(-d) \right),
\]

\[
\approx S(T_1)(N(d + \sigma \sqrt{T_2-T_1}) - ce^{-r(T_2-T_1)} N(d)),
\]

where \( N \) is the distribution function of a standard Gaussian random variable and where

\[
d = -\ln c + (r - \sigma^2/2)(T_2 - T_1) / \sigma \sqrt{T_2 - T_1}.
\]

(b) We should now calculate the value at some arbitrary time \( 0 < t < T_1 \). Using the tower property of conditional expectation and that \( e^{-r(T_1-t)} = e^{-r(T_1-T)} e^{-r(T-T)} \) we obtain

\[
\Pi_{FSO}(t) = \mathbb{E}^Q[e^{-r(T_1-t)} \max(S(T_2) - cS(T_1), 0) | \mathcal{F}_{T_1}],
\]

\[
= \mathbb{E}^Q[e^{-r(T_1-t)} \mathbb{E}^Q[e^{-r(T_2-T_1)} \max(S(T_2) - cS(T_1), 0) | \mathcal{F}_{T_1}] | \mathcal{F}_t],
\]

\[
= \mathbb{E}^Q[e^{-r(T_1-t)} \Pi_{FSO}(T_1) | \mathcal{F}_t],
\]

\[
= \mathbb{E}^Q[e^{-r(T_1-t)} S(T_1)(N(d + \sigma \sqrt{T_2-T_1}) - ce^{-r(T_2-T_1)} N(d)) | \mathcal{F}_t],
\]

\[
= S(t)(N(d + \sigma \sqrt{T_2-T_1}) - ce^{-r(T_2-T_1)} N(d)).
\]
6. (a) Since 
\[ f_X(t) = \frac{\partial}{\partial S} \Pi_{FSO}(t) = (N(d + \sigma \sqrt{T_2 - T_1}) - ce^{-r(T_2 - T_1)} N(d)), \]
\[ h_B(t) = \frac{\Pi_{FSO}(t) - S(t)(N(d + \sigma \sqrt{T_2 - T_1}) - ce^{-r(T_2 - T_1)} N(d))}{B(t)} = 0. \]

The original intention in the exam was just to consider the case \( 0 < t \leq T_1 \). However since some students also answered about what the hedge should be after time \( T_1 \) we present also for clarity and completeness this case. For \( T_1 < t < T_2 \) the contract behaves like an ordinary European call option and the hedge is
\[ h_t(t) = \frac{\partial}{\partial S} \Pi_{FSO}(t) = N(d + \sigma \sqrt{T_2 - t}), \]
\[ h_B(t) = \frac{\Pi_{FSO}(t) - S(t)N(d + \sigma \sqrt{T_2 - t})}{B(t)} = -S(T_1)ce^{-rT_2}N(d), \]
where
\[ \tilde{d} = \frac{\ln(S(t)/c/S(T_1)) + (r - \sigma^2/2)(T_2 - t)}{\sigma \sqrt{T_2 - t}}. \]

6. (a) Since \( f(t, T) = -\frac{\partial}{\partial T} \ln(p(t, T)) \) we get
\[ p(t, T) = e^{-\int_t^T f(t, u) du}. \]

Calculating the dynamics for \( p \), using the dynamics of \( f \) and that \( f(t, t) = r(t) \) and that \( \sigma(t, t) = 0 \), we get
\[ dp(t, T) = dp(t, T) \left( \int_t^T f(t, u) du \right) + p(t, T) \left( \int_t^T f(t, u) du \right)^2 / 2, \]
\[ = p(t, T) \left( f(t, t) dt - \int_t^T df(t, u) du \right) + p(t, T) \left( f(t, t) dt - \int_t^T df(t, u) du \right)^2 / 2, \]
\[ = p(t, T) \left( r(t) - \int_t^T \frac{\partial}{\partial u} \left( \frac{\sigma(t, u)^2}{2} \right) du + \left( \int_t^T \frac{\partial}{\partial u} \sigma(t, u) du \right)^2 / 2 \right) dt \]
\[ + p(t, T) \int_t^T \frac{\partial}{\partial u} \sigma(t, u) du dW(t), \]
\[ = p(t, T) \left( r(t) - \sigma(t, T)^2 / 2 + \sigma(t, t)^2 / 2 + (\sigma(t, T) - \sigma(t, t))^2 / 2 \right) dt \]
\[ + p(t, T)(\sigma(t, T) - \sigma(t, t)) dW(t), \]
\[ = p(t, T) \left( r(t) - \sigma(t, T)^2 / 2 + \sigma(t, T)^2 / 2 \right) dt + p(t, T)\sigma(t, T) dW(t), \]
\[ = r(t)p(t, T) dt + p(t, T)\sigma(t, T) dW(t). \]

(b) Since we have that \( X \) is the ratio of a traded asset and the numeraire, it will be a martingale under the numeraire measure \( \mathbb{Q}^{T_1} \). Therefore it is enough to consider the diffusion part since we know that the drift part should be zero. We thus have
\[
d\mathcal{X}(s) = d\mathcal{P}(s, T_1) / \mathcal{P}(s, T_2),
\]
\[
= \left( \frac{\partial}{\partial \mathcal{P}(s, T_1)} \mathcal{P}(s, T_1) \right) \mathcal{P}(s, T_1) \sigma(s, T_1) \ d\mathcal{W}^{Q_2}(s)
+ \left( \frac{\partial}{\partial \mathcal{P}(s, T_2)} \mathcal{P}(s, T_2) \right) \mathcal{P}(s, T_2) \sigma(s, T_2) \ d\mathcal{W}^{Q_2}(s),
\]
\[
= \mathcal{P}(s, T_1) \sigma(s, T_1) \ d\mathcal{W}^{Q_2}(s) - \mathcal{P}(s, T_2) \sigma(s, T_2) \ d\mathcal{W}^{Q_2}(s),
\]
\[
= \mathcal{X}(s)(\sigma(s, T_1) - \sigma(s, T_2)) \ d\mathcal{W}^{Q_2}(s).
\]

(c) We first note that \( \mathcal{X} \) is a geometric BM with time varying volatility. So we have that
\[
\mathcal{X}(s) = \mathcal{X}(t) \exp \left( -\frac{1}{2} \int_t^s (\sigma(u, T_1) - \sigma(u, T_2))^2 \ du + \int_t^s (\sigma(u, T_1) - \sigma(u, T_2)) \ d\mathcal{W}^{Q_2}(u) \right),
\]
\[
\overset{d}{=} \mathcal{X}(t) \exp \left( -\frac{1}{2} \Sigma^2_{t,s} + \Sigma_{t,s}G \right),
\]
where \( G \) is a standard Gaussian random variable and where
\[
\Sigma^2_{t,s} = \int_t^s (\sigma(u, T_1) - \sigma(u, T_2))^2 \ du.
\]

We will now use this to find the value of the derivative with pay-off \( \max(\mathcal{X}(t) - \mathcal{X}(T_1), 0) \) at maturity \( T_2 \). So we have according to the risk neutral valuation formula under the numeraire measure \( Q_2 \) that
\[
\Pi(t) = \mathcal{P}(t, T_2) \mathbb{E}^{Q_2} \left[ \max(\mathcal{X}(t) - \mathcal{X}(T_1), 0) \mid \mathcal{F}_t \right],
\]
\[
= \mathcal{P}(t, T_2) \mathbb{E}^{Q_2} \left[ \max \left( \mathcal{X}(t) - \mathcal{X}(t) \exp \left( -\frac{1}{2} \Sigma^2_{t,T_1} + \Sigma_{t,T_1}G \right), 0 \right) \mid \mathcal{F}_t \right],
\]
\[
= \mathcal{P}(t, T_2) \int_{-\infty}^\infty \max \left( \mathcal{X}(t) - \mathcal{X}(t) \exp \left( -\frac{1}{2} \Sigma^2_{t,T_1} + \Sigma_{t,T_1}x \right), 0 \right) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \ dx,
\]
\[
= \mathcal{P}(t, T_2) \mathcal{X}(t) \int_{-\infty}^d \left( 1 - \exp \left( -\frac{1}{2} \Sigma^2_{t,T_1} + \Sigma_{t,T_1}x \right) \right) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \ dx,
\]
\[
= \mathcal{P}(t, T_2) \mathcal{X}(t) \left( \int_{-\infty}^d \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \ dx - \int_{-\infty}^{-d} \exp \left( -\frac{1}{2} (\Sigma^2_{t,T_1} - 2\Sigma_{t,T_1}x + x^2) \right) \frac{1}{\sqrt{2\pi}} \ dx \right),
\]
\[
= \mathcal{P}(t, T_2) \mathcal{X}(t)(N(d) - N(d - \Sigma_{t,T_1})),
\]
where \( N \) is the distribution function of a standard Gaussian random variable and where
\[
d = \frac{1}{2} \frac{\Sigma_{t,T_1}}{\Sigma_{t,T_1}} = \frac{\Sigma_{t,T_1}}{2}.
\]

Using the specific form of \( d \), the symmetry of the standard Gaussian distribution and that \( \mathcal{P}(t, T_2) \mathcal{X}(t) = \mathcal{P}(t, T_1) \) we can further simplify the value as
\[
\Pi(t) = \mathcal{P}(t, T_1)(2N(d) - 1).
\]
(d) Using that
\[ p(t, T) = e^{-\int_t^T f(t, u) \, du}, \]
for any pair \((t, T)\) with \(t \leq T\) and that
\[ X(s) = \frac{p(s, T_1)}{p(s, T_2)}, \]
for any \(s \leq T_1\) we get
\[
\max(X(t) - X(T_1), 0) = \max \left( \frac{p(t, T_1)}{p(t, T_2)} - \frac{p(T_1, T_1)}{p(T_1, T_2)}, 0 \right),
\]
\[
= \max \left( e^{-\int_t^{T_1} f(t, u) \, du} - e^{-\int_{T_1}^{T_2} f(T_1, u) \, du}, 0 \right),
\]
\[
= \max \left( e^{\int_{T_2}^{T_1} f(T_1, u) \, du} - e^{\int_{T_1}^{T_2} f(T_1, u) \, du}, 0 \right),
\]
which was what to be shown.