Beyond Black-Scholes

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FMSN25/MAFM24 Valuation of Derivative Assets

October 2, 2018
Stylized facts

- Non-normal daily log-returns
- Aggregational normality
- Long dependence of squared/absolute log-returns
- Heavy tailed log-returns
- Stochastic volatility
OMXS30-index
OMXS30-index (Daily log-returns)

\[ r_t = \log(S_t) - \log(S_{t-\delta}) \]
Are daily log-returns Gaussian?

![normplot log returns OMXS30](image_url)
What Do Real Option Prices Look like?

Optionprices 20110927 10:10:51
What Do Real Option Prices Look like?

Optionprices 20110927 12:55:00

Time to maturity

Strike

0 600 800 1000

0 0.2 0.4 0.6

0 50 100 150 200

0.8 0.6 0.4 0.2

0 200 150 100 50 0
What Do Real Option Prices Look like?

Optionprices 20110928 9:40:00

Graph showing the relationship between time to maturity, strike, and option prices.
What Do Real Option Prices Look like?
What Do Real Option Prices Look like?
Implied volatility

If the Black-Scholes model were true all we need to know is the volatility to price options.
If the Black-Scholes model was true the implied volatility would be constant!

![Graph showing implied volatility against time to maturity and moneyness. The graph illustrates the relationship between implied volatility and these two factors, highlighting the non-constant nature of implied volatility as per the Black-Scholes model.](image-url)
If the Black-Scholes model was true the implied volatility would be constant!

![Implied Volatility Graph](image-url)
If the Black-Scholes model was true the implied volatility would be constant!
If the Black-Scholes model was true the implied volatility would be constant!
How bad is the Black-Scholes fit?

Only 6.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 5.6% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 8.2% of the model prices are within the ASK-BID bounds!
How bad is the Black-Scholes fit?

Only 7.3% of the model prices are within the ASK-BID bounds!
What can we do about this?

We need more advanced models!!

- Stochastic volatility
- Stock models with jumps (Exponential Lévy processes)
- Stock models with jumps and stochastic volatility
- Local volatility models
- Markov switched models
How can we model volatility?

Continuous time stochastic volatility Heston

\[
\begin{align*}
    dV_t &= \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^{(1)} \\
    dS_t &= \mu S_t dt + S_t \sqrt{V_t} \left( \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(2)} \right)
\end{align*}
\]
Heston model is not complete

Equation for Q-dynamics:

\[ r = \mu - g_1(t) \sqrt{V(t)} \rho - g_2(t) \sqrt{V(t)} \sqrt{1 - \rho^2} \]

\[ \frac{\delta}{\delta} = \kappa(\theta - V(t)) - g_1(t) \sqrt{V(t)} \beta \]

How should volatility risk be priced?
No general criteria available since volatility is not explicitely traded.
What about VIX?
Possible $\mathcal{Q}$-dynamics

We can choose $g_1$ and $g_2$ as

$$g_1(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{\Xi(t)}{\rho}, \quad g_2(t) = \frac{\mu - r}{\sqrt{V(t)}} \frac{1 - \Xi(t)}{\sqrt{1 - \rho^2}},$$

$\Xi$ is a “free” parameter. A choice of the form $\Xi(t) = a + bV(t)$ give us nice properties. So e.g. $a = b = 0 \Rightarrow \Xi(t) = 0$ leaves the $V$ dynamics unchanged, i.e. volatility risk is not priced by the market. Another choice is

$$a = \frac{\kappa \theta - \kappa^\mathcal{Q} \theta^\mathcal{Q}}{\mu - r} \frac{\rho}{\beta}, \quad b = \frac{\kappa^\mathcal{Q} - \kappa \rho}{\mu - r} \frac{\rho}{\beta},$$

which gives the $\mathcal{Q}$-dyn

$$dS_0(t) = rS_0(t)dt,$$
$$dS_1(t) = S_1(t)rdt + S_1(t)\sqrt{V(t)}(\rho dW_1^\mathcal{Q}(t) + \sqrt{1 - \rho^2}dW_2^\mathcal{Q}(t)),$$
$$dV(t) = \kappa^\mathcal{Q}(\theta^\mathcal{Q} - V(t))dt + \sigma_V \sqrt{V(t)}dW_1^\mathcal{Q}(t).$$
OMXS30 Heston-volatility - Estimated from option prices
Lévy processes

A process $X$ with following properties is called a Lévy process

- $X_0 = 0$
- Independent increments $X_{t+s} - X_t$ independent of $X_t$ for all $s > 0$ for all $t > 0$
- Stationary increments $X_{s+t} - X_t \overset{d}{=} X_s$ for all $s > 0$ for all $t > 0$
Examples of Lévy processes

- Wiener process
- Poisson
- Compound Poisson
- Merton process = Compound Poisson with Gaussian increments plus a Wiener process with drift [Merton, 1976]
- Gamma process
- Normal Inverse Gaussian (NIG) process [Barndorff-Nielsen, 1997]
- Variance Gamma (VG) process [Madan and Seneta, 1990]
- Carr Geman Madan Yor (CGMY) process [Carr et al., 2002]
- Finite Moment Log Stable (FMLS) process (crash model) [Carr and Wu, 2003]
General Lévy processes

A general Lévy process can be written as

\[ X(t) = \mu t + \sigma W(t) + Z(t) \]

Linear drift \( \mu t \),
Brownian motion with variance \( \sigma^2 \): \( \sigma W(t) \).
Pure jump process \( Z(t) \)
The characteristic function of any one-dimensional Lévy process can be written as

\[ \phi(y, t) = \mathbb{E}[\exp(iyX(t))] = \exp(tK(y)), \]

where

\[ K(y) = i\mu y + (iy)^2\sigma^2/2 + K_z(y) \]

with

\[ K_z(y) = i\gamma y + \int_{\mathbb{R}} (e^{iyx} - 1 - iyxI(|x| < 1))\nu(dx), \]

\( \nu \) is called the Lévy measure.
Lévy measures

Interpretation
The number
\[ \int_{a}^{b} \nu(dx), \]
equals the average number of jumps with sizes between \( a \) and \( b \) per time unit.

General restriction on \( \nu \)
\[ \int_{\mathbb{R}} \min(x^2, 1) \nu(dx) < \infty \]
This is equivalent to that all Lévy processes has finite quadratic variation.
Expectation and variance

**Expectation**

\[ \mathbb{E}[X(t)] = tK'(0)/i = t \left( \mu + \gamma + \int_{|x|>1} x \nu(dx) \right) \]

**Variance**

\[ \mathbb{V}[X(t)] = -tK''(0) = t \left( \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx) \right) \]

But note that neither the variance nor the expectation needs to be finite!

**Moment relations**

The expectation \( \mathbb{E}[|g(X(t))|] \) is finite for all \( t > 0 \) if

\[ \int_{|x|>1} |g(x)| \nu(dx) < \infty, \]

provided that \( |g(x + y)| \leq c |g(x)g(y)| \) for some \( c > 0 \) \( \forall x, y \in \mathbb{R} \).
Exponentially affine stock price models under $\mathbb{Q}$

A stock price model is called exponentially affine if [Duffie et al., 2000]

$$
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy r(T - t) + A(t, T, iy) + B(t, T, iy)V(t)) ,
$$

where $A$ and $B$ does not depend on $S$ (or $V$). Note that $B$ is related to stochastic volatility and is set to zero for models with out stochastic volatility. Almost all recent stock price models fall into this class.

**Examples:** Black-Scholes, Heston, Bates, Merton, VG, CGMY, NIG and NIG-CIR etc ...

**Not in the class:** Constant elasticity of Variance (CEV), Stochastic alpha-beta-rho (SABR) and Local volatility models.
Condition for the discounted price process to be a \( \mathbb{Q} \)-martingale

The discounted price process is a martingale if

\[
\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}S(T) | \mathcal{F}_t] = S(t).
\]

This is true if \( A(t, T, 1) = 0 \) and \( B(t, T, 1) = 0 \).
The Merton model [Merton, 1976]

\[
\frac{dS_t}{S_t} = r S_t dt + \sigma S_t dW_t + S_t (e^{J_t} - 1) dN_t - S_t \lambda (e^{\mu_j + \sigma_j^2/2} - 1) dt
\]

where \( J_t \in N(\mu_j, \sigma_j^2) \), \( N \) is a Poisson process with intensity \( \lambda \).

\[
\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy (T - t) + A(t, T, iy))
\]

\[
A(t, T, iy) = (T - t)((-\sigma^2/2)i y + (iy)^2 \sigma^2/2 + \lambda \left((e^{iy \mu_j + (iy)^2 \sigma_j^2/2} - 1) - iy(e^{\mu_j + \sigma_j^2/2} - 1)\right)
\]

Note that \( S_{t-} = \lim_{s \uparrow t} S_s \).
How bad is the Merton fit?

Only 8.4% of the model prices are within the ASK-BID bounds!
The Heston model [Heston, 1993]

\[
\begin{align*}
    dV_t &= \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t}dW_t^{(1)} \\
    dS_t &= \mu S_t dt + S_t \sqrt{V_t}(\rho dW_t^{(1)} + \sqrt{1 - \rho^2}dW_t^{(2)})
\end{align*}
\]

\[
\mathbb{E}[e^{iy\ln(S(T))}|S(t)] = \exp(iy\ln(S(t)) + iy\rho(T-t) + A(t,T,iy) + B(t,T,iy)V(t))
\]

\[
A(t,T,iy) = \frac{\kappa \theta}{\sigma_v^2} \left((\kappa - \rho \sigma_v iy - d)(T-t) - 2 \log((\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1))/(2d))\right),
\]

\[
B(t,T,iy) = (1 - e^{-d(T-t)}) \frac{(iy)^2 - iy}{(\kappa - \rho \sigma_v iy)(1 - e^{-d(T-t)}) + d(e^{-d(T-t)} + 1)},
\]

\[
d = \sqrt{(\rho \sigma_v iy - \kappa)^2 + \sigma_v^2(iy + y^2)}.
\]
How good is the Heston fit?

About 72% of the model prices are within the ASK-BID bounds!
The Bates model $\approx$ Heston+Merton [Bates, 1996]

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t + S_t (e^{J_t} - 1) dN_t - S_t \lambda (e^{\mu J_t + \sigma^2 J_t / 2} - 1) dt$$

where $J_t \in \text{Norm}(\mu_J, \sigma^2_J)$, $N$ is Poisson a process with intensity $\lambda$ and $V$ is as in Heston.

$$\mathbb{E}[e^{iy \ln(S(T))} | S(t)] = \exp(iy \ln(S(t)) + iy r(T - t) + A(t, T, iy) + B(t, T, iy) V(t))$$

$$A(t, T, iy) = A_{Merton}(t, T, iy)|_{\sigma=0} + A_{Heston}(t, T, iy)$$

$$B(t, Y, iy) = B_{Merton}(t, T, iy) + B_{Heston}(t, T, iy)$$

$$= B_{Heston}(t, T, iy)$$
How good is the Bates fit?

About 80% of the model prices are within the ASK-BID bounds!
The Normal Inverse Gaussian (NIG) model [Barndorff-Nielsen, 1997]

\[ S_t = S_0 \exp(rt + X(t)), \]

where \( X(t) \) is NIG Lévy process.

\[
\mathbb{E}[e^{iy\ln(S(T))}|S(t)] = \exp(iy\ln(S(t)) + iy(T-t) + A(t,T,iy))
\]

\[
A(t,T,iy) = (T-t)\delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2} \right) - iy(T-t)\delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2} \right),
\]

where \( \alpha > |\beta + 1|, \delta > 0. \)
The NIGCIR model [Carr et al., 2003b]

This is a stochastic volatility (stochastic time change) model with jumps

\[
S_t = S_0 \exp(rt + X(I_t))
\]

\[
I_t = \int_0^t V_s ds
\]

where \(X\) is a NIG Lévy process, and \(V\) is as in Heston.

\[
A(t, T, iy) = A_{ICIR}(t, T, A_{NIG}(0, 1, iy)),
\]

\[
B(t, T, iy) = B_{ICIR}(t, T, A_{NIG}(0, 1, iy)),
\]

where \(\mathbb{E}[\exp(z \int_t^T V_s ds) | \mathcal{F}_t] = \exp(A_{ICIR}(t, T, z) + B_{ICIR}(t, T, z) V(t)).\) with

\[
A_{ICIR}(t, T, z) = A_{Heston}(t, T, iy) |(iy+y^2) = -2z, \rho = 0,\]

\[
B_{ICIR}(t, T, z) = B_{Heston}(t, T, iy) |(iy+y^2) = -2z, \rho = 0.\]
Fourier methods for pricing exponentially affine models

Let $\bar{s}_T = \ln(S(T))$, $\bar{k} = \ln(K)$.

- The Fourier transform for the pay-off a European call.

  $\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{s}_T} - e^{\bar{k}}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}$, if $\text{Re}z > 0$.

- The Fourier transform for the pay-off a European put.

  $\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}$, if $\text{Re}z < -1$.

- The Fourier transform for $-\min(S(T), K)$.

  $-\int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z+1)}$, if $-1 < \text{Re}z < 0$. 
Fourier methods for pricing exponentially affine models

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- The Fourier transform for the pay-off a European call.

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- The Fourier transform for the pay-off a European put.

$$\int_{\mathbb{R}} e^{z\bar{k}} \max(e^{\bar{k}} - e^{\bar{s}_T}, 0) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } \text{Re}z < -1.$$

- The Fourier transform for $-\min(S(T), K)$.

$$- \int_{\mathbb{R}} e^{z\bar{k}} \min(e^{\bar{k}}, e^{\bar{s}_T}) d\bar{k} = \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)}, \text{ if } -1 < \text{Re}z < 0.$$
Fourier methods for pricing exponentially affine models

Let $\tilde{s}_T = \ln(S(T))$, $\tilde{k} = \ln(K)$.

- The Fourier transform for the pay-off of a European call.

$$\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{s}_T} - e^{\tilde{k}}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } \Re z > 0.$$  

- The Fourier transform for the pay-off of a European put.

$$\int_{\mathbb{R}} e^{z\tilde{k}} \max(e^{\tilde{k}} - e^{\tilde{s}_T}, 0) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } \Re z < -1.$$  

- The Fourier transform for $-\min(S(T), K)$.

$$-\int_{\mathbb{R}} e^{z\tilde{k}} \min(e^{\tilde{k}}, e^{\tilde{s}_T}) d\tilde{k} = \frac{e^{(z+1)\tilde{s}_T}}{z(z+1)}, \text{ if } -1 < \Re z < 0.$$
Inverse Fourier transform

Now we have that

\[ \max(S_T - K, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)} \bigg|_{z = \bar{z} + iw} \, dw, \quad \bar{z} > 0 \]

Thus we can write

\[
\Pi(t) = \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \max(S_T - K, 0) | S_t \right]
\]

\[
= \mathbb{E}^Q \left[ e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{e^{(z+1)\bar{s}_T}}{z(z + 1)} \bigg|_{z = \bar{z} + iw} \, dw | S_t \right]
\]

\[
= e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \mathbb{E}^Q \left[ e^{(z+1)\bar{s}_T} | S_t \right] \bigg|_{z = \bar{z} + iw} \, dw
\]
Use that $S_T$ comes from an exponentially affine model

Then

$$\mathbb{E}^Q\left[e^{(z+1)\bar{S}_T}|S_t\right] = e^{r(t,T)(t-T)(z+1)+(z+1)\ln(S_t)+A(t,T,z+1)+B(t,T,z+1)V_t}$$

$$= g(t,T,z+1)$$

So that

$$\Pi(t) = e^{-r(t,T)(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z\bar{k}} \frac{g(t,T,z+1)}{z(z+1)} \left|_{z=\bar{z}+iw} \right. d\omega$$

$$= e^{-r(t,T)(T-t)} \frac{1}{\pi} \int_0^{\infty} \text{Re} \left( e^{-z\bar{k}} \frac{g(t,T,z+1)}{z(z+1)} \left|_{z=\bar{z}+iw} \right. \right) d\omega$$
Calculation of the inverse Fourier transform

We can use the fast Fourier transform.

We can use quadrature methods.

\[ \Pi(t) \approx \sum_{j=1}^{N} w_j^{(N)} e^{x_j^{(N)}} e^{-r(t,T)(T-t)} \frac{1}{\pi} \text{Re} \left( e^{-z \bar{k}} \frac{g(t, T, z + 1)}{z(z + 1)} \bigg|_{z=\bar{z} + ix_j^{(N)}} \right), \]

where \( x_j^{(N)}, w_j^{(N)} \) are weights coming from the Gauss-Laguerre quadrature method.
References


Ito’s formula for Lévy processes

\[
df(X(t)) = f'(X(t)) \mu dt + f''(X(t)) \sigma^2 / 2 dt + \sigma f'(X(t)) dW(t) \\
+ f'(X(t-)) dZ(t) \\
+ (f(X(t-)) + \Delta Z(t)) - f(X(t-)) - f'(X(t-)) \Delta Z(t),
\]

where $\Delta Z(t)$ is the jump in $Z$. ▶ Back
Origin of NIG

The original NIG distribution depend on four parameters \((\alpha, \beta, \delta, \mu)\) and it is related to two independent Brownian motions \(W_1\) and \(W_2\). Let \(W_1\) be a Brownian motion starting at \(\mu\) with drift \(\beta\) and let \(W_2\) be a Brownian motion starting at 0 with drift \(\sqrt{\alpha^2 - \beta^2}\). Let \(\tau_\delta = \inf\{s > 0 : W_2(s) > \delta\}\). Now \(X = W_1(\tau_\delta)\) has a NIG distribution with parameters \((\alpha, \beta, \delta, \mu)\) and

\[
\mathbb{E}[e^{iyX}] = \exp(iy\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iy)^2}))
\]

In order to get the right model for stocks we should choose \(\mu = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2})\).