Solution.

1. We can start to look at the interval \((0 \leq S_T \leq K_1)\). Here we do nothing since the pay-off is zero. Moving on to the second interval \((K_1 \leq S_T \leq K_2)\) we see that we can add a European call option with strike \(K_1\) giving \((S_T - K_1)^+ = S_T - K_1\). This does not change the pay-off in the first interval. Moving on to the third interval \((K_2 \leq S_T \leq K_3)\) we see that if we also subtract two European call options with strike \(K_2\) we get \((S_T - K_1)^+ - 2(S_T - K_2)^+ = (S_T - K_1) - 2(S_T - K_2) = 2K_2 - K_1 - S_T = K_3 - S_T\). This leaves the pay-off unchanged in the first two intervals. In the final interval \((K_3 \leq S_T)\) we add a European call option with strike \(K_3\), which gives \((S_T - K_1)^+ - 2(S_T - K_2)^+ + (S_T - K_3)^+ = (S_T - K_1) - 2(S_T - K_2) + (S_T - K_3) = 2K_2 - K_1 - K_3 + S_T - 2S_T + S_T = 2K_2 - K_1 - K_3 = 0\). This again does not change the pay-off in the previous intervals. So let \(\Pi_C(t, H, T)\) be the price at time \(t\) of a European call with strike \(H\) and maturity \(T\). So the price of the butterfly spread at time \(t\), \(\Pi(t)\), is given by

\[
\Pi(t) = \Pi_C^E(t, K_1, T) - 2\Pi_C^E(t, K_2, T) + \Pi_C^E(t, K_3, T).
\]

To see this assume that the price of \(X\) and the static replication differs and some time \(s\) say. Sell the most expensive of the two and buy the cheapest put the rest of the money into the bank account. At maturity the pay-off of the butterfly spread and its replication cancels but we still have money in the bank and thus we have constructed an arbitrage opportunity. Therefore the price of the static replication and the butterfly spread must coincide for all \(0 \leq t \leq T\).

Alternative replication: Using the put call parity on all the European call options we obtain that the price of the butterfly spread can alternatively be written as

\[
\Pi(t) = \Pi_P^E(t, K_1, T) - 2\Pi_P^E(t, K_2, T) + \Pi_P^E(t, K_3, T),
\]

where \(\Pi_P^E(t, H, T)\) is the price at time \(t\) of a European put with strike \(H\) and maturity \(T\).
2. (a) By applying the Ito formula to $Z_t = X_t / Y_t$ we obtain

$$dZ_t = \frac{dX_t}{Y_t} + \frac{0}{2Y_t^2} = \frac{dX_t}{Y_t} + \frac{-X_t dY_t}{Y_t^2} - \frac{2X_t (dY_t)^2}{2Y_t^3}$$

\begin{align*}
&= Z_t \mu dt + Z_t \sigma dW_t - Z_t \alpha dt - Z_t \beta dW_t - Z_t \sigma \beta dt + Z_t \beta^2 dt \\
&= Z_t (\mu - \alpha - \sigma \beta + \beta^2) dt + Z_t (\sigma - \beta) dW_t
\end{align*}

**Alternative approach:** Using that a general GBM $S$ with drift $a$ and volatility $b$ has a solution of the form (for all $0 \leq t_1 \leq t_2$)

$$S(t_2) = S(t_1) e^{(a-b^2/2)(t_1-t_2) + b(W_{t_2} - W_{t_1})},$$

we obtain

$$Z_t = \frac{X_t}{Y_t} = \frac{X_0 e^{(\mu-\sigma \beta/2) t + \sigma W_t - (\alpha-\beta^2/2) t - \beta W_t}}{Y_0}
= \frac{X_0 e^{(\mu-\sigma \beta/2) t + \sigma W_t}}{Y_0}
= \frac{X_0 e^{(\mu-\sigma \beta/2 - (\sigma-\beta)^2/2) t + (\sigma-\beta) W_t}}{Y_0}.
$$

So $Z$ is a GBM with drift $\mu - \alpha - \sigma \beta + \beta^2$ and volatility $\sigma - \beta$.

(b) A geometric Brownian motion is a Martingale if and only if its drift is zero. This gives that the following relation should hold

$$\mu - \alpha - \sigma \beta + \beta^2 = 0.$$ 

This is all takes to solve the problem in this general setting without any strong relation to finance.

**Extra Material outside original problem scope:** If we want to have a more financial touch to the problem, we can think of $X$ and $Y$ as two traded assets with $\mathbb{P}$-dynamics as in the problem. The condition for this market to be free of arbitrage is that

$$\mu = r + \lambda(t) \sigma, \quad \alpha = r + \lambda(t) \beta,$$

where $\lambda(t)$ is the market price of risk per unit volatility. Rephrasing the Martingale condition in this setting we obtain

$$\mu - \alpha - \sigma \beta + \beta^2 = \lambda(t)(\sigma - \beta) - \beta(\sigma - \beta) = (\lambda(t) - \beta)(\sigma - \beta).$$

So then $Z$ is a Martingale if and only if $\lambda(t) \equiv \beta$. This would correspond to that the $\mathbb{P}$-dynamics is the same as the $Y$-numeraire measure dynamics.

3. We have a contract with pay-off

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u \, du$$
at time $T_2$. According to the risk neutral valuation formula we have that $\Pi(t)$, the price at time $t$ is given by

$$
\Pi(t) = \mathbb{E}^Q \left[ \frac{B_t}{B_{T_2}} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u \, du | \mathcal{F}_t \right]
$$

$$
= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^Q \left[ \frac{B_t}{B_{T_2}} S_u | \mathcal{F}_t \right] \, du
$$

$$
= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \frac{B_u}{B_{T_2}} \mathbb{E}^Q \left[ \frac{B_t}{B_u} S_u | \mathcal{F}_t \right] \, du
$$

$$
= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} B_u S_u \, du
$$

$$
= \frac{S_t}{T_2 - T_1} \int_{T_1}^{T_2} e^{-r(T_2-u)} \, du
$$

$$
= \frac{S_t}{T_2 - T_1} \left[ \frac{e^{-r(T_2-u)}}{r} \right]_{T_1}^{T_2}
$$

$$
= \frac{S_t}{T_2 - T_1} \frac{1 - e^{-r(T_2-T_1)}}{r}
$$

$$
= \frac{S_t}{r(T_2 - T_1)} (1 - e^{-r(T_2-T_1)}).
$$

If we let $T_1$ approach $T_2$ from below we get that $\Pi(t)$ approaches $S_t$ which seems reasonable since then the pay-off approaches $S_{T_2}$. ■

4. Using Feynman-Kačs representation formula we obtain

$$
f(t, x) = \mathbb{E}[MI(a < X_T \leq b)|X_t = x]
$$

where $X$ has the following dynamics for $t \leq u \leq T$

$$
dX_u = \mu \, du + \sigma \, dW_u, \ X_t = x.
$$

We see that

$$
X_T = x + \mu(T-t) + \sigma(W_T - W_t) \overset{d}{=} x + \mu(T-t) + \sigma\sqrt{T-t}G,
$$

where $G$ is standard Gaussian random variable. We thus obtain that

$$
f(t, x) = \int_{-\infty}^{\infty} MI(a < x + \mu(T-t) + \sigma\sqrt{T-t}y \leq b) \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy
$$

$$
= \int_{d_2}^{d_1} M \frac{e^{-y^2}}{\sqrt{2\pi}} \, dy
$$

$$
= M(N(d_2) - N(d_1)),
$$

where $N$ is the distribution function of a standard normal and

$$
d_1 = \frac{a - x - \mu(T-t)}{\sigma\sqrt{T-t}},
$$

$$
d_2 = \frac{a - x - \mu(T-t) - \sigma\sqrt{T-t}b}{\sigma\sqrt{T-t}}.
$$
\[ d_2 = \frac{b - x - \mu(T - t)}{\sigma \sqrt{T - t}}. \]

So we have that \( f(t, x) = M(N(d_2) - N(d_1)). \)

**Extra material for checking the solution:**

We start by checking the boundary condition. Now \( f(T, x) = E[MI(a < X_T \leq b)|X_T = x] = MI(a < x \leq b). \) Note however that since \( f(T, x) \) is discontinuous in \( x \) at \( x = a \) and \( x = b \) we will in general have \( \lim_{t \uparrow T} f(t, b) \neq f(T, b) \) and \( \lim_{t \uparrow T} f(t, a) \neq f(T, a) \) which can be seen from the calculations below.

So use that \( c \) can be either \( a \) or \( b \). For \( x < c \) we have that
\[
\lim_{t \uparrow T} \frac{c - x - \mu(T - t)}{\sigma \sqrt{T - t}} = \infty,
\]
for \( c < x \) we have that
\[
\lim_{t \uparrow T} \frac{c - x - \mu(T - t)}{\sigma \sqrt{T - t}} = -\infty,
\]
For \( x = c \) (which happens with probability zero seen from time \( t < T \)) we have
\[
\lim_{t \uparrow T} \frac{-\mu(T - t)}{\sigma \sqrt{T - t}} = \lim_{t \uparrow T} \frac{-\mu \sqrt{T - t}}{\sigma} = 0.
\]
Note that \( N(\infty) = 1, \ N(-\infty) = 0 \) and \( N(0) = 1/2 \). This gives that \( \lim_{t \uparrow T} f(t, x) = MI(a < x < b) + M/2(I(x = b) + I(x = a)) \). We can in principle modify the final condition so that \( f(T, x) = MI(a < x < b) + M/2(I(x = b) + I(x = a)) \) and since the points \( X_T = a \) and \( X_T = b \) has probability zero given \( \mathcal{F}_t \) this will not effect \( f(t, x) \) for \( t < T \).

To check that the solution satisfies the PDE we start by calculating the partial derivatives. Note that \( N'(z) = n(z) \) and that \( N''(z) = n'(z) = -zn(z). \)

\[
\begin{align*}
\frac{\partial}{\partial t} f(t, x) &= Mn(d_2) \left( \frac{d_2}{2(T-t)} + \frac{\mu}{\sigma \sqrt{T-t}} \right) - Mn(d_1) \left( \frac{d_1}{2(T-t)} + \frac{\mu}{\sigma \sqrt{T-t}} \right) \\
\mu \frac{\partial}{\partial x} f(t, x) &= -\mu Mn(d_2) - n(d_1) \\
\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} f(t, x) &= \frac{\sigma^2}{2} Mn(d_2) - n(d_1) + n(d_1) \\
&= M\frac{n(d_2) - n(d_1)}{2(T-t)} \\
&= M\frac{n(d_2) - n(d_1)}{2(T-t)}
\end{align*}
\]

Putting all this together we get that
\[
\begin{align*}
\frac{\partial}{\partial t} f(t, x) + \mu \frac{\partial}{\partial x} f(t, x) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} f(t, x) \\
&= Mn(d_2) \left( \frac{d_2 - d_2}{2(T-t)} + \frac{\mu - \mu}{\sigma \sqrt{T-t}} \right) \\
&- Mn(d_1) \left( \frac{d_1 - d_1}{2(T-t)} + \frac{\mu - \mu}{\sigma \sqrt{T-t}} \right) \\
&= 0.
\end{align*}
\]

Thus we have that \( f(t, x) = M(N(d_2) - N(d_1)) \) solves the PDE. ■
5. (a) The risk neutral valuation formula states that the value of a simple claim with maturity \( T \) and pay-off \( \Phi(S_T) \) at time \( t \) is given as

\[
\Pi(t) = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} \Phi(S_T) | \mathcal{F}_t \right]
\]

Applying this to the European power put, i.e. derivative with pay off \( \max(K - S_T, 0)^2 \) and maturity \( T \), using that the conditional distribution of \( S_T \) given \( S_t = s \) is given by

\[
S_T | \{ S_t = s \} \overset{d}{=} se^{(r - \frac{\sigma^2}{2}) T + \sigma \sqrt{T} G},
\]

for a standard GBM where \( G \) is standard Gaussian r.v. and \( \tau = T - t \), we obtain

\[
\Pi(t) = e^{-rt} \mathbb{E}^Q \left[ \max \left( K - se^{(r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} G}, 0 \right)^2 \right]
\]

\[
= e^{-rt} \int_{-\infty}^{\infty} \max \left( K - se^{(r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} y}, 0 \right)^2 \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy
\]

\[
= e^{-rt} \int_{-\infty}^{d} \left( K - se^{(r - \frac{\sigma^2}{2}) \tau + \sigma \sqrt{\tau} y} \right)^2 \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy
\]

\[
= e^{-rt} K^2 \int_{-\infty}^{d} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy - e^{-rt} 2Ks \int_{-\infty}^{d} e^{\frac{y^2 - 2\sigma \sqrt{\tau} y + \sigma^2 \tau}{2}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy
\]

\[
+ e^{-rt} s^2 \int_{-\infty}^{d} e^{2\sigma \tau + \sigma^2 y} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy
\]

\[
= e^{-rt} K^2 N(d) - 2Ks N(d - \sigma \sqrt{\tau}) + e^{(r+\sigma^2)\tau} s^2 N(d - 2\sigma \sqrt{\tau}),
\]

where \( N \) is the distribution function for a standard Gaussian r.v. and

\[
d = \frac{\ln \left( \frac{K}{s} \right) - \left( r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.
\]

(b) We use the standard delta-hedge to obtain the replicating portfolio. We thus get

\[
h_S(t) = \frac{\partial}{\partial s} \left( e^{-rt} K^2 N(d) - 2Ks N(d - \sigma \sqrt{\tau}) + e^{(r+\sigma^2)\tau} s^2 N(d - 2\sigma \sqrt{\tau}) \right).
\]
We now look at Eq. (*) and realise that the integrand is zero at the point \( y = d \) so the the term coming from the \( s \)-dependence of \( d \) vanishes. This is due to the fact that the pay-off is zero at the point \( S_T = K \). Using this we obtain

\[
\begin{align*}
\frac{\partial}{\partial s} \left( e^{-rT} K^2 N(d) - 2K s N(d - \sigma \sqrt{T}) + e^{(r+\sigma^2)T} s^2 N(d - 2\sigma \sqrt{T}) \right) \\
= -2K N(d - \sigma \sqrt{T}) + 2e^{(r+\sigma^2)T} s N(d - 2\sigma \sqrt{T}),
\end{align*}
\]

\[
\begin{align*}
B_i \\
= e^{-rT} K^2 N(d) - e^{r(T-2t)+\sigma^2T} s^2 N(d - 2\sigma \sqrt{T}).
\end{align*}
\]

(b) In order to find the fair value of the derivative we first rewrite Eq. (**)

\[
\left( \frac{p(t, S)}{p(t, T)} \right) (1 + \tau_1 L_{T_1}[T_1, T_2]) (1 + \tau_2 L_{T_2}[T_2, T_3]) - (1 + \tau_1 L_{T_1}[T_1, T_2]) (1 + \tau_2 L_{T_1}[T_2, T_3])^{+} \\
= \frac{p(T_1, T_2)}{p(T_1, T_3)} \left( \frac{p(T_2, T_2)}{p(T_2, T_3)} - \frac{p(T_1, T_2)}{p(T_1, T_3)} \right)^{+},
\]

which was to shown.

(b) In order to find the fair value of the derivative we first rewrite Eq. (**)
where

Pi(\(T_1\)) = \(\mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T,u) du - f(T_1,u) du} - 1 \right)^+ | \mathcal{F}_{T_1} \right] \)

= \(\mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T,u) du} - f(T_1,u) du \right) + | \mathcal{F}_{T_1} \right] \)

= \(\mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T_1,u) du} - 1 \right)^+ | \mathcal{F}_{T_1} \right] \)

= \(\mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T_1,u) du} - 1 \right)^+ | \mathcal{F}_{T_1} \right] \).

We now use that the forward rate \(f(s,u)\) has the following \(Q_{T_3}\) dynamics

\[ df(s,u) = \frac{1}{2} \frac{\partial}{\partial u} \left( \int_u^{T_3} \sigma(s,x) dx \right)^2 ds + \sigma(s,u) dW^{Q_{T_3}}(s). \]

Plugging this in we obtain

\[ \Pi(T_1) = \mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T_1,u) du} - 1 \right)^+ | \mathcal{F}_{T_1} \right] \]

\[ = \mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T_1,u) du} - 1 \right)^+ | \mathcal{F}_{T_1} \right] \]

\[ = \mathbb{E}^{Q_{T_3}} \left[ \left( e^{\int_{T_2}^{T_3} f(T_1,u) du} - 1 \right)^+ | \mathcal{F}_{T_1} \right] \]

\[ = \mathbb{E}^{Q_{T_3}} \left[ \left( e^{-\frac{\sigma^2}{2} + \sqrt{\sigma^2 G} - 1} \right)^+ | \mathcal{F}_{T_1} \right], \]

where \(G\) is a standard Gaussian r.v. and where

\[ V = \int_{T_1}^{T_2} \left( \int_{T_2}^{T_3} \sigma(s,u) du \right)^2 ds. \]

We finally obtain

\[ \Pi(T_1) = \mathbb{E}^{Q_{T_3}} \left[ \left( e^{-\frac{\sigma^2}{2} + \sqrt{\sigma^2 G} - 1} \right)^+ | \mathcal{F}_{T_1} \right] \]

\[ = \int_{-\infty}^{\infty} \left( e^{-\frac{\sigma^2}{2} + \sqrt{\sigma^2 y} - 1} \right)^+ \frac{e^{-\frac{y^2}{2\pi}}}{\sqrt{2\pi}} dy \]

\[ = \int_{\sqrt{\sigma^2}}^{\infty} \left( e^{-\frac{\sigma^2}{2} + \sqrt{\sigma^2 y} - 1} \right)^+ \frac{e^{-\frac{y^2}{2\pi}}}{\sqrt{2\pi}} dy \]

\[ = \int_{\sqrt{\sigma^2}}^{\infty} \frac{e^{-\frac{1}{2}(y^2 - 2\sqrt{\sigma^2 y} + \sigma^2)}}{\sqrt{2\pi}} dy - \int_{\sqrt{\sigma^2}}^{\infty} \frac{e^{-\frac{y^2}{2\pi}}}{\sqrt{2\pi}} dy \]

\[ = \left( \frac{\sigma^2}{2} + \sqrt{\sigma^2} - (\frac{\sigma^2}{2} - \sigma^2) \right) \]

\[ = N(\sqrt{\sigma^2}/2) - N(-\sqrt{\sigma^2}/2) \]

\[ = 2N(\sqrt{\sigma^2}/2) - 1, \]

where \(N\) is the distribution function for a standard Gaussian r.v. and where

\[ V = \int_{T_1}^{T_2} \left( \int_{T_2}^{T_3} \sigma(s,u) du \right)^2 ds. \]