1. Your bank offers you a derivative called a long butterfly spread. The derivative has maturity $T$ and the following pay-off function:

\[
\begin{align*}
0 & \quad S(T) \leq K_1, \\
S(T) - K_1 & \quad K_1 \leq S(T) \leq K_2, \\
K_3 - S(T) & \quad K_2 \leq S(T) \leq K_3, \\
0 & \quad K_3 \leq S(T),
\end{align*}
\]

where $K_3 > K_2 > K_1 > 0$ and $K_2 - K_1 = K_3 - K_2$. Show how you can replicate this pay-off by using standard contracts. You are allowed to use the stock $S$, European put and call options, and a zero-coupon bond each with maturity $T$. Express the price for $0 < t < T$ of the long butterfly spread using a linear combination of the prices of standard contracts.

2. Let the processes $X_t$ and $Y_t$ be given by the following SDE:s:

\[
\begin{align*}
\quad & dX_t = \mu_X t \, dt + \sigma_X t \, dW_t \\
\quad & dY_t = \alpha_Y t \, dt + \beta_Y t \, dW_t
\end{align*}
\]

where $W$ is a standard Brownian motion, $\alpha$, $\beta$, $\mu$ and $\sigma$ are real-valued constants.

(a) Find the dynamics for 

\[ Z_t = \frac{X_t}{Y_t}. \] 

(b) Find the relation $\alpha$, $\beta$, $\mu$ and $\sigma$ must satisfy for $Z_t$ to be a Martingale.

3. Assume that $S_t$ for $t \geq 0$ follows the standard Black-Scholes model. Find the fair value at time $t$ with $0 < t < T_1$ for the derivative $X$ which pays

\[ \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u \, du \]

at time $T_2$, where $T_2 > T_1$. 

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4. Solve the PDE

$$\frac{\partial f(t, x)}{\partial t} + \mu \frac{\partial f(t, x)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2} = 0$$

$$f(T, x) = \begin{cases} 
M & a < x \leq b, \\
0 & \text{otherwise},
\end{cases}$$

for \( t \in [0, T], \ x \in \mathbb{R} \) using the Feynman-Kac representation, where \( M > 0, \ a < b, \ \mu \) and \( \sigma \) are real-valued constants.

5. Consider a derivative with maturity \( T \) and pay-off \( \Phi(S_T) = \max(K - S_T, 0)^2 \) in the Black-Scholes market.

   (a) Find the price-formula for the derivative (European power-put) for \( 0 \leq t \leq T \). (12p)

   (b) Calculate the replicating portfolio (delta-hedge) for the derivative, that is find a self-financing portfolio consisting of the stock and the bank account that hedges the derivative. (8p)

6. Consider the LIBOR market on the tenor \([T_1, T_2, T_3]\), where \( T_1 < T_2 < T_3 \). You want to invest in the LIBOR market at time \( T_1 \), but you are not sure whether it is best to fix the interest rate over both the periods \([T_1, T_2]\) and \([T_2, T_3]\) at \( T_1 \) or to fix \([T_1, T_2]\) at \( T_1 \) and then wait until \( T_2 \) to fix the period \([T_2, T_3]\). The reason for this uncertainty is that you do not know whether the LIBOR rate will go up or down if you wait until \( T_2 \) to fix the second period. Define \( \tau_1 = (T_2 - T_1) \) and \( \tau_2 = (T_3 - T_2) \). In the first case we get a pay-off \((1 + \tau_1 L_{T_1}[T_1, T_2])(1 + \tau_2 L_{T_2}[T_2, T_3])\) at time \( T_3 \) and in the second we get a pay-off \((1 + \tau_1 L_{T_1}[T_1, T_2])(1 + \tau_2 L_{T_2}[T_2, T_3])\) at time \( T_3 \). To protect yourself against taking the wrong decision you want to buy a derivative which covers this risk. So you buy a derivative with pay-off

\[
((1 + \tau_1 L_{T_1}[T_1, T_2])(1 + \tau_2 L_{T_2}[T_2, T_3]) - (1 + \tau_1 L_{T_1}[T_1, T_2])(1 + \tau_2 L_{T_1}[T_2, T_3]))^+.
\]

This is equivalent to the product of a spot LIBOR contract on \([T_1, T_2]\) and an at the money caplet on \([T_2, T_3]\). Use that for general \( 0 \leq t \leq S \leq T \) that

\[
(1 + (T - S) L_t[S, T]) = \frac{p(t, S)}{p(t, T)} = e^\int_S^T f(t, u) du,
\]

where \( p(t, S), p(t, T) \) are ZCB:s with maturity \( S \) and \( T \) respectively and \( f(t, u) \) is the forward rate. Assume that we have the following Gaussian HJM dynamics for the forward rate under the \( p(t, T) \) numeraire measure \( \mathbb{Q}^{T_3} \), for \( 0 \leq s \leq u \leq T_3 \)

\[
df(s, u) = \frac{1}{2} \left( \int_u^{T_3} \sigma(s, x) dx \right)^2 ds + \sigma(s, u) dW^{\mathbb{Q}^{T_3}}(s)
\]

where \( \sigma \) is a deterministic function and where \( W^{\mathbb{Q}^{T_3}} \) is a standard \( \mathbb{Q}^{T_3} \) Brownian motion.

(a) Show that you can rewrite the pay-off in (1) as (5p)

\[
\frac{p(T_1, T_1)}{p(T_1, T_2)} \left( \frac{p(T_2, T_2)}{p(T_2, T_3)} - \frac{p(T_1, T_2)}{p(T_1, T_3)} \right)^+.
\]

(b) Find the fair value of the derivative defined in (1) at time \( T_1 \) using the dynamics in (2).(15p)

**Good luck!**