Valuation of derivative assets
Lecture 9

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Girsanov transformation

Theorem (Å: Thm 9.7 p. 220-221)

Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\{W_t^\mathbb{P}\}_{t \geq 0}$ is a (d-dim) Brownian motion w.r.t. $\mathcal{F}_t$. Let $g_t$ be a (d-dim) process adapted to $\mathcal{F}_t$ for $t \in [0, T]$ which satisfies

$$
\mathbb{E}^\mathbb{P}\left[\exp\left(\frac{1}{2} \int_0^T |g_t|^2 \, dt\right)\right] < \infty, \quad (\text{Novikov condition}).
$$

Define the process $L_t$ by

$$
L_t = \exp\left(-\int_0^t g_s^* \, dW_s^\mathbb{P} - \frac{1}{2} \int_0^t |g_s|^2 \, ds\right), \quad 0 \leq t \leq T.
$$

Define a new probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\mathbb{Q}(A) = \mathbb{E}^\mathbb{P}[1_A L_T]$ for $A \in \mathcal{F}_T$.

Then $W_t^\mathbb{Q} = W_t^\mathbb{P} + \int_0^t g_s \, ds$ is a standard (d-dim) $\mathbb{Q}$-BM on $[0, T]$. 

The new dynamics after change of measure

Suppose that the market \((N+1)\) assets have the \(\mathbb{P}\)-dynamics

\[
\begin{align*}
\text{d}B_t &= r(t)B_t \, \text{d}t, \\
B_0 &= 1, \\
\text{d}S_t &= \text{diag}(S_t)\mu(t, S_t) \, \text{d}t + \text{diag}(S_t)\sigma(t, S_t) \, \text{d}W^\mathbb{P}_t, \\
S_0 &= s.
\end{align*}
\]

Using the Girsanov kernel \(g_t\) we get the \(\mathbb{Q}\)-dynamics

\[
\begin{align*}
\text{d}B_t &= r(t)B_t \, \text{d}t, \\
B_0 &= 1, \\
\text{d}S_t &= \text{diag}(S_t)(\mu(t, S_t) - \sigma(t, S_t)g_t) \, \text{d}t + \text{diag}(S_t)\sigma(t, S_t) \, \text{d}W^\mathbb{Q}_t, \\
S_0 &= s.
\end{align*}
\]
The likelihood ratio process $L$

Applying the Ito formula to

$$L_t = \exp \left( - \int_0^t g_s^* \, dW^\mathbb{P}_s - \frac{1}{2} \int_0^t |g_s|^2 \, ds \right)$$

we get that

$$dL_t = \left(-\frac{1}{2}|g_t|^2 + \frac{1}{2}|g_t|^2\right)L_t \, dt - L_t g_t^* \, dW^\mathbb{P}_t$$

So knowing the dynamics of $L$ we can read off the Girsanov kernel $g$. (This will be used on slide 9.)
Numeraires

Definition (Numeraire)

A numeraire is the basic unit of currency on the market. Any strictly positive asset of the form

\[ N(t) = N(0) + \int_0^t \sum_{i=0}^n \alpha_i(t) \, dS_i(u), \]

can be used as a numeraire.

That means that \( N \) is a strictly positive self-financing portfolio on the market \( S_0, S_1, \ldots, S_n \).

Numeraires are used as discounting factors.
The numeraire measure $Q^N$

First note that $Q = Q^0$ is the numeraire measure for the numeraire $S_0 = B$ (bank account).
What happens if we want to use $S_1$ as the numeraire instead? What is the corresponding numeraire-measure $Q^1$?

Note that the values of all contingent claims should remain unchanged!
The numeraire measure $Q^1$

We should have that $Q^1 \sim Q^0$ (and thus also $Q^1 \sim P$). Let

$$L_T = \frac{dQ^1}{dQ^0}$$

on $\mathcal{F}_T$. We must then have that

$$\Pi(0; X) = S_0(0)E^{Q^0} \left[ \frac{X}{S_0(T)} | \mathcal{F}_0 \right]$$

$$= S_1(0)E^{Q^1} \left[ \frac{X}{S_1(T)} | \mathcal{F}_0 \right]$$

$$= S_1(0)E^{Q^0} \left[ \frac{XL_T}{S_1(T)} | \mathcal{F}_0 \right]$$

for all $\mathcal{F}_T$-claims $X$ with $E^{Q^0}[|X|] < \infty$. 
This then gives that

\[ \frac{S_0(0)}{S_0(T)} = \frac{L_TS_1(0)}{S_1(T)} \]

and thus

\[ L_T = \frac{S_1(T)S_0(0)}{S_0(T)S_1(0)}. \]

and since \( S_1(t)/S_0(t) \) is a \( \mathbb{Q}^0 \)-martingale we get that

\[ L_t = \mathbb{E}^{\mathbb{Q}^0}[L_T|\mathcal{F}_t] = \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)}. \]
The numeraire measure $\mathbb{Q}^1$ cont 2

Under $\mathbb{Q}^0$ we have (with $S(t) = [S_1(t), \ldots, S_n(t)]^*$)

\[
\begin{align*}
    dS_0(t) &= r(t)S_0(t) \, dt \\
    dS(t) &= \text{diag}(S(t)) \, 1_n r(t) \, dt + \text{diag}(S(t)) \sigma(t, S(t)) \, dW^{\mathbb{Q}^0}(t)
\end{align*}
\]

This gives that

\[
\begin{align*}
    dL_t &= d \left( \frac{S_1(t)}{S_0(t)} \right) \frac{S_0(0)}{S_1(0)} \\
    &= r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \, dt + \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \sigma_1.(t, S_t) \, dW^{\mathbb{Q}^0}(t) \\
    &\quad - r(t) \frac{S_1(t)S_0(0)}{S_0(t)S_1(0)} \, dt \\
    &= L_t \sigma_1.(t, S(t)) \, dW^{\mathbb{Q}^0}(t).
\end{align*}
\]

So the Girsanov kernel $g(t) = -\sigma_{1.}^*(t, S(t))$ takes us from $\mathbb{Q}^0$ to $\mathbb{Q}^1$.
The new dynamics under $\mathbb{Q}^1$ (and arbitrary $\mathbb{Q}^k$ $1 \leq k \leq n$)

Using the Girsanov kernel $g_1(t) = -\sigma_{1.}^*(t, S(t))$ we get

$$
dS_0(t) = r(t)S_0(t) \, dt$$
$$
dS(t) = \text{diag}(S(t)) (1_n r(t) + \sigma(t, S(t)) \sigma_{1.}^*(t, S(t))) \, dt$$
$$
+ \text{diag}(S(t)) \sigma(t, S(t)) \, dW^{\mathbb{Q}^1}(t)
$$

With the same type of argument we get for $\mathbb{Q}^k$ that $g_k(t) = -\sigma_{k.}^*(t, S(t))$ and thus the $\mathbb{Q}^k$ dynamics

$$
dS_0(t) = r(t)S_0(t) \, dt$$
$$
dS(t) = \text{diag}(S(t)) (1_n r(t) + \sigma(t, S(t)) \sigma_{k.}^*(t, S(t))) \, dt$$
$$
+ \text{diag}(S(t)) \sigma(t, S(t)) \, dW^{\mathbb{Q}^k}(t)
$$
The forward measure $\mathbb{Q}^T$ 

If the short rate $r(t)$ is stochastic then $S_0(t)/S_0(T) = e^{-\int_t^T r(s) \, ds}$ is a random variable. This may cause some complications for valuation of derivatives. Suppose we can use a bond that pays out one unit of currency at maturity $T$ as a numeraire instead. This derivative is called a zero coupon bond (ZCB). The value at time $t$ here denoted $p(t,T)$ is given by:

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_0(t)}{S_0(T)} \mathbf{1}_{\mathcal{F}_t}\right] = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r(s) \, ds} | \mathcal{F}_t]$$

Note $p(T,T) = 1$ since $\mathbb{E}_{\mathbb{Q}}[e^{-\int_T^T r(s) \, ds} | \mathcal{F}_T] = \mathbb{E}_{\mathbb{Q}}[1 | \mathcal{F}_T] = 1$. 

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Suppose we have a Black-Scholes type of model for $S_1$ but with $r(\cdot)$ stochastic. So assume $\mathbb{Q}^0$-dynamics:

\[
\begin{align*}
\frac{dS_0}{S_0}(t) &= r(t) dt, \\
\frac{dS_1}{S_1}(t) &= S_1(t) r(t) dt + S_1(t) \sigma_1 dt
\end{align*}
\]

where $W^{\mathbb{Q}^0}$ is $d$-dim $\mathbb{Q}^0$-BM and $\sigma_1$ deterministic $d$-dim row-vector. Further assume that $p(t,T)$ has $\mathbb{Q}^0$-dynamics

\[
\frac{dp}{p}(t,T) = p(t,T) r(t) dt + p(t,T) v(t,T) dt
\]

where $v(t,T)$ deterministic is a $d$-dim row-vector-valued function. This gives with the same arguments as above that the corresponding Girsanov kernel $g_T(t)$ is $-v^*(t,T)$. 

We thus get the $\mathbb{Q}^T$ dynamics

\[
\begin{align*}
    \text{d}S_0(t) &= r(t)S_0(t) \, \text{d}t, \\
    \text{d}S_1(t) &= S_1(t)(r(t) + \sigma_1. v^*(t,T)) \, \text{d}t + S_1(t)\sigma_1. \, \text{d}W^\mathbb{Q}^T(t), \\
    \text{d}p(t,T) &= p(t,T)(r(t) + v(t,T)v^*(t,T)) \, \text{d}t + p(t,T)v(t,T) \, \text{d}W^\mathbb{Q}^T(t)
\end{align*}
\]

Let $X(t) = S_1(t)/p(t,T)$, then $X(T) = S_1(T)/p(T,T) = S_1(T)$. This is now a $\mathbb{Q}^T$-martingale with dynamics

\[
\text{d}X(t) = X(t)(\sigma_1. - v(t,T)) \, \text{d}W^\mathbb{Q}^T(t) \overset{d}{=} X(t)\tilde{\sigma}(t) \, \text{d}\tilde{W}^\mathbb{Q}^T(t),
\]

where $\tilde{\sigma}(t) = |\sigma_1. - v(t,T)|$ and $\tilde{W}^\mathbb{Q}^T(t)$ is a 1-dim $\mathbb{Q}^T$-BM.

To price derivatives with maturity $T$ we can view them as written on $X(T)$ rather than $S_1(T)$. So

\[
\mathbb{E}^\mathbb{Q} \left[ \frac{S_0(t)}{S_0(T)} \Phi(S_1(T)) \bigg| \mathcal{F}_t \right] = \frac{p(t,T)}{p(T,T)} \mathbb{E}^\mathbb{Q}^T \left[ \Phi(S_1(T)) \bigg| \mathcal{F}_t \right] = p(t,T)\mathbb{E}^\mathbb{Q}^T \left[ \Phi(X(T)) \bigg| \mathcal{F}_t \right].
\]
Pricing of European call under stochastic interest rate

Assume that we have the dynamics on the previous slide. We then have that

\[ X(T) = X(t) e^{\int_t^T \frac{\tilde{\sigma}^2(u)}{2} \, du + \int_t^T \tilde{\sigma}(u) \, d\tilde{W}^Q(u)} = X(t) e^{-\frac{\Sigma_{t,T}^2}{2} + \Sigma_{t,T}G}, \]

where \( G \in N(0, 1) \) and \( \Sigma_{t,T}^2 = \int_t^T \tilde{\sigma}^2(u) \, du = \int_t^T |\sigma_1 - \nu(u, T)|^2 \, du. \)

With almost the same calculation (put \( r = 0 \) and replace \( \sigma \sqrt{T - t} \) by \( \Sigma_{t,T} \)) as in the derivation of the Black-Scholes formula we get

\[ p(t, T) \mathbb{E}^{Q^T} [(X(T) - K)^+ | X(t)] = p(t, T) (X(t) N(d_1) - KN(d_2)) \]

\[ = S(t) N(d_1) - p(t, T) KN(d_2), \]

where

\[ d_1 = \frac{\ln(S(t)/(K p(t, T))) + \Sigma_{t,T}^2/2}{\Sigma_{t,T}}, \quad d_2 = \frac{\ln(S(t)/(K p(t, T))) - \Sigma_{t,T}^2/2}{\Sigma_{t,T}}. \]
Preparation for the computer exercise (Heston model)

If we look at real stock prices we see that the volatility is not constant.

**Heston model, \(\mathbb{P}\)-dynamics:**

\[
\begin{align*}
\text{d}S_0(t) &= rS_0(t) \text{d}t, \\
\text{d}S_1(t) &= S_1(t)\mu \text{d}t + S_1(t)\sqrt{V(t)}(\rho \text{d}W_1^\mathbb{P}(t) + \sqrt{1 - \rho^2} \text{d}W_2^\mathbb{P}(t)), \\
\text{d}V(t) &= \kappa(\theta - V(t)) \text{d}t + \beta\sqrt{V(t)} \text{d}W_1^\mathbb{P}(t)
\end{align*}
\]

What about \(\mathbb{Q}\)-dyn?

\[
\mu - g_1(t)\rho\sqrt{V(t)} - g_2(t)\sqrt{1 - \rho^2}\sqrt{V(t)} = r \\
\kappa(\theta - V(t)) - g_1(t)\beta\sqrt{V(t)} = ?
\]

The problem is that volatility is not a traded asset! So we have no unique solution and thus the market is incomplete.
Possible $\mathbb{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathbb{Q} \theta^\mathbb{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathbb{Q} - \kappa \rho}{\mu - r} \frac{\rho}{\beta},$$

which gives the $\mathbb{Q}$-dyn

$$\begin{align*}
    dS_0(t) &= rS_0(t) \, dt, \\
    dS_1(t) &= S_1(t)r \, dt + S_1(t)\sqrt{V(t)}(\rho \, dW_1^\mathbb{Q}(t) + \sqrt{1 - \rho^2} \, dW_2^\mathbb{Q}(t)), \\
    dV(t) &= \kappa^\mathbb{Q}(\theta^\mathbb{Q} - V(t)) \, dt + \beta \sqrt{V(t)} \, dW_1^\mathbb{Q}(t)
\end{align*}$$
Solution for the Heston model?

We have that

\[ S(T) = S(t) e^{\int_t^T (r - \frac{Vu}{2}) \, du} + \int_t^T \sqrt{V(u)} (\rho \, dW_1^P(u) + \sqrt{1 - \rho^2} \, dW_2^P(u)). \]

The problem is that there is no closed form solution for \( V \).

Valuation are usually done by:

1. Fourier methods (Tuesday 13-15 in MH309A)
2. Monte Carlo methods (Thursday)
3. Numerical PDE methods (Outside the scope of this course)
Simulation of the Heston model

\[ S(0) = 100, \ \mu = 0.04, \ V(0) = 0.3, \ \kappa = 3, \ \theta = 0.3, \ \beta = 0.7, \ \rho = -0.6 \]